THE MULTIPLICITY FUNCTION OF A LOCAL RING

BY

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ABSTRACT. Let $A$ be a local ring with maximal ideal $m$. Let $f \in A$, and define $\mu_A(f)$ to be the multiplicity of the $A$-module $A/Af$ with respect to $m$. Under suitable conditions $\mu_A(fg) = \mu_A(f) + \mu_A(g)$. The relationship of $\mu_A$ to reduction of $A$, normalization of $A$ and a quadratic transform of $A$ is studied. It is then shown that there are positive integers $n_1, \ldots, n_s$ and rank one discrete valuations $\nu_1, \ldots, \nu_s$ of $A$ centered at $m$ such that $\mu_A(f) = n_1 \nu_1(f) + \cdots + n_s \nu_s(f)$ for all regular elements $f$ of $A$.

Let $A$ be a nonnull noetherian local ring with maximal ideal $m$. Let $d$ be the (Krull) dimension of $A$, the maximal length of a chain of prime ideals of $A$, excluding $A$. Let $k$ be the residue field $A/m$, and let $G_mA$ be the associated graded ring of $A$ with respect to $m$.

Let $f \in A$. If $A/Af$ is of dimension $d - 1$ define $\beta_A(f)$ to be $e_m(A/Af)$, the multiplicity of the $A$-module $A/Af$ relative to $m$ in dimension $d - 1$ ([6, p. V-2] or the multiplicity of the local ring $A/Af$ ([7, p. 294], or [3, p. 75])). If $A/Af$ is of dimension $d$, define $\beta_A(f)$ to be $\infty$. Call $\beta_A(f)$ the multiplicity of $f$ (at $m$ in $A$).

If $A$ is a regular local ring, $\mu_A$ is known to be the order valuation of $A$ [3, 40.2, p. 154]. If $A$ is entire $\beta_A(fg) = \beta_A(f) + \beta_A(g)$ (Proposition 1, §1). The order function $\nu_A$ of $A$ [7, p. 249] satisfies $\nu_A(f + g) \geq \min \{\nu_A(f), \nu_A(g)\}$, and (Proposition 2, §1) $\nu_A$ is a valuation if and only if $\mu_A$ is a multiple of $\nu_A$.

If the ideal $(0)$ is unmixed in $A$, $\mu_A$ is found to extend to the components of $A$ (Lemma 2, §2). If $A$ is of dimension one, $\mu_A$ is found to extend to the normalization of $A$ (Lemma 3, §2). The extension of $A$ to the first neighborhood ring of $A$ (a quadratic transform of $A$) is found to preserve $\mu_A$ (Lemma 4, §3).

This is used to prove the theorem of §4, that there are positive integers $n_1, \ldots, n_s$ and discrete rank one valuations $\nu_1, \ldots, \nu_s$ of $A$ centered at $m$ such that for every regular element $f$ of $A$

$$
\mu_A(f) = n_1 \nu_1(f) + \cdots + n_s \nu_s(f).
$$

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The valuations \( v_1, \ldots, v_s \) arise from (dimension one) normalization of the first neighborhood ring of \( A \), and each \( n_i \) is the product of the length of a primary component of \((0)\) in \( A \) of dimension \( d \), the multiplicity of a \( d \)-dimensional component of the tangent cone of \( A \) at the origin, the index of a normalization and another factor arising from a nonfinite normalization of an entire local ring of dimension one.

Let \( p \) be a prime ideal of the noetherian ring \( A \). The \textit{depth} of \( p \) will denote throughout the Krull dimension of \( A/p \).

1. Elementary properties of \( \mu_A \). For an \( A \)-module \( M \) let \( l_A(M) \) denote the length of \( M \) as an \( A \)-module. If \( p \) is a prime ideal of \( A \) and if \( \mathfrak{U} \) is an ideal of \( A \) let

\[
\lambda_p(\mathfrak{U}) = l_A_p(A_p/\mathfrak{U} \mathfrak{U}).
\]

**Proposition 1.** Let \( f \) and \( g \) be two elements of a local ring \( A \), and assume either that \( f \) is a regular element of \( A \) or that \( \mu_A(f) = \infty \). Then

\[
\mu_A(fg) = \mu_A(f) + \mu_A(g).
\]

**Proof.** If \( \mu_A(f) = \infty \), then \( f \) and \( fg \) are contained in a prime ideal of \( A \) of depth \( d \), and \( \mu_A(fg) = \infty \).

Let \( f \) be a regular element of \( A \) and assume that \( \mu_A(g) \) is finite. By [6, p. V-3], for any \( h \in A \) such that \( \mu_A(h) \) is finite,

\[
\mu_A(h) = \sum_p \lambda_p(Ah) e_m(A/p)
\]

where the sum ranges over all prime ideals \( p \) of \( A \) of depth \( d - 1 = \dim A - 1 \),

\[
0 \to A|f|/Afg \to A|f|g/fg \to A|f|/Af \to 0
\]

is exact, \( A|f|/Afg \cong A/Ag \) as \( A \)-modules, \( \lambda_p(Afg) = \lambda_p(Af) + \lambda_p(Ag) \), and the proposition follows.

**Remark.** Let \( A = k[x, y]_{(x, y)} = k[X, Y]/(X^2, XY) \). By direct computation \( \mu_A(y) = 3 \) and \( \mu_A(y^2) = 5 \). Thus \( \mu_A(fg) \) need not be \( \mu_A(f) + \mu_A(g) \) if neither \( f \) nor \( g \) is regular and if both \( \mu_A(f) \) and \( \mu_A(g) \) are finite.

**Proposition 2.** Let \( A \) be an entire local ring and suppose the order function \( v_A \) of \( A \) is a valuation. Then

\[
\mu_A = \em(A) v_A.
\]

**Proof.** \( G_m A \) is entire, and if \( f \) is a nonzero element of \( A \), \( f \) is superficial of degree \( v_A(f) \). Thus [7, Lemma 4, p. 286], \( \mu_A(f) = \em(A/|Af|) = \em(A) \cdot v_A(f) \).

**Corollary.** If \( A \) is a regular local ring then \( \mu_A \) is the order valuation.
Remark. Let $A$ be an entire local ring of dimension one and suppose the order function $v_A$ of $A$ is a valuation. Then $\mathcal{G}_m A$ is an entire graded ring over $k = A/m$ of dimension one which must be the polynomial ring in one variable over $k$, $\dim_k m/m^2 = 1$, $A$ is therefore a regular local ring, and $\mu_A = v_A$.

