

## CLASSIFICATION OF SIMPLY CONNECTED FOUR- DIMENSIONAL $RR$ -MANIFOLDS

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**ABSTRACT.** Let  $(M, g)$  be a Riemannian manifold. We assume that there is a mapping  $s: M \rightarrow I(M)$ , where  $I(M)$  is the group of isometries of  $(M, g)$ , such that  $s_x = s(x)$ ,  $\forall x \in M$ , has  $x$  as a fixed isolated point, then  $(M, g)$  is called a Riemannian  $s$ -manifold. If the tensor field  $S$  on  $M$  defined by the relation  $S_x = (ds_x)_x$ ,  $\forall x \in M$ , is differentiable and invariant by each isometry  $s_x$ , then the manifold  $(M, g)$  is called a regularly  $s$ -symmetric Riemannian manifold. The aim of the present paper is to classify simply connected four-dimensional regularly  $s$ -symmetric Riemannian manifolds.

**Introduction.** Let  $(M, g)$  be a connected Riemannian manifold. We assume that there is a family of isometries  $\{s_x: x \in M\}$  on  $(M, g)$  such that to each point  $x \in M$  corresponds an isometry  $s_x$  which has this point fixed isolated. The isometry  $s_x$  is called Riemannian symmetry at  $x$  or simply symmetry at  $x$ . The family of Riemannian symmetries  $\{s_x: x \in M\}$  is called Riemannian  $s$ -structure or simply  $s$ -structure on  $(M, g)$ .

Every Riemannian  $s$ -structure  $\{s_x: x \in M\}$  on  $(M, g)$  determines a tensor field  $S$  of type  $(1, 1)$  defined by  $S_x = (ds_x)_x$ . This tensor field is called symmetry tensor field of the  $\{s_x: x \in M\}$ . If  $S$  is smooth then the  $s$ -structure  $\{s_x: x \in M\}$  is called smooth. If the tensor field  $S$  of a smooth  $s$ -structure  $\{s_x: x \in M\}$  is invariant by each  $s_x$ , then it is called regular. A Riemannian manifold  $(M, g)$  with a regular  $s$ -structure is called regularly  $s$ -symmetric Riemannian manifold or briefly  $RR$ -manifold and is denoted by  $(M, g, s)$ . If there is an integer  $k$  such that  $s_x^k = \text{id}$ ,  $\forall x \in M$ , then the  $RR$ -manifold is called regularly  $k$ -symmetric Riemannian manifold and the positive integer  $k$  is called regular order of the  $s$ -structure  $\{s_x: x \in M\}$ .

The aim of the present paper is the classification of all simply connected regularly  $s$ -symmetric Riemannian manifolds of four dimensions. There are exactly two categories of such manifolds. The first category contains the Riemannian symmetric spaces of four dimensions, which are known [9, pp. 283-289].

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The second category contains only one *RR*-manifold whose regular order is four, which is given in §7.

2. In this section we give some additional definitions and a few known results.

Let  $(M, g, s)$  be an *RR*-manifold with regular  $s$ -structure  $\{s_x: x \in M\}$ . It was mentioned above that to this  $s$ -structure  $\{s_x: x \in M\}$  we can associate a tensor field  $S$  of type  $(1, 1)$ . If  $S$  is a parallel tensor field, then the  $s$ -structure  $\{s_x: x \in M\}$  is called parallel regular  $s$ -structure on  $(M, g)$ .

The following theorem is known [6].

**THEOREM (I).** *Any regularly  $s$ -symmetric Riemannian manifold is regularly  $k$ -symmetric for some  $k$ , i.e. of finite regular order.*

From the above theorem we conclude that the classification of simply connected four-dimensional regularly  $s$ -symmetric Riemannian manifolds is reduced to the classification of *RR*-manifolds of finite regular order.

In the classification below we assume the regular order  $k \geq 3$  because if  $k = 2$ , then the *RR*-manifolds are Riemannian symmetric spaces, which have been classified [9, pp. 283–289].

We also assume that the regular  $s$ -structure on  $M$  is not parallel, because if it is parallel then the manifold is a locally symmetric space and since it is simply connected, then  $M$  is a symmetric space [7, p. 457].

Let  $(M, g, s)$  be an *RR*-manifold with regular  $s$ -structure  $\{s_x: x \in M\}$ . The following are known [6].

**LEMMA (I).** *Let  $(M, g, s)$  be an *RR*-manifold, and let  $B$  be an  $S$ -invariant tensor field of type  $(1, 2)$  on  $(M, g, s)$ . Let  $\theta_1, \dots, \theta_n$  be the eigenvalues of  $S$  at a point  $P \in M$  and  $U_1, \dots, U_n$  a corresponding basis of eigenvectors in the complex vector space  $M_P^{\mathbb{C}} = M_P \otimes \mathbb{C}$ . Then:*

- (a) *If  $\theta_i \theta_j$  is not an eigenvalue, then  $B(U_i, U_j) = 0$ .*
- (b) *If  $\theta_i \theta_j$  is an eigenvalue  $\theta_k$ , then  $B(U_i, U_j)$  belongs to the eigenspace corresponding to  $\theta_k$ .*

**COROLLARY (I).** *Let  $(M, g, s)$  be an *RR*-manifold with a nonparallel regular  $s$ -structure  $\{s_x: x \in M\}$ . If  $\theta_1, \dots, \theta_n$  are the eigenvalues of  $S$  at a point  $P \in M$ , then in the set of all products  $\{\theta_i \theta_j, 1 \leq i < j \leq n\}$  there is at least one eigenvalue.*

For a simply connected four-dimensional homogeneous Riemannian manifold the following theorem is known [4, p. 363].

**THEOREM (II).** *Let  $M = G/H$  be a simply connected four-dimensional homogeneous Riemannian manifold. If  $\dim H > 1$  then  $M$  is a symmetric space.*

For any RR-manifold we have the theorem [7, p. 452],

**THEOREM (III).** *Every RR-manifold  $(M, g, s)$  is a homogeneous space, i.e.  $M = G/H$ .*

It is known that  $\dim M = \dim G - \dim H$ , where  $G$  is a transitive Lie group of isometries of  $M$  and  $H$  the isotropy subgroup of  $G$  at a point of  $M$ . If  $\dim M = 4$ , then the maximal dimensions of  $G$  and  $H$  are ten and six, respectively and their minimal dimensions are four and zero respectively.

In the classification below we shall study only the case  $\dim H \leq 1$ , because if  $\dim H \geq 2$ , the manifold  $M$  is symmetric space and the classification of such spaces has been done.

3. Let  $(M, g, s)$  be a simply connected four-dimensional RR-manifold with nonparallel regular  $s$ -structure  $\{s_x: x \in M\}$  of order  $k$ . The manifold  $M$  is a homogeneous space, i.e.  $M = G/H$ . Let  $\bar{\nabla}$  be the canonical connection of the second kind on the homogeneous space  $M = G/H$ . Let  $\bar{R}, \bar{T}$  be the curvature and torsion tensor fields of  $\bar{\nabla}$ , respectively.

**PROPOSITION (I).** *If  $S$  is the symmetry tensor field on  $(M, g, s)$ , then the eigenvalues of  $S$  are of the form  $\varphi, \bar{\varphi}, \varphi^2, \bar{\varphi}^2$ .*

**PROOF.** Let  $P$  be a point of the manifold  $M$ . Then  $S_P$  is an orthogonal transformation on the tangent space  $T_P(M)$ . By complexification of  $T_P(M)$  we obtain the complex vector space  $T_P^{\mathbb{C}}(M) = \mathbb{C} \otimes T_P(M)$  and from the linear transformation  $S_P$  we have another linear transformation on  $T_P^{\mathbb{C}}(M)$ , which is denoted by  $S_P^{\mathbb{C}}$ .

The dimension of the vector space  $T_P^{\mathbb{C}}(M)$  is four over  $\mathbb{C}$ . Therefore there are four eigenvalues of  $S_P^{\mathbb{C}}$ , which have the form

$$(3.1) \quad \theta_1 = \varphi_1 = \cos \frac{2\pi\nu}{k} + i \sin \frac{2\pi\nu}{k} = e^{2\pi i\nu/k},$$

$$\theta_2 = \bar{\varphi}_1 = \cos \frac{2\pi\nu}{k} - i \sin \frac{2\pi\nu}{k} = e^{-2\pi i\nu/k},$$

$$(3.2) \quad \theta_3 = \varphi_2 = \cos \frac{2\pi\lambda}{k} + i \sin \frac{2\pi\lambda}{k} = e^{2\pi i\lambda/k},$$

$$\theta_4 = \bar{\varphi}_2 = \cos \frac{2\pi\lambda}{k} - i \sin \frac{2\pi\lambda}{k} = e^{-2\pi i\lambda/k}.$$

Since the regular  $s$ -structure  $\{s_x: x \in M\}$  on RR-manifold  $(M, g, s)$  is not parallel, we conclude that the set

$$I = \{\theta_i\theta_j: 1 \leq i < j \leq 4\} = \{\varphi_1\bar{\varphi}_1 = \varphi_2\bar{\varphi}_2 = 1, \varphi_1\varphi_2, \varphi_1\bar{\varphi}_2, \bar{\varphi}_1\varphi_2, \bar{\varphi}_1\bar{\varphi}_2\}$$

contains at least one eigenvalue of  $S_P^{\mathbb{C}}$ .