The following proposition gives a geometric definition of $\mu_A$. The local ring $A$ is said to be affine if it is the homomorphic image of a localization of a polynomial ring over a field.

Proposition 3. Let $A$ be an entire affine local ring which has an infinite residue field $k = A/m$. Then $A$ is the homomorphic image of an affine regular local ring $B$. Let $p$ be the kernel of this homomorphism of $B$ onto $A$, which is local, and notice that $B$ is equicharacteristic with residue field $k$. Let $d$ be the dimension of $A$. Then for every regular element $f$ of $A$,

$$\mu_A(f) = \min_{f_1, \ldots, f_d-1} \{ i(Z(B/p) \cdot Z(B/Bf_1) \cdots Z(B/Bf_{d-1}) \cdot Z(B/Bf), m) \}$$

where the minimum is taken over all $f_1, \ldots, f_{d-1} \in A$ for which the intersection is proper. For the definition and notation of the right-hand side of the equation see [1] and [6, §V–C].

Remark. By applying Lemma 2, §2 to $\mu_A(f) = e_{(f_1, \ldots, f_{d-1})}(A)$, by the additivity of $Z(B/p)$ and the linearity of $i(\cdot, m)$, the hypothesis that $A$ be entire may be dropped from Proposition 3.

Remark. This proposition does not necessarily hold if the residue field is finite. For let $k$ be the field of $p^n$ elements, and let $A = k[X_1, X_2]$. Letting $\mu'$ denote the formula of the right-hand side of the equality of the proposition, $\mu'(X_2(\Pi_{a \in k}(X_1 - aX_2))) = p^n + 2$, whereas $\mu_A(X_2(\Pi_{a \in k}(X_1 - aX_2))) = p^n + 1$.

Proof of Proposition 3.

$$\mu_A(f) = e_{(f_1, \ldots, f_{d-1})}(A/\mathfrak{A})$$

for some $f_1, \ldots, f_{d-1} \in m$ [7, Theorem 22, p. 294]

$$= \min_{f_1, \ldots, f_{d-1}} \{ e_{(f_1, \ldots, f_{d-1})}(A/\mathfrak{A}) \}$$

where $(f_1, \ldots, f_{d-1})$ is an open ideal of $A/\mathfrak{A}$ [7, Lemma 2, p. 285]. The elements $f_1, \ldots, f_{d-1}$ have representatives in $B$ and in $A$, and consider $f_1, \ldots, f_{d-1}$ to be in either $B$, $A$ or $A/\mathfrak{A}$.

Let $N$ be the maximal ideal of $B$, let $\hat{B}$ be the $N$-adic completion of $B$, and let $\hat{p} = \hat{B}p$. $\hat{A} = \hat{B}/\hat{p}$. $\hat{B} \simeq k[[X_1, \ldots, X_n]]$ for some $n$. Let $(\hat{f}_1, \ldots, \hat{f}_{d-1})$ be an open ideal of $A/\hat{A}$.
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\[ e(f_1, \ldots, f_{d-1})(A/Af) = e(f_1, \ldots, f_{d-1}, f)(A) \]

([4, p. 300] for \((0) : A A f) = (0)\)

\[ = e(f_1, \ldots, f_{d-1}, f)(\mathcal{B}[\hat{p}]) \]

\[ = e(f_1 \otimes 1, \ldots, f_{d-1} \otimes 1, f \otimes 1) \]

\[ (((\mathcal{B} \otimes_k \mathcal{B}[\hat{p}])(X_1 \otimes 1 - 1 \otimes X_1, \ldots, X_n \otimes 1 - 1 \otimes X_n)) \]

\[ = e(X_1 \otimes 1 - 1 \otimes X_1, \ldots, X_n \otimes 1 - 1 \otimes X_n, f_1 \otimes 1, \ldots, f_{d-1} \otimes 1, f \otimes 1)(\mathcal{B} \otimes_k \mathcal{B}[\hat{p}]) \]

[4, p. 300], for \(X_1 \otimes 1 - 1 \otimes X_1, \ldots, X_n \otimes 1 - 1 \otimes X_n\) is a prime sequence in \(\mathcal{B} \otimes_k \mathcal{B}[\hat{p}]\) as will be shown below. As will also be shown below, \(f_1 \otimes 1, \ldots, f_{d-1} \otimes 1, f \otimes 1\) is a prime sequence in \(\mathcal{B} \otimes_k \mathcal{B}[\hat{p}]\). The above equality may now be continued.

\[ e(f_1, \ldots, f_{d-1})(A/Af) \]

\[ = e(X_1 \otimes 1 - 1 \otimes X_1, \ldots, X_n \otimes 1 - 1 \otimes X_n)(\mathcal{B}[f_1, \ldots, f_{d-1}, f)(\mathcal{B}[\hat{p}]) \]

[4, p. 300] \[= \chi(B(f_1, \ldots, f_{d-1}, f), B[p]) \]

[6, p. V-12]

\[ = \psi(Z(B[p]) \cdots Z(B[Bf]) \cdots Z(B[Bf_d]) \cdots Z(Bf), m) \]

[6, p. V-20].

It must be shown that \(X_1 \otimes 1 - 1 \otimes X_1, \ldots, X_n \otimes 1 - 1 \otimes X_n\) is a prime sequence in

\(\mathcal{B} \otimes_k \mathcal{A} \approx (\cdots ((\mathcal{A}[[X_1]])[[X_2]]) \cdots )[[X_n]].\)

By induction, it follows from the fact that \(X_1 - \alpha\) is a regular element of \(R[[X_1]]\) for any \(\alpha \in R\) where \(R\) is a noetherian ring.