In order to define the order of the nonparallel regular  $s$ -structure  $\{s_x: x \in M\}$  we distinguish the following cases:

(i) There are two distinct eigenvalues of  $S_P^C$ , that is,  $\theta_1 = \theta_3 = \varphi_1 = \varphi_2$  and  $\theta_2 = \theta_4 = \bar{\varphi}_1 = \bar{\varphi}_2$ . Therefore the set  $I$  contains the elements  $I = \{\varphi_1\bar{\varphi}_1 = \varphi_2\bar{\varphi}_2 = 1, \varphi_1^2, \bar{\varphi}_1^2\}$  which has at least one eigenvalue of  $S_P^C$ . Therefore we have  $\varphi_1^2 = \bar{\varphi}_1$  from which we obtain  $\bar{\varphi}_1^2 = \varphi_1, \varphi_1^3 = 1$  and  $\bar{\varphi}_1^3 = 1$ . Hence the order of  $\{s_x: x \in M\}$  is three and the conditions of the proposition are satisfied, i.e.  $\varphi = \varphi_1, \bar{\varphi} = \bar{\varphi}_1, \varphi^2 = \bar{\varphi}_1, \bar{\varphi}^2 = \varphi_1$ .

(ii) We assume that the distinct eigenvalues of  $S_P^C$  are three. Therefore the two equal eigenvalues must be  $-1$ , i.e.  $\varphi_2 = \bar{\varphi}_2 = -1$ . In this case the set  $I = \{1, -\varphi_1, -\bar{\varphi}_1\}$  must contain at least one eigenvalue of  $S_P^C$ . Hence we have  $-\varphi_1 = \bar{\varphi}_1$  from which we obtain  $\varphi_1 = i, \bar{\varphi}_1 = -i$ . Therefore if the order of the nonparallel  $s$ -structure  $\{s_x: x \in M\}$  is four, then the eigenvalues of  $S_P^C$  are  $\varphi_1 = i, \bar{\varphi}_1 = -i, \varphi_2 = \bar{\varphi}_2 = -1$  which satisfy the conditions of the proposition i.e.  $\varphi_1 = \varphi = i, \bar{\varphi}_1 = \bar{\varphi} = -i, \varphi_2 = \varphi^2 = -1, \bar{\varphi}_2 = \bar{\varphi}^2 = -1$ .

(iii) Finally, we assume that all eigenvalues of  $S_P^C$  are distinct. Hence the collection  $I$  consists of the following elements  $\{\varphi\bar{\varphi}_1 = \varphi_2\bar{\varphi}_2 = 1, \varphi_1\varphi_2, \varphi_1\bar{\varphi}_2, \bar{\varphi}_1\varphi_2, \bar{\varphi}_1\bar{\varphi}_2\}$ . Now we suppose that  $\varphi_1\bar{\varphi}_2 = \bar{\varphi}_1 \rightarrow \bar{\varphi}_2 = \bar{\varphi}_1^2$  which, by means of (3.1) and (3.2), takes the form  $e^{2\pi i(\nu-\lambda)/k} = e^{-2\pi i\nu/k}$  from which we have

$$(3.3) \quad \lambda = -kp + 2\nu, \quad \text{where } p \in \mathbb{Z}.$$

We can assume that  $k > \lambda > \nu \geq 1$  and therefore the eigenvalues of  $S_P^C$  are all distinct.

The relations (3.2) by means of (3.3) take the form

$$(3.4) \quad \begin{aligned} \theta_3 = \varphi_2 &= \cos \frac{4\pi\nu}{k} + i \sin \frac{4\pi\nu}{k} = e^{4\pi i\nu/k}, \\ \theta_4 = \bar{\varphi}_2 &= \cos \frac{4\pi\nu}{k} - i \sin \frac{4\pi\nu}{k} = e^{-4\pi i\nu/k}. \end{aligned}$$

If we denote by  $\varphi = \varphi_1$ , then from (3.1) and (3.4) we obtain that the eigenvalues of  $S_P^C$  are of the form  $\varphi = \varphi_1, \bar{\varphi} = \bar{\varphi}_1, \varphi^2 = \varphi_2, \bar{\varphi}^2 = \bar{\varphi}_2$ . If the order of  $\{s_x\}$  is equal to five, then the set  $I$  contains all the eigenvalues of  $S_P^C$ , that is  $\{\varphi_1\varphi_2 = \bar{\varphi}_2, \varphi_1\bar{\varphi}_2 = \bar{\varphi}_1, \bar{\varphi}_1\varphi_2 = \varphi_1, \varphi_1\bar{\varphi}_2 = \varphi_2\}$ .

Finally, if the order of  $\{s_x\}$  is greater than five, then the set  $I$  contains the following eigenvalues  $(\varphi_1\bar{\varphi}_2 = \bar{\varphi}_1, \bar{\varphi}_1\varphi_2 = \bar{\varphi}_1)$  of  $S_P^C$ .

If we make other assumptions, other than  $\varphi_1\bar{\varphi}_2 = \bar{\varphi}_1$ , i.e.  $\varphi_1\varphi_2 = \bar{\varphi}_1$ , we obtain the same results. We always distinguish two cases when the order of  $\{s_x\}$  is five and when the order of  $\{s_x\}$  is greater than five. Q.E.D.

Therefore the classification of all simply connected four-dimensional  $RR$ -manifolds  $(M, g, s)$  is reduced to studying the following cases: (i)  $\text{order}(\{s_x\}) = 3$ ,

(ii) order  $(\{s_x\}) \geq 6$ , order  $(\{s_x\}) = 5$  and finally order  $(\{s_x\}) = 4$ , which will be studied separately below.

4. We assume that the nonparallel regular  $s$ -structure  $\{s_x\}$  of the  $RR$ -manifold  $(M, g, s)$  has order 3. If  $P \in M$ , then the linear transformation  $S_P^C$  on  $T_P^C(M)$  has two distinct eigenvalues.

**PROPOSITION (II).** *If  $(M, g, s)$  is a simply connected four-dimensional  $RR$ -manifold whose regular  $s$ -structure  $\{s_x: x \in M\}$  has order 3, then the Riemannian manifold  $(M, g)$  is a symmetric space.*

**PROOF.** The  $RR$ -manifold  $(M, g, s)$  is a homogeneous space  $M = G/H$ . Let  $T_0(M)$  be the tangent space of  $M$  at its origin  $0$ . From the symmetry tensor field  $S$  we obtain a linear transformation  $S_0^C$  on  $T_0(M)$  whose two distinct eigenvalues are conjugate complex numbers. We also obtain a linear transformation  $S_0$  on  $T_0(M)$ . Therefore there exists an orthonormal base  $\{f_1, f_2, f_3, f_4\}$  of  $T_0(M)$  such that  $S_0$  can be written in matrix form.

$$S_0 = \begin{pmatrix} \cos \frac{2\pi}{3} & -\sin \frac{2\pi}{3} & 0 & 0 \\ \sin \frac{2\pi}{3} & \cos \frac{2\pi}{3} & 0 & 0 \\ 0 & 0 & \cos \frac{2\pi}{3} & -\sin \frac{2\pi}{3} \\ 0 & 0 & \sin \frac{2\pi}{3} & \cos \frac{2\pi}{3} \end{pmatrix}.$$

Let  $L(H)$  be the Lie algebra of all real endomorphisms  $A$  on  $T_0(M)$  which, as derivations on the tensor algebra of the vector space  $T_0(M)$ , satisfy the relations

$$A(S_0) = A(g_0) = A(\bar{T}_0) = A(\bar{R}_0) = 0.$$

Since we have  $\bar{\nabla} \bar{T} = \bar{\nabla} \bar{R} = \bar{\nabla} D = \bar{\nabla} S = 0$  where  $D$  is the difference tensor field of type  $(1, 2)$  on  $M$  defined by the relation [1, p. 137]

$$D(Y, X) = (\nabla S)(S^{-1}Y, (I - S)^{-1}X) = (\nabla_{(I-S)^{-1}X} S)(S^{-1}Y),$$

we conclude that  $\bar{R}_0(X, Y) \in L(H)$ ,  $\forall X, Y \in T_0(M)$ .