It must also be shown that \(f \otimes 1, f_1 \otimes 1, \ldots, f_{d-1} \otimes 1\) is a prime sequence in \(\mathcal{B} \otimes_k \mathcal{A}\). \((f, f_1, \ldots, f_{d-1})\) has height \(d\) in \(B\), so \(f, f_1, \ldots, f_{d-1}\) is a prime sequence in \(B\). Let \(R\) and \(S\) be two rings containing as a subring the field \(k\), and let \(\alpha\) be a regular element of \(R\). \(0 \rightarrow R \rightarrow R \rightarrow R \otimes_k S \rightarrow S\) is exact, and \(\alpha \otimes 1\) is a regular element of \(R \otimes_k S\). It follows immediately that \(f \otimes 1, f_1 \otimes 1, \ldots, f_{d-1} \otimes 1\) is a prime sequence of \(B \otimes_k A\). If \(R\) is a Zariski ring and if \(\widehat{R}\) is the completion of \(R\), then \(f_1, \ldots, f_{d-1}\) is a prime sequence in \(R\) if and only if \(f_1, \ldots, f_d\) is a prime sequence in \(\widehat{R}\) [7, Chapter VIII, §5]. \(A\) and \(B\) are affine over \(k\), so \(B \otimes_k A\) is noetherian, and \(B \otimes_k A\) is a Zariski ring with completion \(\mathcal{B} \otimes_k \mathcal{A}\). Thus \(f \otimes 1, f_1 \otimes 1, \ldots, f_{d-1} \otimes 1\) is a prime sequence in \(\mathcal{B} \otimes_k \mathcal{A}\).

2. The behavior of \(\mu_A\) under reduction of \(A\) and integral extension of \(A\). Let \(A\) be a nonimbedded local ring (the associated prime ideals of \(0\) in \(A\) are all
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Let $IA$ be the integral closure of $A$ contained in $QA$, the total quotient ring of $A$. The minimal (height zero) prime ideals of $A$, $IA$ and $QA$ are in a bijective correspondence. Let $N$ be a minimal prime ideal of $A$. Then $\lambda_N(0) = \lambda_{(IA)N}(0) = \lambda_{(QA)N}(0)$, and $IA/N = IA/IN$ where $IN = (IA)N$. $IA \simeq A'_1 \oplus \cdots \oplus A'_n$ where $I(A'_i) = A'_i$ and $A'_i$ has a unique minimal prime ideal $N'_i$.

$$A'_1 \oplus \cdots \oplus A'_{i-1} \oplus N'_i \oplus A'_{i+1} \oplus \cdots \oplus A'_n = IN_i$$

for $i = 1, \ldots, n$ are the minimal prime ideals of $IA$. Thus a maximal ideal of $IA$ contains a unique minimal prime ideal.

**Lemma 1.** Let $A$ be a dimension one nonimbedded local ring with maximal ideal $m$. Let $IA$ be the integral closure of $A$ in its total quotient ring $QA$. There are only a finite number of prime ideals $m_1, \ldots, m_s$ of $IA$ lying over $m$, and the indices $[IA/m_i : A/m]$ are finite for $i = 1, \ldots, s$. Let $A_i = (IA)_{m_i}$.

If $f$ is an element of $A$,

$$l_A(A/af) = \sum_{i=1}^{s} n_i \lambda_{N'_i}(0) [IA/m_i : A/m] l_{A_i}(A_i/af)$$

the $n_i$ being positive integers depending only upon $A/N$ where $N$ is the nil radical of $A$.

If $IA/IN$ is a noetherian $A$-module, then $n_i = 1$ for $i = 1, \ldots, s$. The $n_i$ may be greater than one, for in Nagata’s example [3, E 3.2, p. 206], $s = 1$ and $n_1 = p$.

**Proof.** It may be assumed that $f$ is a regular element of $A$, for otherwise both sides of the equality are infinite. Let $B$ be a finite $A$-submodule of $IA$, and let $a \in A$ be regular and such that $aB \subset A$.

$$l_A(B/af) = l_A(Ba/Baf) = l_A(A/af) - l_A(A/Ba) - l_A(Baf/Aaf)$$

$$= l_A(A/af) + l_A(A/af) - l_A(A/Ba) - l_A(Ba/Af)$$

$$= l_A(A/af).$$

By [3, Theorem 21.2, p. 70], or by the first part of the proof of [7, Theorem 24, p. 297],

$$l_A(A/af) = \sum_{i=1}^{s} [B/p_i : A/m] l_B(B/p_i/B_{p_i}f)$$

where $p_1, \ldots, p_s$ are the prime ideals of $B$ lying over $m$. There are a finite number of prime ideals in $IA$ lying over $m$, for $s_B \leq l_A(A/af)$. Let $m_1, \ldots, m_s$ be the maximal ideals of $IA$. Note that

$$l_A(\text{dir lim}_i M_i) \leq \max_i \{l_A(M_i)\},$$

$IA/m_i = \text{dir lim}_B B/B \cap m_i$ and $[IA/m_i : A/m]$ is finite.
Let \( \alpha_i \in IA \) be such that \( \alpha_i \in m_i \) and \( \alpha_i \notin \bigcup_{j \neq i} m_j \). Let \( \beta_1, \ldots, \beta_t \in IA \) be such that

\[
[A[\beta_1, \ldots, \beta_t]/(m_i \cap A[\beta_1, \ldots, \beta_t]) : A/m] = [IA/m_i : A/m]
\]

for \( i = 1, \ldots, s \). Let \( A' = A[\alpha_1, \ldots, \alpha_s, \beta_1, \ldots, \beta_t] \). By the formula above, letting \( A \) be \( A' \cap m_i \), it can be assumed that \( s = 1 \) and \([IA/m_i : A/m] = 1\). Then for a finite extension \( B \subset IA \) of \( A \), \( l_A(A/Af) = l_B(B/Bf) \). The nil radical \( N \) of \( A \) is now a prime ideal.

First assume that \( I(A/N) \) is a noetherian \( A/N \)-module. By a finite extension of \( A \) in \( IA \) it can be assumed that \( A/N \) is normal, and thus that \( A/N \) is a regular local ring of dimension one [3, Theorem 33.2, p. 115 and Theorem 21.4, p. 40]. Let \( x \in m/N \) generate \( m/N \) in \( A/N \). Let

\[
(0) = N_0 \subset N_1 \subset \cdots \subset N_{t-1} = NA \subset N_t = A_N
\]

be a composition series of \( A_N \) over \( A_N \), and let \( n_i = A \cap N_i \). \( n_i/n_{i-1} \) is a principal \( A/N \)-module: If \( \alpha_1, \ldots, \alpha_q \in n_i/n_{i-1} \) are nonzero and generate \( n_i/n_{i-1} \) as an \( A \) or \( A/N \)-module, there are \( v, v_j \in A \sim N \) such that \( v_j \alpha_j = v_0 \alpha_1 \) for \( j = 1, \ldots, q \) (for there is a bijective correspondence between the ideals of \( A_N \) and their contractions in \( A \)). Viewed as \( A/N \)-modules, \( \alpha_i = u_jx^j\alpha_1 \) where \( u_j \) is a unit in \( A/N \) and where \( t_j \) is an integer. Let \( t_k = \min \{t_1, \ldots, t_q\} \).