From the relation  $A(S_0) = 0$  we obtain that the linear transformations  $A, S_0$  commute. Therefore  $A$  can be represented by a matrix as follows:

$$A = \begin{pmatrix} \alpha_1 & \alpha_2 & \beta_1 & \beta_2 \\ -\alpha_2 & \alpha_1 & -\beta_2 & \beta_1 \\ \gamma_1 & \gamma_2 & \delta_1 & \delta_2 \\ -\gamma_2 & \gamma_1 & -\delta_1 & \delta_2 \end{pmatrix}.$$

For the orthonormal base  $\{f_1, f_2, f_3, f_4\}$  of  $T_0(M)$  we have  $g_0(\bar{f}_i, \bar{f}_i) = 1$ , and  $g_0(\bar{f}_i, \bar{f}_j) = 0$ ,  $i \neq j$ . From these relations and since  $A(g_0) = 0$  we conclude that the matrix  $A$  takes the form

$$A = \begin{pmatrix} 0 & a & c & b \\ -a & 0 & -b & c \\ -c & b & 0 & d \\ -b & -c & -d & 0 \end{pmatrix}.$$

If we set

$$(4.1) \quad V_1 = f_1 + if_2, \quad V_2 = f_3 + if_4,$$

then the vectors  $V_1, \bar{V}_1, V_2, \bar{V}_2$  form a base of  $T_0^{\mathbb{C}}(M)$ .

If we apply Lemma (I) in our case we conclude that

$$(4.2) \quad \bar{T}_0(V_1, \bar{V}_1) = 0, \quad \bar{T}_0(V_1, V_2) = \lambda \bar{V}_1 + \mu \bar{V}_2,$$

$$(4.3) \quad \bar{T}_0(V_1, \bar{V}_2) = 0, \quad \bar{T}_0(V_2, \bar{V}_2) = 0.$$

The relations (4.2) and (4.3) by means of (4.1) and after some estimates give

$$(4.4) \quad \bar{T}_0(f_1, f_2) = 0, \quad \bar{T}_0(f_3, f_4) = 0,$$

$$(4.5) \quad \bar{T}_0(f_1, f_3) = -\bar{T}_0(f_2, f_4) = \frac{1}{2}(\zeta f_1 + \xi f_2 + \sigma \xi_3 + \tau f_4),$$

$$(4.6) \quad \bar{T}_0(f_1, f_4) = \bar{T}_0(f_2, f_3) = \frac{1}{2}(\xi f_1 - \zeta f_2 + r f_3 - \sigma f_4).$$

From the relations  $A(\bar{T}_0) = 0$  and (4.4), (4.5), (4.6) we obtain

$$(4.7) \quad (2a + d)\zeta = -b\sigma + cr, \quad (2a + d)\xi = br - c\sigma,$$

$$(4.8) \quad (a + 2d)\sigma = -b\zeta - c\xi, \quad (a + 2d)r = -b\xi + c\zeta.$$

The Lie algebra  $L(G)$  of  $G$  can be decomposed as follows:  $L(G) = L(H) \oplus T_0(M)$  whose Lie bracket is defined by

$$(4.9) \quad [X, Y] = -\bar{T}_0(X, Y) - \bar{R}_0(X, Y), \quad X, Y \in T_0(M),$$

$$(4.10) \quad [A, X] = AX, \quad [A, B] = AB - BA, \quad A, B \in L(H), X \in T_0(M).$$

From the relations (4.4), (4.5), (4.6), (4.9), (4.10) and from the fact that  $\dim L(H) \leq 1$  we have

$$(4.11) \quad 2[f_1, f_2] = 2\alpha e,$$

$$(4.12) \quad 2[f_1, f_3] = -(\xi f_1 + \xi f_2 + \sigma f_3 + \tau f_4) + 2\beta e,$$

$$(4.13) \quad 2[f_1, f_4] = -(\xi f_1 - \xi f_2 + \tau f_3 - \sigma f_4) + 2\gamma e,$$

$$(4.14) \quad 2[f_2, f_3] = -(\xi f_1 - \xi f_2 + \tau f_3 - \sigma f_4) + 2\delta e,$$

$$(4.15) \quad 2[f_2, f_4] = (\xi f_1 + \xi f_2 + \sigma f_3 + \tau f_4) + 2\vartheta e,$$

$$(4.16) \quad 2[f_3, f_4] = 2\nu e,$$

$$(4.17) \quad [f_1, e] = af_2 + cf_3 + bf_4, \quad [f_2, e] = -af_1 - bf_3 + cf_4,$$

$$(4.18) \quad [f_3, e] = -cf_1 + bf_2 + df_4, \quad [f_4, e] = -bf_1 - cf_2 - df_4,$$

where  $e$  is the unit vector of  $L(H)$ .

The above Lie brackets must satisfy the Jacobi identity. From this we obtain, except the relations (4.7) and (4.8), the following ones

$$(4.19) \quad -2a\beta + 2c\alpha + \xi\sigma - \zeta\tau = 0, \quad 2a\delta + 2b\alpha + \xi\tau + \zeta\sigma = 0,$$

$$(4.20) \quad b\beta + c\delta = 0, \quad 2b\delta - 2c\beta + 2d\alpha + \sigma^2 + \tau^2 = 0,$$

$$(4.21) \quad -2a\vartheta + 2cv + \xi\sigma - \zeta\tau = 0, \quad -2a\gamma + 2bv + \xi\tau + \zeta\sigma = 0,$$

$$(4.22) \quad b\vartheta - c\gamma = 0, \quad -2b\gamma - 2c\vartheta + 2dv + \sigma^2 + \tau^2 = 0,$$

$$(4.23) \quad 2cv - 2d\beta + \xi\sigma - \zeta\tau = 0, \quad 2bv - 2d\gamma + \xi\tau + \zeta\sigma = 0,$$

$$(4.24) \quad b\beta - c\gamma = 0, \quad 2av - 2b\gamma - 2c\beta + \xi^2 + \zeta^2 = 0,$$

$$(4.25) \quad 2cv - 2d\vartheta + \xi\sigma - \zeta\tau = 0, \quad 2bv + 2d\delta + \xi\tau + \zeta\sigma = 0,$$

$$(4.26) \quad b\vartheta + c\delta = 0, \quad 2av + 2b\beta - 2c\vartheta + \xi^2 + \zeta^2 = 0,$$

$$(4.27) \quad b(\beta + \vartheta) - c(\gamma - \delta) = 0,$$

$$(4.28) \quad \alpha\delta + b(\alpha - v) + d\gamma = 0, \quad -a\vartheta + c(\alpha - v) + d\beta = 0,$$

$$(4.29) \quad a\beta + c(\alpha - v) - d\vartheta = 0, \quad a\gamma - b(\alpha - v) + d\delta = 0,$$

$$(4.30) \quad \tau(\beta - \vartheta) - \sigma(\gamma + \delta) = 0, \quad \sigma(\beta - \vartheta) + \tau(\gamma + \delta) = 0,$$

$$(4.31) \quad \xi(\beta - \vartheta) - \zeta(\gamma + \delta) = 0, \quad \zeta(\beta - \vartheta) + \xi(\gamma + \delta) = 0.$$

Since  $\bar{\nabla}$  is the canonical connection of the second kind on  $M$ , then we have [5, Vol. II, p. 193]

$$(4.32) \quad \bar{R}_0(X, Y)Z = -[[X, Y]_b, Z], \quad X, Y, Z \in T_0(M).$$

The formula (4.32) for the vectors  $f_1, f_2, f_3$  and by means of (4.11) and the first of (4.18) becomes

$$\bar{R}_0(f_1, f_2)f_3 = \alpha(-cf_1 + bf_2 + df_3);$$

applying the derivation  $A$  to this we obtain

$$\begin{aligned} \bar{R}_0(A(f_1), f_2)f_3 + \bar{R}_0(f_1, A(f_2))f_3 + \bar{R}_0(f_1, f_2)A(f_3) \\ = \alpha(-cA(f_1) + bA(f_2) + dA(f_3)) \end{aligned}$$

from which by virtue of (4.11)–(4.18) we have

$$(4.33) \quad \alpha b = 0, \quad b[b(\beta + \vartheta) - c(\gamma - \delta)] = 0,$$

$$(4.34) \quad c[b(\beta + \vartheta) - c(\gamma - \delta)] = 0, \quad d[b(\beta + \vartheta) - c(\gamma - \delta)] = 0.$$