Let \( n_i/n_{i-1} = A\alpha_k \). So there are \( a_1, \ldots, a_t \in N \) with \( n_i = (a_1, \ldots, a_t) \). For \( i = 1, \ldots, t \),

\[
0 \to \frac{n_i + Af}{n_{i-1} + Af} \to \frac{A}{n_{i-1} + Af} \to \frac{A}{n_i + Af} \to 0
\]

is exact. Map \( A \to (n_i + Af)/(n_{i-1} + Af) \) by \( y \mapsto ya_i + (f, a_1, \ldots, a_{i-1}) \). Suppose \( ya_i \in (f, a_1, \ldots, a_{i-1}) \). There are \( c, c_1, \ldots, c_{i-1} \in A \) such that \( cf = c_1a_1 + \cdots + c_{i-1}a_{i-1} - ya_i \), \( y \notin N \) and \( n_i \) is \( N \)-primary because it is the contraction of an \( A_N \)-primary ideal, so \( c \in (a_1, \ldots, a_i) \). Thus there is an element \( b \) of \( A \) such that \( ya_i - ba_j \in (a_1, \ldots, a_{i-1}) \), \( a_i \notin (a_1, \ldots, a_{i-1}) \) which is \( N \)-primary, so \( y - bf \in N \). Hence

\[
(n_i + Af)/(n_{i-1} + Af) \simeq A/(N + Af),
\]

and

\[
l_A(A/Af) = \lambda_N(0) l_{A/N}(A/(N + Af)) = \lambda_N(0) l_{IA/IN}(IA/IA \cdot f).
\]

Now drop the assumption that \( I(A/N) \) is a finite \( A/N \)-module. Let \( \hat{A} \) be the \( m \)-adic completion of \( A \). \( l_{\hat{A}}(A/Af) = l_{\hat{A}}(\hat{A}/\hat{A}f) \). The pair \( A, m \) is a Zariski ring, so \( (A/N)^\wedge \simeq \hat{A}/\hat{N} \). \( \hat{A} \) and \( \hat{N} \) are unmixed [7, Chapter VIII, §4]. Letting \( M_j \) be a minimal prime ideal of \( \hat{A} \), \( I(\hat{A}/M_j) \) is a finite \( \hat{A}/M_j \)-module [3, Theorem 32.1, p. 112]. By the finite case above
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\[ l_{\mathcal{A}}(\mathcal{A}/\mathcal{A}f) = \sum_{i} \lambda_{M_{i}}(0) l_{\mathcal{A}/M_{i}}((\mathcal{A}/M_{i}))/((\mathcal{A}/M_{i})f). \]

A \subseteq \mathcal{A} \subseteq \mathcal{A}_{M_{i}} canonically. Let

\[(0) = N_{0} \subseteq N_{1} \subseteq \cdots \subseteq N_{t-1} = A_{N} N \subseteq N_{t} = A_{N}

be a composition series of A_{N}. N_{t} \otimes_{A_{N}} \mathcal{A}_{M_{i}} can be refined into a composition series for A_{M_{i}}. Now N_{j}/N_{j-1} \cong A_{N}/A_{N} N_{j}, this completion and localization are exact, so N_{j}/N_{j-1} \otimes_{A_{N}} A_{M_{i}} are all isomorphic for i = 1, \ldots, t of length

\[ \lambda_{M_{j}}(N_{j}) = l_{(\mathcal{A}/N)_{M_{j}}/(\mathcal{A}/N)f}, \]

and \( \lambda_{M_{j}}(0) = \lambda_{N}(0) \lambda_{M_{j}}(N_{j}) \). Thus

\[ l_{\mathcal{A}}(\mathcal{A}/\mathcal{A}f) = \lambda_{N}(0) l_{\mathcal{A}/N}((\mathcal{A}/N}/(\mathcal{A}/N)f), \]

and it follows that

\[ l_{\mathcal{A}}(A/\mathcal{A}f) = \lambda_{N}(0) l_{A/IN}(A/(N + A)f). \]

\( I(A/N) = I/A/IN \), and \( I/A/IN \) is a regular local ring of dimension one [3, Theorem 33.2, p. 115 and Theorem 12.4, p. 40]. Let \( x \) be a generator of the maximal ideal \( m_{1} \) of \( I \) and let \( u \) be a unit in \( IA \) such that for some integer \( n \), \( f = ux^{n} \). By a finite extension of \( A \) it may be assumed that \( u \) and \( x \) are elements of \( A \). To finish the proof, notice that \( l_{I}(I/IAx) = 1 \) and \( IN \subseteq (IA)x \) so that

\[ \frac{l_{A/IN}(A/(A/N)/(A/N)f)}{l_{IA}(I/IAx)} = l_{A/IN}(A/(A/N)-(A/N)x). \]

Let \( n_{1} = l_{A/IN}(A/(A/N)-(A/N)x). \)

**Lemma 2.** Let \( A \) be a local ring with maximal ideal \( m \), let \( N_{1}, \ldots, N_{n} \) be the prime ideals of \( A \) of depth \( d = \dim A \). For every regular element \( f \) of \( A \)

\[ \mu_{A}(f) = \sum_{i=1, \ldots, n} \lambda_{N_{i}}(0) \mu_{A/N_{i}}(f + N_{i}). \]

**Proof.** If \( \dim A = 0 \), the formula holds trivially. Let \( p \) be a prime ideal of \( A \) of depth \( d - 1 \) and containing \( f \). Then \( B = A_{p} \) is of dimension one and is nonimbedded, for \( f \) is a regular element. Note that if \( N_{i} \subseteq p \), then \( \lambda_{N_{i}}(0) = \lambda_{BN_{i}}(0) \). By Lemma 1, applied to \( B \) and to \( B/BN_{i} \) for \( N_{i} \subseteq p \),

\[ l_{B}(B/Bf) = \sum_{N_{i} \subseteq p} \lambda_{N_{i}}(0) l_{B/BN_{i}}((B/BN_{i})/(B/BN_{i})f), \]

and by [6, p. V-3],

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\[ \mu_A(f) = \sum_p l_p(A/A_f) e_m(A/p) = \sum_i \lambda_{N_i}(0) l_{p|N_i}(A/N_i) e_m(A/p) = \sum_{i=1}^s \lambda_{N_i}(0) \mu_{A/N_i}(f + N_i). \]