The same formula (4.32) for the vectors  $f_1, f_2, f_4$  and by means of (4.11) and the second of (4.18) takes the form

$$(4.35) \quad \bar{R}_0(f_1, f_2)f_4 = -\alpha(bf_1 + cf_2 + df_3).$$

If we apply the derivation  $A$  to (4.35) and after some calculations we obtain

$$(4.36) \quad \alpha c = 0, \quad b[b(\beta + \vartheta) - c(\gamma - \delta)] = 0,$$

$$(4.37) \quad c[b(\beta + \vartheta) - c(\gamma - \delta)] = 0, \quad d[b(\beta + \vartheta) - c(\gamma - \delta)] = 0.$$

From the relations (4.7) and (4.8) by virtue of the first of (4.33) and (4.36) and if  $\alpha \neq 0$  we conclude that

$$(4.38) \quad (2a + d)\xi = (2a + d)\zeta = (\alpha + 2d)r = (a + 2d)\sigma = 0.$$

If  $2a + d \neq 0$  and  $a + 2d \neq 0$ , then from (4.38) we have  $\xi = \zeta = r = \sigma = 0$ . If  $2a + d = 0$  and  $a + 2d \neq 0$ , then (4.38) implies  $\xi = \zeta = 0$ . Therefore the second of (4.24) becomes  $2av = 0$ , which implies  $v = 0$ , because if  $a = 0$ , then  $d = 0$  and hence  $a + 2d = 0$  which contradicts our assumption  $a + 2d \neq 0$ .

From the second of (4.20) and (4.22) we conclude that  $d(\alpha - v) = 0$  and since  $\alpha \neq 0$  and  $v = 0$  we have  $d = 0$  and from  $2a + d = 0$  we take  $a = 0$ , which contradicts our assumption  $2d + a \neq 0$ . Therefore the assumption  $2a + d = 0$  and  $a + 2d \neq 0$  is not valid.

If  $2a + d = 0$  and  $a + 2d = 0$ , then we obtain  $a = d = 0$  and from the second of (4.20) and (4.24) we have  $\xi = \zeta = r = \sigma = 0$ .

Now, we assume that  $\alpha = 0$ . The systems of equations (4.30) and (4.31) have solutions different from zero, i.e.  $r = \sigma = 0$  and  $\xi = \zeta = 0$ , if the following relations are satisfied:

$$(4.39) \quad \beta - \vartheta = 0, \quad \gamma + \delta = 0.$$

From the assumption  $\alpha = 0$ , the relations (4.28), (4.29), the second of (4.20) and (4.22) and the relations (4.39) we conclude that  $bv = cv = dv = 0$ . If  $v \neq 0$  we obtain again  $b = c = d = 0$  and from this  $\xi = \zeta = r = \sigma = 0$ .

If  $v = 0$  and because  $\alpha = 0$ , then (4.28) and (4.29) by virtue of (4.39) take the form

$$\beta(a - d) = 0, \quad \gamma(a - d) = 0.$$

If  $a \neq 0$ , then  $\beta = \gamma = \delta = \vartheta = 0$ , then from the second of (4.22) and (4.24) we have  $\xi = \zeta = r = \sigma = 0$ .

If we apply the derivation  $A$  to the relations

$$\bar{R}_0(f_1, f_3)f_4 = \beta[f_4, e], \quad \bar{R}_0(f_1, f_3)f_2 = \beta[f_2, e]$$

which are obtained from (4.32) for the pairs  $(X = f_1, Y = f_2, Z = f_4)$  and  $(X = f_1, Y = f_3, Z = f_4)$  respectively, after some calculations we have

$$(4.41) \quad \beta c = 0, \quad b(a\delta - bv + b\alpha + d\vartheta) = 0,$$

$$(4.42) \quad c(a\delta - bv + b\alpha + d\vartheta) = 0, \quad d(a\delta - bv + b\alpha + d\vartheta) = 0,$$

$$(4.43) \quad \beta c = 0, \quad a(a\delta - bv + b\alpha + d\gamma) = 0,$$

$$(4.44) \quad b(a\delta - bv + b\alpha + d\gamma) = 0, \quad c(a\delta - bv + b\alpha + d\gamma) = 0.$$

If  $c \neq 0$  and since  $\alpha = v = 0$ , then from (4.41), (4.42), (4.43) and (4.44) we obtain  $\beta = 0$ ,  $a\delta + d\vartheta = 0$  and  $a\delta + d\gamma = 0$ . We assume that  $\gamma \neq 0$ , because the case  $\gamma = 0$  has been studied. If  $\gamma \neq 0$ , then we have  $a = d = 0$  and hence  $\xi = \zeta = r = \sigma = 0$ .

If  $c \neq 0$  and  $\beta \neq 0$ , then from the first of (4.40) we conclude that  $a = d$ . Since  $a\delta + d\vartheta = 0$  and  $a = d \neq 0$  we have  $\delta + \vartheta = 0$ . The relation (4.27) by virtue of  $\delta + \vartheta = 0$  and first of (4.39) becomes  $\beta b = 0$  and since  $\beta \neq 0$  we have  $b = 0$ . From this we take  $\xi = \zeta = r = \sigma = 0$ .

From the above we conclude that in every case we have  $\xi = \zeta = r = \sigma = 0$ . Therefore the manifold  $M = G/H$  with the canonical connection of the second kind is an affine symmetric space. Hence the  $RR$ -manifold  $(M, g, s)$  is a symmetric Riemannian manifold.

5. Let  $(M, g, s)$  be a simply connected four-dimensional  $RR$ -manifold whose regular  $s$ -structure  $\{s_x\}$  has order greater than or equal to six. Therefore the linear transformation  $S_0^C$  on the vector space  $T_0^C(M) = T_0(M) \otimes \mathbb{C}$  has four distinct eigenvalues, where  $T_0(M)$  is the tangent space of  $M = G/H$  at its origin  $0$ .

PROPOSITION (III). *Let  $(M, g, s)$  be a simply connected four-dimensional RR-manifold whose regular  $s$ -structure  $\{s_x\}$  has order greater than five. Then the Riemannian manifold  $(M, g)$  is a symmetric space.*

PROOF. There exists a linear transformation  $S_0$  on  $T_0(M)$  from which we obtain the linear transformation  $S_0^C$  on  $T_0^C(M)$ . Therefore there exists an orthonormal base  $\{f_1, f_2, f_3, f_4\}$  of  $T_0(M)$  such that  $S_0$  can be represented by the matrix.

$$S_0 = \begin{pmatrix} \cos \frac{2\pi\nu}{k} & -\sin \frac{2\pi\nu}{k} & 0 & 0 \\ \sin \frac{2\pi\nu}{k} & \cos \frac{2\pi\nu}{k} & 0 & 0 \\ 0 & 0 & \cos \frac{4\pi\nu}{k} & -\sin \frac{4\pi\nu}{k} \\ 0 & 0 & \sin \frac{4\pi\nu}{k} & \cos \frac{4\pi\nu}{k} \end{pmatrix}, \quad k \geq 6.$$

For the relations  $A(S_0) = A(g_0) = 0$ , which are valid for every  $A \in L(H)$ , we conclude that  $A$  can be represented as a matrix with respect to the  $\{f_1, f_2, f_3, f_4\}$  by the following form

$$(5.1) \quad A = \begin{pmatrix} 0 & -c & 0 & 0 \\ c & 0 & 0 & 0 \\ 0 & 0 & 0 & -p \\ 0 & 0 & p & 0 \end{pmatrix}.$$

If we set

$$(5.2) \quad V_1 = f_1 + if_2, \quad V_2 = f_3 + if_4,$$

then  $V_1, V_2, \bar{V}_1, \bar{V}_2$  form a base of  $T_0^C(M)$ , which are the eigenvectors of  $S_0^C$  whose eigenvalues are  $\varphi, \bar{\varphi}, \varphi^2, \bar{\varphi}^2$  respectively, where  $\varphi = \cos(2\pi\nu/k) + i \sin(2\pi\nu/k)$ . From Lemma (I) we obtain

$$(5.3) \quad \bar{T}_0(V_1, \bar{V}_1) = 0, \quad \bar{T}_0(V_1, V_2) = 0,$$

$$(5.4) \quad \bar{T}_0(V_1, \bar{V}_2) = \lambda \bar{V}_1, \quad \bar{T}_0(\bar{V}_1, V_2) = \bar{\lambda} V_1,$$

$$(5.5) \quad \bar{T}_0(\bar{V}_1, \bar{V}_2) = 0, \quad \bar{T}_0(V_1, \bar{V}_2) = 0.$$

From the form of  $A$ , given by (5.1), we have

$$(5.6) \quad A(f_1) = -cf_2, \quad A(f_2) = cf_1, \quad A(f_3) = -pf_4, \quad A(f_4) = pf_3.$$

From the relations (5.2) by virtue of (5.6) we obtain

$$(5.7) \quad A(V_1) = -cV_1, \quad A(\bar{V}_1) = c\bar{V}_1, \quad A(V_2) = -piV_2, \quad A(\bar{V}_2) = pi\bar{V}_2.$$

Since  $A(T_0) = 0$  and applying the derivation  $A$  to the first of (5.4) we take

$$\bar{T}_0(A(V_1), \bar{V}_2) + \bar{T}_0(V_1, A(\bar{V}_2)) = \lambda A(\bar{V}_1),$$

which by means of (5.7) and after some estimates implies  $p = 2c$ . Therefore the matrix (5.1) and the relations (5.6) take the form

$$(5.8) \quad A = c \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \end{pmatrix},$$

$$(5.9) \quad A(f_1) = -cf_2, \quad A(f_2) = cf_1, \quad A(f_3) = -2cf_4, \quad A(f_4) = 2cf_3.$$

The relations (5.3), (5.4) and (5.5) by virtue of (5.2) and after some calculations give

$$(5.10) \quad \bar{T}_0(f_1, f_3) = \bar{T}_0(f_2, f_4) = \frac{1}{2}(\zeta f_1 + \xi f_2),$$

$$(5.11) \quad \bar{T}_0(f_2, f_3) = -\bar{T}_0(f_1, f_4) = \frac{1}{2}(\zeta f_1 - \xi f_2),$$

$$(5.12) \quad \bar{T}_0(f_1, f_2) = \bar{T}_0(f_3, f_4) = 0,$$

where  $\zeta = \operatorname{Re}(\lambda)$  and  $\xi = \operatorname{Im}(\lambda)$ .

Since  $\dim(L(H)) \leq 1$ , then the relations (4.9) and (4.10) by means of (5.6), (5.10), (5.11) and (5.12) imply

$$(5.13) \quad 2[f_1, f_2] = \alpha e, \quad 2[f_1, f_3] = \zeta f_1 + \xi f_2 + \beta e,$$

$$(5.14) \quad 2[f_1, f_4] = -\xi f_1 + \zeta f_2 + \gamma e, \quad 2[f_2, f_3] = \xi f_1 - \zeta f_2 + \delta e,$$

$$(5.15) \quad 2[f_2, f_4] = \zeta f_1 + \xi f_2 + \vartheta e, \quad 2[f_3, f_4] = \nu e,$$

$$(5.16) \quad [f_1, e] = -cf_2, \quad [f_2, e] = cf_1,$$

$$(5.17) \quad [f_3, e] = -2cf_4, \quad [f_4, e] = 2cf_3,$$

where

$$e = \begin{pmatrix} 0 & -c & 0 & 0 \\ c & 0 & 0 & 0 \\ 0 & 0 & 0 & -2c \\ 0 & 0 & 2c & 0 \end{pmatrix}$$

for some fixed  $c$ .

These Lie brackets must satisfy the Jacobi identity from which we take

$$(5.18) \quad c(2\gamma + \delta) = 0, \quad c(2\beta - \vartheta) = 0, \quad c(\gamma + 2\delta) = 0,$$

$$(5.19) \quad c(\beta - 2\vartheta) = 0, \quad c\alpha = 0, \quad cv + \xi^2 + \zeta^2 = 0.$$

If  $c = 0$ , then the system of the equations (5.18) and (5.19) is satisfied when  $\xi = \zeta = 0$  and hence the Riemannian manifold  $(M, g)$  is a symmetric space.

If  $c \neq 0$  and  $v \neq 0$ , then from the system of equations (5.18) and (5.19) we have  $\alpha = \beta = \gamma = \delta = \vartheta = 0$  and therefore the relations (5.13), (5.14), and (5.15) take the form

$$(5.20) \quad [f_1, f_2] = 0, \quad 2[f_1, f_3] = \zeta f_1 + \xi f_2,$$

$$(5.21) \quad 2[f_1, f_4] = -\xi f_1 + \zeta f_2, \quad 2[f_2, f_3] = \xi f_1 - \zeta f_2,$$

$$(5.22) \quad 2[f_2, f_4] = \zeta f_1 + \xi f_2, \quad 2[f_3, f_4] = v e.$$

If  $\dim(L(H)) = 1$ , then  $L(G)$  is a Lie algebra of five dimensions. Let  $L(K)$  be an ideal of  $L(G)$ . Therefore we have  $[\lambda, \mu] \in L(K)$  for every  $\lambda \in L(K)$  and for every  $\mu \in L(G)$ . From the relations (5.16), (5.17), (5.20), (5.21) and (5.22) and after some estimates we conclude that  $L(K) = \{0\}$ . Therefore the Lie algebra  $L(G)$  is simple. This is impossible because  $\dim(L(G)) = 5$ . Therefore the assumption  $\dim(L(H)) = 1$  is false and hence  $\dim(L(H)) = 0$ . From this we conclude that  $v = 0$  which implies  $\xi = \zeta = 0$ . This completes the proof of the proposition, i.e.  $(M, g)$  symmetric space.

6. We assume that  $RR$ -manifold  $(M, g, s)$  is simply connected and four dimensions whose regular  $s$ -structure  $\{s_x\}$  has order five. Therefore the linear transformation  $S_0^C$  on  $T_0^C(M)$  has also four distinct eigenvalues. However this case is a little different from the previous one, because the set  $I$  contains all the eigenvalues of  $S_0^C$ .

**PROPOSITION (IV).** *We consider a simply connected four-dimensional  $RR$ -manifold  $(M, g, s)$  whose regular  $s$ -structure  $\{s_x\}$  has order five. Then the Riemannian manifold  $(M, g)$  is a symmetric space.*

PROOF. It is known that there exists an orthonormal base  $\{f_1, f_2, f_3, f_4\}$  of  $T_0(M)$  such that the linear transformation  $S_0$  on  $T_0(M)$  can be represented by the matrix

$$S_0 = \begin{pmatrix} \cos \frac{2\pi}{5} & -\sin \frac{2\pi}{5} & 0 & 0 \\ \sin \frac{2\pi}{5} & \cos \frac{2\pi}{5} & 0 & 0 \\ 0 & 0 & \cos \frac{4\pi}{5} & -\sin \frac{4\pi}{5} \\ 0 & 0 & \sin \frac{4\pi}{5} & \cos \frac{4\pi}{5} \end{pmatrix}.$$

Since we have  $A(S_0) = A(g_0) = 0$  for every  $A \in L(H)$  we conclude that  $A$  can be represented by the matrix

$$(6.1) \quad A = \begin{pmatrix} 0 & -b & 0 & 0 \\ b & 0 & 0 & 0 \\ 0 & 0 & 0 & -d \\ 0 & 0 & d & 0 \end{pmatrix}$$

with respect to the basis  $\{f_1, f_2, f_3, f_4\}$  of  $T_0(M)$ .

From the form of the matrix  $A$  we have

$$(6.2) \quad A(f_1) = bf_2, \quad A(f_2) = -bf_1, \quad A(f_3) = df_4, \quad A(f_4) = -df_3.$$

If we put

$$(6.3) \quad V_1 = f_1 + if_2, \quad V_2 = f_3 + if_4,$$

then by means of (6.2) and after some estimates we obtain

$$(6.4) \quad A(V_1) = -ibV_1, \quad A(\bar{V}_1) = ib\bar{V}_1, \quad A(V_2) = idV_2, \quad A(\bar{V}_2) = id\bar{V}_2.$$

From the form of the eigenvalues of  $S_0^C$  and by means of Lemma (I) we have

$$(6.5) \quad \bar{T}_0(V_1, \bar{V}_1) = 0, \quad \bar{T}_0(V_1, V_2) = \lambda\bar{V}_1, \quad \bar{T}_0(V_1, \bar{V}_2) = \mu\bar{V}_1,$$

$$(6.6) \quad \bar{T}_0(V_2, \bar{V}_2) = 0, \quad \bar{T}_0(\bar{V}_1, V_2) = \bar{\mu}V_1, \quad \bar{T}_0(\bar{V}_1, \bar{V}_2) = \bar{\lambda}V_2,$$

which by virtue of (6.3) and after some calculations imply

$$(6.7) \quad \bar{T}_0(f_1, f_2) = \bar{T}_0(f_3, f_4) = 0,$$

$$(6.8) \quad \bar{T}_0(f_1, f_3) = \frac{1}{2}(rf_1 + \sigma f_2 + \zeta f_3 + \xi f_4),$$

$$(6.9) \quad \bar{T}_0(f_1, f_4) = \frac{1}{2}(-\sigma f_1 + rf_2 + \xi f_3 - \zeta f_4),$$

$$(6.10) \quad \bar{T}_0(f_2, f_3) = \frac{1}{2}(\sigma f_1 - rf_2 + \xi f_3 - \zeta f_4),$$

$$(6.11) \quad \bar{T}_0(f_2, f_4) = \frac{1}{2}(rf_1 + \sigma f_2 - \zeta f_3 - \xi f_4),$$

where  $r = \operatorname{Re}(\mu)$ ,  $\sigma = \operatorname{Im}(\mu)$ ,  $\zeta = \operatorname{Re}(\lambda)$  and  $\xi = \operatorname{Im}(\lambda)$ .