**Lemma 3.** Let \( A \) be a dimension one local ring with maximal ideal \( m \), let \( m_1, \ldots, m_s \) be the prime ideals of \( I_A \) lying over \( m \), and let \( A_i = I_A m_i \). For every regular element \( f \) of \( A \),

\[ \mu_A(f) = \sum_{i=1}^s \lambda_{N_i}(0)n_i[I_A/m_i : A/m]\mu_A(f) \]

for some positive integers \( n_1, \ldots, n_s \) where \( N_i \) is the minimal prime ideal of \( A_i \).

This is a restatement of Lemma 1. (If \( A \) is imbedded, the only regular elements of \( A \) are the units, and the formula holds trivially.)

**Remark.** Lemma 3 does not necessarily hold if the dimension of \( A \) is greater than one. Let

\[ A = k[w, x, y, z]((w, x, y, z)) = k[W, X, Y, Z]((W, X, Y, Z)) \]

where \( k \) is a field. By direct computation \( \mu_A(x) = 9 \) and \( \mu_A(y) = 6 \).

Thus \( \mu_{IA}(x) = \mu_{IA}(y) = 3 \). By the Corollary of Proposition 2, \( \mu_{IA} = v \) where \( v \) is the order valuation of \( k[s, t]((s, t)) \) having valuation ring \( k(s/t)[t]((t)) \). \( \mu_A = v + w \) where \( w \) is the valuation having valuation ring \( k(t/s^2)[s]((s)) \). (See §4.)

3. The first neighborhood ring of \( A \): a quadratic transform of \( A \) which is compatible with \( \mu_A \). Let \( G_m A \) be the associated graded ring of \( A \) with respect to \( m \). Let \( m = (x_1, \ldots, x_n) \). The natural homomorphisms

\[ A[X_1, \ldots, X_n] \rightarrow k[X_1, \ldots, X_n] \rightarrow G_m A \]

(where \( k = A/m \)) will be used. Let \( A[X] \) denote \( A[X_1, \ldots, X_n] \), and let \( k[X] \) denote \( k[X_1, \ldots, X_n] \). I will denote the ideal \((X_1, \ldots, X_n)\) of \( A[X] \), \( k[X] \), and \( G_m A \).

A familiarity with Northcott’s *The neighborhoods of a local ring* [5] is assumed. For the definition of the first neighborhood ring \( P \) of \( A \), see [5, p. 361]. Let \( \mathfrak{p}_1, \ldots, \mathfrak{p}_r \) be the height one prime ideals of \( \mathfrak{P} \) lying over \( m \), and let \( \mathfrak{p}_i \) be the prime ideal of \( G_m A \) corresponding to \( \mathfrak{p}_i \) [5, Propositions 1–4]. The preimage of \( \mathfrak{p}_i \) in \( k[X] \) will also be denoted by \( \mathfrak{p}_i \). For the definition of a superficial element of \( A \) see [5, p. 362], [3, p. 72 and Theorem 30.1, p. 103], or [7, p. 285].
Lemma 4. Let $A$ be an entire local ring with maximal ideal $m$ and an infinite residue field $k$. Let $\mathfrak{p}$ be the first neighborhood ring of $A$, let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the height one prime ideals of $\mathfrak{p}$ lying over $m$, let $\mathfrak{p}_i = \mathfrak{p}_i^\prime$, and let $\mathfrak{p}_i$ be the prime ideal of $\mathfrak{g}_m^A$ corresponding to $\mathfrak{p}_i$. Then

$$\mu_A(f) = e_f(\mathfrak{g}_m^A/\mathfrak{p}_1^i)\mu_{\mathfrak{p}_1}(f) + \cdots + e_f(\mathfrak{g}_m^A/\mathfrak{p}_r^i)\mu_{\mathfrak{p}_r}(f)$$

for all $f \in A$.

Proof. The equality is easily shown to hold for a superficial element of $A$. Let $f \in A$ be superficial of degree $s$. $\mu_A(f) = e_m(A/\text{Af}) = s e_m(A)$ [7, Lemma 4, p. 286], and

$$\mu_A(f) = s(e_f(k[X]/\mathfrak{p}_1)e_{\mathfrak{p}_1}(\mathfrak{p}_1/\mathfrak{p}_1^i m) + \cdots + e_f(k[X]/\mathfrak{p}_r)e_{\mathfrak{p}_r}(\mathfrak{p}_r/\mathfrak{p}_r^i m))$$

[5, formula E, p. 370]. Let $x$ be a superficial element of $A$ of degree one. $f/x^s \in \mathfrak{p}_i$, $\mathfrak{p}_i^i m = \mathfrak{p}_i x$ for $i = 1, \ldots, r$, and

$$\mu_A(f) = s(e_f(k[X]/\mathfrak{p}_1)\mu_{\mathfrak{p}_1}(x) + \cdots + e_f(k[X]/\mathfrak{p}_r)\mu_{\mathfrak{p}_r}(x))$$

$$= e_f(k[X]/\mathfrak{p}_1)\mu_{\mathfrak{p}_1}(f) + \cdots + e_f(k[X]/\mathfrak{p}_r)\mu_{\mathfrak{p}_r}(f).$$

The proof of the equality in general will occupy the rest of this section.

First let $\dim A \geq 2$. The proof will proceed by fixing the element $f \in A$ and blowing up $A$ to a one-dimensional ring $B$ such that $\mathfrak{p}_i^i = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r$ is an integral extension of $B$ and such that $\mathfrak{g}_m^B(B/\text{Af})$ is nearly a linear section of $\mathfrak{g}_m^A(A/\text{Af})$.