Since  $A(T_0) = 0$  for every  $A \in L(H)$  and applying the derivations  $A$  to the second and third of (6.5) we take

$$\bar{T}_0(A(V_1), V_2) + \bar{T}_0(V_1, A(V_2)) = \lambda A(\bar{V}_2),$$

$$\bar{T}_0(A(V_1), \bar{V}_2) + \bar{T}_0(V_1, A(\bar{V}_2)) = \mu A(\bar{V}_1),$$

which by means of (6.4) and after some estimates give  $\lambda(2d + b) = 0$ ,  $\mu(2b + d) = 0$  from which we have  $\lambda = \mu = 0$  or  $\lambda = 0$ ,  $\mu \neq 0$  and  $d = -2b$  or  $\lambda \neq 0$ ,  $\mu = 0$  and  $b = -2d$  or finally  $\lambda \neq 0$ ,  $\mu \neq 0$  and therefore  $d = b = 0$ .

If  $\lambda = \mu = 0$ , then the Riemannian manifold  $(M, g)$  is a symmetric space. If  $\lambda = 0$  and  $\mu \neq 0$ , then we are in the case of §5 and hence  $(M, g)$  is a symmetric space.

Now we assume that  $\lambda \neq 0$ ,  $\mu = 0$  and  $b = -2d$  then (6.2), (6.8), (6.9), (6.10) and (6.11) take the form

$$(6.12) \quad A(f_1) = -2df_2, \quad A(f_2) = 2df, \quad A(f_3) = -df_4, \quad A(f_4) = -df_3,$$

$$(6.13) \quad \bar{T}_0(f_1, f_3) = \frac{1}{2}(\zeta f_3 + \xi f_4), \quad \bar{T}_0(f_1, f_4) = \frac{1}{2}(\xi f_3 - \zeta f_4),$$

$$(6.14) \quad \bar{T}_0(f_2, f_3) = \frac{1}{2}(\xi f_3 - \zeta f_4), \quad \bar{T}_0(f_2, f_4) = -\frac{1}{2}(\zeta f_3 + \xi f_4).$$

Since  $\dim(L(H)) \leq 1$ , then from (4.9), (4.10), (6.7), (6.8), (6.9), (6.10) and (6.11) we obtain

$$[f_1, f_2] = \alpha e, \quad [f_1, f_3] = -\frac{1}{2}(\zeta f_3 + \xi f_4) + \beta e,$$

$$[f_1, f_4] = -\frac{1}{2}(\xi f_3 - \zeta f_4) + \gamma e, \quad [f_2, f_3] = -\frac{1}{2}(\xi f_3 - \zeta f_4) + \delta e,$$

$$[f_2, f_4] = \frac{1}{2}(\xi f_3 + \zeta f_4) + \vartheta e, \quad [f_3, f_4] = \nu e,$$

$$[f_1, e] = 2df_2, \quad [f_2, e] = -2df_1, \quad [f_3, e] = -df_4, \quad [f_4, e] = df_3,$$

where  $e$  is a vector of  $L(H)$ .

If we use the same arguments as in §5 we obtain in this case that, the Riemannian manifold  $(M, g)$  is a symmetric space.

Finally, if  $\lambda \neq 0$  and  $\mu \neq 0$ , then  $b = d = 0$  and hence  $\bar{R}_0 = 0$ . Therefore the relation (4.9) becomes  $[X, Y] = -\bar{T}_0(X, Y)$  which by means of (6.5) and (6.6) implies

$$(6.15) \quad [V_1, \bar{V}_1] = 0, \quad [V_1, V_2] = \lambda \bar{V}_2, \quad [V_1, \bar{V}_2] = \mu \bar{V}_1,$$

$$(6.16) \quad [V_2, \bar{V}_2] = 0, \quad [\bar{V}_1, \bar{V}_2] = \bar{\lambda} V_2, \quad [\bar{V}_1, V_2] = \bar{\mu} V_1.$$

These Lie brackets must satisfy the Jacobi identity. Therefore we have

$$[[V_1, \bar{V}_1], V_2] + [[V_2, V_1], \bar{V}_1] + [[\bar{V}_1, V_2], V_1] = 0$$

which by virtue of (6.15) and (6.16) and after some estimates takes the form  $\lambda \bar{\lambda} V_2 = 0$  or  $\lambda = 0$  similarly we have

$$[[V_2, \bar{V}_2], \bar{V}_1] + [[\bar{V}_1, V_1], \bar{V}_2] + [\bar{V}_2, \bar{V}_1], V_1] = 0$$

which by means of (6.15) and (6.16) and after some calculations we obtain  $\mu \bar{\mu} V_1 = 0$  or  $\mu = 0$ .

This completes the proof of Proposition (IV).

7. It has been proved that a simply connected four-dimensional RR-manifold  $(M, g, s)$  whose regular  $s$ -structure  $\{s_x\}$  has order different than four is a symmetric space. Now we assume that the regular  $s$ -structure  $\{s_x\}$  has order four and therefore the linear transformation  $S_0^C$  on the vector space  $T_0^C(M)$  has three distinct eigenvalues which are  $-1, i$  and  $-i$ . The eigenvalue  $-1$  is double.

We also have a linear transformation  $S_0$  on the tangent space  $T_0(M)$  of  $M = G/H$  at its origin  $0$ . Hence there exists an orthonormal base  $\{f_1, f_2, f_3, f_4\}$  of  $T_0(M)$  such that the linear transformation  $S_0$  can be represented by the matrix

$$S_0 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

PROPOSITION (V). *Let  $(M, g, s)$  be a simply connected four-dimensional RR-manifold whose regular  $s$ -structure  $\{s_x\}$  has order four. Then there are four vector fields  $X, Y, Z, W$  on  $M$  satisfying the relations*

$$(i) \quad \begin{aligned} \bar{T}(X, Z) &= -X, \quad \bar{T}(Y, Z) = Y, \quad \bar{T}(X, Y) = 0, \\ \bar{T}(X, W) &= -X, \quad \bar{T}(Y, W) = Y, \quad \bar{T}(W, Z) = 0, \end{aligned}$$

$$(ii) \quad g(X, X) = g(Y, Y) = 1, \quad g(Z, Z) = \frac{1}{|\gamma|^2}, \quad g(W, W) = \frac{1}{|\delta|^2}.$$

PROOF. From the form of the linear transformation  $S_0$  we have

$$(7.1) \quad S_0(f_1) = -f_2, \quad S_0(f_2) = f_1, \quad S_0(f_3) = -f_3, \quad S_0(f_4) = -f_4.$$

If we set

$$(7.2) \quad V_1 = f_1 + if_2, \quad U_2 = f_3, \quad W_2 = f_4$$

then by means of (7.1) we obtain

$$(7.3) \quad S_0(V_1) = -iV_1, \quad S_0(U_2) = -U_2, \quad S_0(W_2) = -W_2.$$

From Lemma (I) and the form of the eigenvalues of  $S_0^{\mathbb{C}}$  we have

$$(7.4) \quad \bar{T}_0(V_1, V_1) = 0, \quad \bar{T}_0(\bar{V}_1, \bar{V}_1) = 0, \quad \bar{T}_0(U_2, W_2) = 0,$$

$$(7.5) \quad \bar{T}_0(V_1, \bar{V}_1) = 0, \quad \bar{T}_0(V_1, U_2) = \gamma\bar{V}_1, \quad \bar{T}_0(V_1, W_2) = \delta\bar{V}_1,$$

$$(7.6) \quad \bar{T}_0(\bar{V}_1, U_2) = \bar{\gamma}V_1, \quad \bar{T}_0(\bar{V}_1, W_2) = \bar{\delta}V_1.$$

The Lie algebra  $L(H)$  consists of linear transformations  $A$  on  $T_0(M)$  which satisfy  $A(S_0) = A(g_0) = A(\bar{R}_0) = A(\bar{T}_0) = 0$ .