Let $\nu_A$ be the order function of $A$ with respect to $m$. Let $x$ be a superficial element of $A$ of degree one, let $m = (x_1, \ldots, x_n)$ and let $\Pi$ be a form of degree one in $A[x_1, \ldots, x_n]$ with $x = \Pi(x_1, \ldots, x_n)$. $\Pi$ will also denote its image modulo $m$ in $k[x_1, \ldots, x_n]$. Consider the diagram,

$$\begin{array}{ccc}
A[x_1, \ldots, x_n] & \xrightarrow{\rho} & k[x_1, \ldots, x_n] \\
\downarrow{\chi} & & \downarrow{\psi} \\
A & \xrightarrow{\sigma} & \mathfrak{g}_m^A
\end{array}$$

where $\sigma(\xi) = (\xi + m^v_A(\xi) + 1)/m^v_A(\xi) + 1$, $\psi$ is the canonical homomorphism and $k = A/m$, $\chi$ is the homomorphism with $\chi(X_i) = x_i$ and $\chi|_A = \text{id}_A$, and $\rho(F)$ is the leading form modulo $m$ of $F$. $\sigma(Af)$ is an ideal of $\mathfrak{g}_m^A$, but $\sigma$ need not be a homomorphism. Let $\tau Af = \psi^{-1}\sigma(Af)$, let $\omega Af = \chi^{-1}(Af) = (X_1 - x_1, \ldots, X_n - x_n, f)$, and let $\sigma Af$ denote $\sigma(Af)$.

$\rho(\omega Af) = \tau Af$. First notice that if $E \in \omega Af$ and $\deg E = \nu_A(\chi E) = s$ then $\psi E = \psi(E + m[X] + f^{s+1}) = E(x_1, \ldots, x_n) + m^{s+1}$. Secondly notice that $\psi^{-1}(0) = \tau Af \subset \rho(\omega Af)$. If $E \in \omega Af$ and if $\psi E = 0$ then $\rho E \in \psi^{-1}(0) \subset$
\( \rho(\omega A f) \). If \( E \in \omega A f \) and if \( \psi \rho E \neq 0 \) then \( \deg E = v_A(\chi E), \psi \rho E = \sigma \chi E \), and \( \rho E \in \tau A f \). Hence \( \rho(\omega A f) \subset \tau A f \). Let \( e \in A f \). Let \( E \in \omega A f \) be such that \( \deg E = v_A(e) \) and \( \chi E = e \). Then \( \sigma e = \psi \rho E, \rho E \in \psi^{-1}(\sigma e), \) and \( \tau A f \subset \rho(\omega A f) \).

Let \( p \) be an isolated prime ideal of \( \tau A 0 \). Then depth \( p = \dim A - \text{height } p \geq 2 \) and depth(\( p, \Pi \)) \( \geq 1 \).

Choose \( \Theta \) to be a form of degree one in \( A [X] = A [X_1, \ldots, X_n] \) such that \( y = \Theta(x_i) \) is a superficial element of \( A \) and a superficial element of \( A/\omega A f \), such that \( \Theta \) is contained in no isolated prime ideal of \( (\rho, \Pi) \) for any isolated prime ideal \( p \) of \( \tau A 0 \), and such that \( y \) is contained in no associated prime ideal of \( A x \) other than possibly \( m \). Each condition is viewed as a condition on form ideals in \( k[\bar{X}] \). Let \( \Theta \) also denote its image modulo \( m \) in \( k[\bar{X}] \).

Let \( u = y/x \). Let \( P \) be the kernel of the canonical homomorphism of \( A [U] \) onto \( A [u] \) where \( A [U] \) is the polynomial ring in one variable and \( U \) maps to \( u \). \( P \cap A = (0) \), and it follows that \( P \) is of height one in \( A [U] \). Letting \( D_A \) denote the set of prime ideals of \( A \) which occur as an imbedded prime ideal of a proper principal ideal of \( A \) (see \([2, \S 6]\)), \( Q \in D_{A [U]} \) if and only if \( Q \cap A \in D_A \) and \( Q = (Q \cap A) \cdot A [U] \). \( y - x U \) is prime in \( A [U] \) if and only if \( x, y \) form a prime sequence in \( A \), but this is the case if and only if \( m \notin D_A \). If \( m \notin D_A \) then \( P = (y - x U) \), and \( P \subset m [U] \). If \( m \in D_A \) then \( P \) and \( m [U] \) are the associated prime ideals of \( (y - x U) \). For if \( Q \) is an associated prime ideal of \( (y - x U) \) of height greater than one then \( x, y \in Q \cap A \) and \( Q = m [U] \). If \( Q \) is of height one, either \( Q \cap A = q \neq (0), \) in which case \( Q = q [U] \) and \( x, y \in q \) which contradicts the choice of \( y \), or \( Q \cap A = (0) \) in which case \( Q = (Q A)[U] \cdot (y - x U) = P \). It again follows that \( P \subset m [U] \). So \( A [u]/m [u] \cong k [u] \), and \( \bar{u} = u + m \cdot A [u] \) is transcendental over \( k \).

Let \( S = A [u] \sim mA [u] \) and let \( B = S^{-1} A [u] \). \( B/mB \cong k(\bar{u}) \) a simple transcendental extension of \( k \). \( \dim A [U] = \dim A + 1 \), the kernel \( P \) of the homomorphism \( A [U] \rightarrow A [u] \) is height one, \( m [U] \) is of height equal to \( \dim A \), and \( \dim B = \dim A - 1 \). Consider \( G_{mA} B \) and the commutative diagram

\[
\begin{array}{ccc}
A[X_1, \ldots, X_n] & \xrightarrow{\rho} & B[X_1, \ldots, X_n] \\
\downarrow \rho & & \downarrow \rho \\
k[X_1, \ldots, X_n] & \xrightarrow{\psi} & k(\bar{u})[X_1, \ldots, X_n] \\
G_mA & \xrightarrow{\phi} & G_{mA} B \\
\end{array}
\]

where \( \phi \) is the canonical homomorphism induced by the inclusion \( A \subset B \). Define \( \sigma, \tau \) and \( \omega \) for \( B \) as was done for \( A \). Notice that \( \omega A f \subset \omega B f \), so \( \tau A f \subset \tau B f \). \( \Theta - \bar{u} \Pi \in \omega B f \). Let \( q \) be an associated prime ideal of \( \tau A f \) which is not \( I = (X_1, \ldots, X_n) \). If \( \Theta - \bar{u} \Pi \in k(\bar{u}) \cdot q \), then \( \Theta - \bar{u} \Pi \in k[\bar{u}] \cdot q \) and \( \Theta \in q \), which is
a contradiction to the superficiality of \( \vartheta \). Therefore \( \Theta - \bar{u} \Pi \notin k(\bar{u})q \), and \( \Theta - \bar{u} \Pi \) is superficial as an element of \( k(\bar{u})[X]/k(\bar{u}) \cdot \tau Af \).

Now \( \mu_A(f) = e_f(k[X]/\tau Af) \) and \( \mu_B(f) = e_f(k(\bar{u})[X]/\tau Bf) \). These modules are homogeneous and their lengths over \( k[X] \) or \( k(\bar{u})[X] \) are their dimensions over \( k \) or \( k(\bar{u}) \). Thus \( \mu_A(f) = e_f(k(\bar{u})[X]/k(\bar{u}) \cdot \tau Af) \). By Lemmas 3 and 4 of [7, pp. 285–286], if \( \dim A > 2 \),

\[
e_f(k(\bar{u})[X]/k(\bar{u}) \cdot \tau Af) = e_f(k(\bar{u})[X]/(\tau Af, \Theta - \bar{u} \Pi)),
\]

and if \( \dim A = 2 \),

\[
e_f(k(\bar{u})[X]/k(\bar{u}) \cdot \tau Af) = e_f(k(\bar{u})[X]/(\tau Af, \Theta - \bar{u} \Pi))
\]

\[-1^{k(\bar{u})[X]}((\tau \cdot \Theta - \bar{u} \Pi)/(\tau \cdot \tau Af))\]

for all large enough \( n \) and \( c \) with \( n > c \). Because \( \Theta - \bar{u} \Pi \) is contained in no associated prime ideal of \( k(\bar{u}) \cdot \tau Af \) other than possibly \( I \), the homogeneous parts of like degree of \( k(\bar{u}) \cdot \tau Af \) and of \( (k(\bar{u}) \cdot \tau Af: \Theta - \bar{u} \Pi)/k(\bar{u}) \cdot \tau Af \) are equal for sufficiently large degree. So for large enough \( n \) and \( c \), over \( k(\bar{u}) \)

\[(\tau \cdot \tau Af: \Theta - \bar{u} \Pi)/(\tau \cdot \tau Af) = (k(\bar{u}) \cdot \tau Af: \Theta - \bar{u} \Pi)/k(\bar{u}) \cdot \tau Af,\]

and for \( \dim A = 2 \),

\[
\begin{align*}
e_f(k(\bar{u})[X]/k(\bar{u}) \cdot \tau Af) &= e_f(k(\bar{u})[X]/(\tau Af, \Theta - \bar{u} \Pi)) \\
\quad &\quad - \dim_{k(\bar{u})}(k(\bar{u}) \cdot \tau Af: \Theta - \bar{u} \Pi)/k(\bar{u}) \cdot \tau Af.
\end{align*}
\]

Let

\[
\alpha = \dim_{k(\bar{u})}\tau Bf/(\tau Af, \Theta - \bar{u} \Pi)
\]

and

\[
\beta = \dim_{k(\bar{u})}(k(\bar{u}) \cdot \tau Af: \Theta - \bar{u} \Pi)/k(\bar{u}) \cdot \tau Af.
\]

It is to be shown that \( \alpha = \beta \). Then \( \alpha \) is finite, for \( \beta \) is finite by the superficiality of \( \Theta - \bar{u} \Pi \), and it follows that if \( \dim A > 2 \), \( \mu_A(f) = \mu_B(f) \). If \( \dim A = 2 \) it follows from \( \alpha = \beta \) that \( \mu_A(f) = \mu_B(f) \).

If \( \mathcal{U} \) is a set of polynomials in \( X_1, \ldots, X_n \), let \( \mathcal{U}(d) \) be the set of all elements of \( \mathcal{U} \) which have no nonzero homogeneous component of degree strictly less than \( d \), and let \( \mathcal{U}_d \) be the set of all homogeneous elements of \( \mathcal{U} \) of degree \( d \).

Let \( S = A[U] \sim m[U] \), and let \( A(U) \) denote \( S^{-1}A[U] \). Let \( \tau(P, f) = \rho(P, \omega A(U)f) \) and \( \tau(\Theta - U \Pi, f) = \rho(\Theta - U \Pi, \omega A(U)f) \). Consider

\[
\begin{array}{ccc}
A(U)[X] & \xrightarrow{\rho} & k(U)[X] \\
\downarrow{\psi} & \quad & \downarrow{\psi} \\
B[X] & \xrightarrow{\rho} & k(\bar{u})[X]
\end{array}
\]
where \( \rho(\alpha) \) is the leading form in \( X_1, \ldots, X_n \) of \( \alpha \) modulo \( mA(U)[X] \) or \( mB[X] \), where \( \psi(U) = u \) and \( \psi|_{A\{X\}} = id_{A\{X\}} \), and where \( \overline{\psi}(U) = \overline{u} \) and \( \overline{\psi}_{k\{X\}} = id_{k\{X\}} \). Because \( P \subset (P, \omega A(U)f) \),

\[
\overline{\psi}_\tau(P, f) = \rho \psi(P, \omega A(U)f) = \tau Bf.
\]

Note that \( \overline{\psi} : k(U)[X] \rightarrow k(\overline{u})[X] \) is an isomorphism over the isomorphism \( k(U) \cong k(\overline{u}) \) induced by \( \overline{\psi} \). Let

\[
\gamma = \dim_{k(U)} \tau(P, f) / (\Theta - U\Pi, f) = \dim_{k(\overline{u})} \tau Bf / \overline{\psi}_\lambda(\Theta - U\Pi, f).
\]

Then

\[
\dim_{k(U)} \tau(f, \Theta - U\Pi) / (\tau Af, \Theta - U\Pi) = \alpha - \gamma.
\]

Let \( H \) be \( \rho((\omega A(U)f)^\wedge, A(U)[X]) \Theta - U\Pi \) where \( \wedge \) denotes the \( I \)-adic completion. Let \( Q \) be an associated prime ideal of \( \omega A(U)f \). \( (X_1 - x_1, \ldots, X_n - x_n) \subset Q \), so \( Q \subset (mA(U), I) \). \( A(U)[X]_{m\overline{A}(U)f} \) with the \( I \)-adic topology is a Zariski ring with completion \( A(U)[[X]] \). Hence

\[
((\omega A(U)f)^\wedge, A(U)[X]) \Theta - U\Pi = (\omega A(U)f, A(U)[X]) \Theta - U\Pi
\]

[7, Corollary 4, p. 266], and \( H = \rho(\omega A(U)f) : \Theta - U\Pi \). So \( \overline{\psi} H \subset (k(\overline{u}) \cdot \tau Af : \Theta - U\Pi) \). Let

\[
\delta = \dim_{k(U)} H / k(U) \cdot \tau Af.
\]

Then

\[
\dim_{k(U)} (k(U) \cdot \tau Af : \Theta - U\Pi) / H = \beta - \delta.
\]

It is to be first shown that \( \alpha - \gamma = \beta - \delta \).