From the relations  $A(S_0) = A(g_0) = 0$  we conclude that  $A$  can be represented by the matrix

$$A = \begin{pmatrix} 0 & b & 0 & 0 \\ -b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

with respect to the base  $\{f_1, f_2, f_3, f_4\}$  of  $T_0(M)$ .

From the form of linear transformation  $A$  we obtain

$$(7.7) \quad A(f_1) = -bf_2, \quad A(f_2) = bf_1, \quad A(f_3) = 0, \quad A(f_4) = 0.$$

From (7.2) by virtue of (7.7) we conclude that

$$(7.8) \quad A(V_1) = ibV_1, \quad A(\bar{V}_1) = -ib\bar{V}_1, \quad A(U_2) = 0, \quad A(W_2) = 0.$$

Since  $A(\bar{T}_0) = 0$ , then from the second of (7.5) we have

$$\bar{T}_0(A(V_1), U_2) + \bar{T}_0(V_1, A(V_2)) = A(V_1)$$

which by means of (7.8) becomes  $bi\bar{T}_0(V_1, U_2) = -\gamma bi\bar{V}_1$  which implies  $b = 0$ . Therefore  $A = 0$  and  $L(H) = \{0\}$ .

The complexification of the tangent space  $T_0(M)$  gives the four-dimensional complex vector space  $T_0^{\mathbb{C}}(M)$  which can be written  $T_0^{\mathbb{C}}(M) = L^{\mathbb{C}} \oplus N^{\mathbb{C}}$ , where  $L^{\mathbb{C}}$  is the vector subspace of  $T_0^{\mathbb{C}}(M)$  which is spanned by the vectors  $V_1$  and  $\bar{V}_1$  and  $N^{\mathbb{C}}$  the complexification of the vector space which is the eigenspace of  $S_0$  with eigenvalue  $-1$ .

On the vector space  $L^{\mathbb{C}}$  we consider a linear transformation  $B_{U_2}$  defined by

$$B_{U_2}: V_1 \rightarrow \bar{T}_0(V_1, U_2) = \gamma \bar{V}_1, \quad B_{U_2}: \bar{V}_1 \rightarrow \bar{T}_0(\bar{V}_1, U_2) = \bar{\gamma} V_1$$

which has the property such that  $B_{U_2}^2: V_1 \rightarrow \gamma \bar{\gamma} V_1 = |\gamma|^2 V_1$ .

If we set  $U_2/|\gamma| = Z$ , then we have another linear transformation  $B_Z$  on  $L^{\mathbb{C}}$  with the property  $B_Z^2 = \text{id}$ , whose eigenvalues are  $-1$  and  $+1$ . Let  $X, Y$  be the corresponding real unit eigenvectors determined uniquely up to sign.

Similarly from the third of (7.5) we obtain another linear transformation  $B_W$  on  $L^{\mathbb{C}}$  from which we construct a new linear transformation  $B_W$  on  $L^{\mathbb{C}}$  such that  $B_W^2 = \text{id}$ . Therefore the eigenvalues of  $B_W$  are  $1$  and  $-1$ . Hence  $B_W$  has the same real unit eigenvectors  $X, Y$ .

Since  $\bar{R} = 0$  and  $M$  simply connected, we can extend the vectors  $X, Y, Z, W$  to parallel vector fields on  $M$ , which will be denoted by the same symbols. From the properties of the linear transformations  $B_Z$  and  $B_W$  we obtain

$$(7.9) \quad \bar{T}(X, Z) = -X, \quad \bar{T}(Y, Z) = Y, \quad \bar{T}(X, W) = -X, \quad \bar{T}(Y, W) = Y.$$

The vectors  $X, Y$  form a basis of the vector space  $L$ . Therefore the vectors  $V_1, \bar{V}_1$  can be written

$$V_1 = \alpha_1 X + \beta_1 Y + i(\alpha_2 X + \beta_2 Y), \quad \bar{V}_1 = \alpha_1 X + \beta_1 Y - i(\alpha_2 X + \beta_2 Y),$$

where

$$(7.10) \quad \alpha_1 X + \beta_1 Y = f_1, \quad \alpha_2 X + \beta_2 Y = f_2.$$

The first of (7.4) by means of the first of (7.2) can be written  $\bar{T}_0(f_1 + if_2, f_1 - if_2) = 0$  from which we have  $\bar{T}_0(f_1, f_2) = 0$ , that by virtue of (7.10) becomes

$$(7.11) \quad (\alpha_1 \beta_2 - \beta_1 \alpha_2) \bar{T}_0(X, Y) = 0.$$

The third of the relations (7.4) can be written

$$(7.12) \quad \bar{T}_0(U_2/|\gamma|, W_2/|\delta|) = \bar{T}_0(Z, W) = 0.$$

From (7.11) and (7.12) we have

$$(7.13) \quad \bar{T}(X, Y) = \bar{T}(Z, W) = 0.$$

(ii) The vectors  $X, Y, Z, W$  form an orthonormal base of the tangent space  $T_0(M)$  with respect to the inner product on it defined by  $g_0$ . Therefore we obtain

$$(7.14) \quad g_0(X, X) = g_0(Y, Y) = 1, \quad g_0(U_2, U_2) = g_0(W_2, W_2) = 1.$$

The last two relations of (7.14) by means of  $U_2/|\delta| = Z$  and  $W_2/|\delta| = W$  become

$$(7.15) \quad g_0(Z, Z) = 1/|\gamma|^2, \quad g_0(W, W) = 1/|\delta|^2.$$

Therefore (7.14) and (7.15) for the vector fields  $X, Y, Z, W$  take the form

$$g(X, X) = g(Y, Y) = 1, \quad g(Z, Z) = 1/|\gamma|^2, \quad g(W, W) = 1/|\delta|^2.$$

**PROPOSITION (VI).** *Let  $(M, g, s)$  be a simply connected four-dimensional RR-manifold whose regular  $s$ -structure  $\{s_x: x \in M\}$  has order 4. Then there are four vector fields  $X, Y, Z, W$  satisfying*

$$[X, Z] = [X, W] = X, \quad [Y, Z] = [Y, W] = -Y, \quad [X, Y] = [W, Z] = 0,$$

$$SX = \epsilon Y, \quad SY = \epsilon' X, \quad SZ = -Z, \quad SW = -W,$$

where  $\epsilon, \epsilon' = \pm 1$  and  $S$  is the symmetry tensor field.

**PROOF.** It is known that the following formula holds

$$(7.16) \quad \bar{T}(X_1, Y_1) = \bar{\nabla}_{X_1} Y_1 - \bar{\nabla}_{Y_1} X_1 - [X_1, Y_1].$$

Since  $\bar{R} = 0$ ,  $\bar{\nabla}_{X_1} Y_1 = \bar{\nabla}_{Y_1} X_1 = 0$ . Therefore from (7.16) we obtain

$$(7.17) \quad \bar{T}(X, Y) = -[X, Y].$$

The relations (7.9) and (7.13) by virtue of (7.17) imply

$$(7.18) \quad [X, Z] = X, \quad [Y, Z] = -Y, \quad [X, Y] = 0,$$

$$(7.19) \quad [X, W] = X, \quad [Y, W] = -Y, \quad [W, Z] = 0.$$

The symmetry tensor field  $S$  of  $\{s_x\}$  has order 4 and for each  $P \in M$ ,  $S_P$  is an orthogonal transformation on  $T_P(M)$  which satisfies the relations

$$S_P(Z_P) = -Z_P, \quad S_P(W_P) = -W_P, \quad \forall P \in M$$

from which we obtain

$$(7.20) \quad SZ = -Z, \quad SW = -W.$$

It is known that the tensor field  $S$  preserves the tensor field  $\bar{T}$ . Therefore from the first and the second of (7.9) we obtain

$$S(\bar{T}(X, Z)) = \bar{T}(SX, SZ) = -SX, \quad S(\bar{T}(Y, Z)) = \bar{T}(SY, SZ) = SY,$$

which by means of (7.20) take the form

$$(7.21) \quad \bar{T}(SX, Z) = SX, \quad \bar{T}(SY, Z) = -SY.$$

Similarly from the third and the fourth of (7.9) by virtue of (7.20) we conclude that

$$(7.22) \quad \bar{T}(SX, W) = SX, \quad \bar{T}(SY, W) = -SY.$$

From the relations (7.9) by means of (7.21) and (7.22) we have

$$(7.23) \quad SX = \epsilon Y, \quad SY = \epsilon' X,$$

where  $\epsilon, \epsilon' = \pm 1$ .

**THEOREM (IV).** *Let  $(M, g, s)$  be a simply connected four-dimensional RR-manifold whose regular  $s$ -structure  $\{s_x: x \in M\}$  has order four. Then for the manifold  $(M, g)$  we have  $M = R^4(x_1, x_2, x_3, x_4)$  provided with the Riemannian metric*

$$ds^2 = e^{2(x_3+x_4)} dx_1^2 + e^{-2(x_3+x_4)} dx_2^2 + \frac{1}{\lambda^2} dx_3^2 + \frac{1}{\mu^2} dx_4^2,$$

$\lambda, \mu$  constants  $\neq 0$ .

*It can also be represented as a Lie group  $G = G_1 \times R_*$  where  $G_1$  is isomorphic to the Lie group of all hyperbolic motions of an oriented affine plane and  $R_* = R - \{0\}$  provided with a special left-invariant Riemannian metric.*

**PROOF.** Since  $L(H) = \{0\}$ , we conclude that the Lie algebra  $L(G)$  of  $G$  is four dimensions whose Lie bracket satisfies the relations (7.18) and (7.19). The adjoint group  $\text{Int}(L(G))$  of  $L(G)$  is generated by the elements of the form  $e^{2dX_1}, X_1 \in L(G)$  [3, p. 117].

The group  $\text{Int}(L(G))$  can be identified with the group of all matrices of the form

$$W = \begin{pmatrix} e^{-\gamma} & 0 & \lambda & 0 \\ 0 & e^{\gamma} & \mu & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

If we obtain as a Lie group  $G_1$  the set of all matrices of the form  $W$ , then  $G_1$  is isomorphic to the Lie group  $G_2$  consisting of matrices of the form

$$Y = \begin{pmatrix} e^{-\gamma} & 0 & \lambda \\ 0 & e^{\gamma} & \mu \\ 0 & 0 & 1 \end{pmatrix}.$$

The Lie group  $G_2$  is isomorphic to the Lie group of all hyperbolic motions of an oriented affine plane:

$$u' = e^{-\gamma}u + \lambda, \quad v' = e^{\gamma}v + \mu.$$

From the above and [3, p. 119] we conclude that the Lie group  $G = G_1 \times R_*$ . From the form of the Lie group  $G$ ,  $\dim G = 4$  and the results given in [4, p. 362] we conclude that  $G$  is diffeomorphic to  $R^4(x_1, x_2, x_3, x_4)$ .

The Lie algebra  $L(G)$  of left vector fields on  $G$  has a base consisting of the following vector fields:

$$(7.25) \quad X = e^{-(x_3+x_4)} \frac{\partial}{\partial x_1}, \quad Y = e^{(x_3+x_4)} \frac{\partial}{\partial x_2}, \quad Z = \frac{\partial}{\partial x_3}, \quad W = \frac{\partial}{\partial x_4}.$$

These vector fields satisfy the equations

$$(7.26) \quad [X, Z] = X, \quad [Y, Z] = -Y, \quad [X, Y] = 0,$$

$$(7.27) \quad [X, W] = X, \quad [Y, W] = -Y, \quad [Z, W] = 0.$$

We define a left invariant Riemannian metric  $g$  on  $G$  by the conditions

$$(7.28) \quad g(X, Y) = g(X, Z) = g(X, W) = g(Y, Z) = g(Y, W) = g(Z, W) = 0,$$

$$(7.29) \quad g(X, X) = g(Y, Y) = 1, \quad g(Z, Z) = \frac{1}{\lambda^2}, \quad g(W, W) = \frac{1}{\mu^2}, \quad \lambda, \mu \neq 0.$$

The metric  $g$  with respect to the coordinate system  $(x_1, x_2, x_3, x_4)$

$$(7.30) \quad ds^2 = e^{2(x_3+x_4)} dx_1^2 + e^{-2(x_3+x_4)} dx_2^2 + \frac{1}{\lambda^2} dx_3^2 + \frac{1}{\mu^2} dx_4^2.$$

At each point  $P$  of the manifold  $M$  we have a linear transformation  $S_P$  on  $T_P(M)$  which is defined by

$$S_P(X_P) = \epsilon Y_P, \quad S_P(Y_P) = \epsilon X_P, \quad S_P(Z_P) = -Z_P, \quad S_P(W_P) = -W_P.$$

The linear transformation  $S_P$  defines a regular  $s$ -structure  $\{s_x\}$  on the manifold  $M$  whose order is four.

On the manifold  $M = G$  we define the canonical  $(-)$ -connection  $\bar{\nabla}$  for which we have

$$(7.31) \quad \bar{T}(U, V) = -[U, V], \quad \bar{\nabla}\bar{T} = \bar{R} = 0, \quad \forall U, V \in L(G).$$

The vector fields  $X, Y, Z, W$  are parallel with respect to the connection  $\bar{\nabla}$ .

From the relations (7.21), (7.22), (7.28) and (7.29) by means of (7.20) and (7.23) we obtain that  $\bar{T}$  and  $g$  are invariants by  $S$  and an addition  $S$  and  $g$  are parallel with respect to the connection  $\bar{\nabla}$ .

From the known theorem [6] we obtain that  $S$  gives rise to a nonparallel regular  $s$ -structure  $\{s_x\}$  of order four on the Riemannian manifold  $(G, g)$  and since

the manifold  $M$  is simply connected we conclude that, this is isometric to the manifold  $(G, g)$ .

From the fact that the vector fields  $X, Y, Z, W$  are parallel with respect to the connection  $\bar{\nabla} = \nabla - D$ , then we conclude that

$$\nabla_V(U) = D(V, U), \quad V, U \in \{X, Y, Z, W\}.$$

From this relation and the formula

$$2g(D(V, U'), V') = g(\bar{T}(V, U), V') + g(\bar{T}(V, V'), U) + g(\bar{T}(U, V'), V),$$

$U, V, V' \in \{X, Y, Z, W\}$  and after some calculations we obtain

$$(7.32) \quad \begin{aligned} \nabla_X X &= -\lambda^2 Z - \mu^2 W, & \nabla_X Y &= 0, & \nabla_X Z &= \nabla_X W = X, \\ \nabla_Y X &= 0, & \nabla_Y Y &= \lambda^2 Z + \mu^2 W, & \nabla_Y Z &= \nabla_Y W = -Y, \\ \nabla_Z X &= \nabla_Z Y = \nabla_Z Z = \nabla_Z W = 0, \\ \nabla_W X &= \nabla_W Y = \nabla_W Z = \nabla_W W = 0. \end{aligned}$$

Therefore the Gauss curvature in the basic 2-directions are given by  $\sigma(X, Y) = \lambda^2 + \mu^2$ ,  $\sigma(V, U) = 0$ , where  $V, U \in \{X, Y, Z, W\}$  but it is not simultaneously  $U = X$  and  $V = Y$  or  $U = Y$  and  $V = X$ . It is easily seen that the only tangent 2-planes with sectional curvature  $\lambda^2 + \mu^2$  are those in the distribution spanned by  $\{X, Y\}$ .

Therefore this family must be preserved by any isometry  $I$  of  $(M, g)$ . From this we obtain

$$I_*Z = \epsilon_1 Z, \quad I_*W = \epsilon_2 W, \quad I_*X = X \cos \vartheta + Y \sin \vartheta, \quad I_*Y = \epsilon(-X \sin \vartheta + Y \cos \vartheta),$$

where  $\epsilon_1, \epsilon_2$  and  $\epsilon = \pm 1$  and the parameter  $\vartheta$  is a real function on  $M$ .

Also each isometry  $I$  of  $(M, g)$  is an affine transformation, i.e.  $\nabla_{I_*U} I_*V = I_*(\nabla_U V)$  for any two vector fields  $U, V$  on  $M$ . By examining the cases  $(U = X, V = Z)$  ( $U = X, V = W$ ),  $(U = Y, V = Z)$  and  $(U = Y, V = W)$  and by means of (7.32) and after some calculations we obtain

$$\begin{aligned} I_*X &= \delta X, & I_*Y &= \delta' Y, & I_*Z &= Z, & I_*W &= W, \\ I_*X &= \delta Y, & I_*Y &= \delta' X, & I_*Z &= -Z, & I_*W &= -W, \end{aligned}$$

where  $\delta, \delta' = \pm 1$ .

The isotropy subgroup of  $G$  at any point  $P \in M$  is finite of order 8 and contains exactly four symmetries. The manifold  $(M, g)$  with this metric is not symmetric.

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