Let \( M \in A(U)[X_1, \ldots, X_n] \) be homogeneous of degree \( d \) such that \( M + mA(U)[X] \in \tau(\Theta - U\Pi, f) \). The following four assertions follow easily from the fact that \( x_i - X_i \in \omega A(U)f \). There is an integer \( h \leq d - 1 \) and forms \( H_i \in A(U)[X] \) of degree \( i = h, \ldots, d - 1 \) such that

\[
(\Theta - U\Pi)(H_h + \cdots + H_{d-1}) + M \in \omega A(U)f + A(U)[X]_{(d+1)}.
\]

If \( M - M' \in mA(U)[X]_d \), then

\[
(\Theta - U\Pi)(H_h + \cdots + H_{d-1}) + M' \in \omega A(U)f + A(U)[X]_{(d+1)}.
\]

If \( H_h - H'_h \in mA(U)[X]_h \), there are forms \( H'_i \in A(U)[X] \) for \( i = h + 1, \ldots, d - 1 \) such that

\[
(\Theta - U\Pi)(H'_h + \cdots + H'_{d-1}) + M \in \omega A(U)f + A(U)[X]_{(d+1)}.
\]

If \( F \in A(U)[X]_d \) and if \( F + mA(U)[X] \in k[X] \cdot \tau Af \), then
\begin{equation}
(\Theta - U \Pi)(H_h + \cdots + H_{d-1}) + (M + F) \in \omega A(U)[X] + A(U)[X]_{(d+1)}.
\end{equation}

Note that \( H_h + mA(U)[X] \in (k(U) \cdot \tau A f : \Theta - U \Pi) \). Let \( h(M) < \deg M \) be the maximal degree of all such \( H_h \) as above. Let \( H(M) \) be the set of all such \( H_h \) as above with \( h = h(M) \). \( M + mA(U)[X] \in (\tau A f, \Theta - U \Pi) \) if and only if \( h(M) = \deg M - 1 \) which is true if and only if \( |h(M)| \subset H(M) \) (which in this case is \( A(U)[X]_{h(M)} \)). If \( b \in A(U) \sim mA(U) \), \( bH(M) = H(bM) \). If \( H \in H(M) \) then
\begin{equation}
(H + mA(U)[X]_{h(M)}) + H_{h(M)} \subset H(M)/mA(U)[X]_{h(M)}
\end{equation}
and \( H(M) \) will be considered as a subset of \( (k(U) \cdot \tau A f : \Theta - U \Pi)/i \).

A \( k(U) \)-linear injection of \( \tau(f, \Theta - U \Pi)/(\tau A f, \Theta - U \Pi) \) into \( (k(U) \cdot \tau A f : \Theta - U \Pi)/i \) is to be defined. Let \( M_1, \ldots, M_a \in A(U)[X] \) be forms such that their residues modulo \( mA(U)[X] \) are in \( \tau(f, \Theta - U \Pi) \), such that their residues in \( \tau(f, \Theta - U \Pi)/(\tau A f, \Theta - U \Pi) \) are linearly independent over \( k(U) \), such that \( h(M_i) < h(M_{i+1}) \) and such that if \( h(M_i) = h(M_{i+1}) \) then \( \deg M_i \geq \deg M_{i+1} \). Choose \( \eta_i \in H(M_i) \). Suppose \( \eta_i, \ldots, \eta_{t-1} \) are linearly independent over \( k(U) \), and suppose \( \eta_i = \alpha_1 \eta_1 + \cdots + \alpha_{t-1} \eta_{t-1} \) where \( \alpha_i \in A(U) \). The \( \alpha_i \) are nonzero only for those \( M_i \) with \( h(M_i) = h(M_i) \). \( (H_{M_i}) = h(M_{t-1}) \), for \( \eta_i \neq 0 \). Let \( M_1, \ldots, M_{t-1} \) be exactly those \( M_i \) with \( i < t \), \( h(M_i) = h(M_i) \) and \( \deg M_i = \deg M_t \). Then \( h(M_t - \alpha_1 M_1 - \cdots - \alpha_{t-1} M_{t-1}) > h(M_i) \), so replace \( M_t \) by \( M_t - \alpha_1 M_1 - \cdots - \alpha_{t-1} M_{t-1} \), choose a new \( \eta_t \), and reorder \( M_t, \ldots, M_a \). With a finite number of repetitions of the above process \( \eta_1, \ldots, \eta_{t-1} \) will be linearly independent, for at worst \( h(M_i) \) will eventually be greater than \( h(M_{t-1}) \), and linear independence will follow. Thus \( a \leq \beta - \delta \), and \( a - \gamma \leq \beta - \delta \).

A construction analogous to the above is used to derive the opposite inequality. Let \( H \in A(U)[X]_d \) with \( H + mA(U)[X] \in (k(U) \cdot \tau A f : \Theta - U \Pi) \). Let \( m(H) \) be the maximal integer \( m \) such that there exists a form \( M \) of degree \( m \) and forms \( H_i \) of degree \( i = d + 1, \ldots, m - 1 \) such that
\begin{equation}
(\Theta - U \Pi)(H + H_{d+1} + \cdots + H_{m-1}) + M \in \omega A(U)f + A(U)[X]_{(m+1)}
\end{equation}
and \( M + mA(U)[X] \notin (\tau A f, \Theta - U \Pi) \). If such a maximum does not exist then \( H + mA(U)[X] \notin H \), and if \( H + mA(U)[X] \notin H \), then \( m(H) \geq \deg H + 1 \). Let \( M(H) \) be the set of all such \( M \) of degree \( m(H) \). \( M(bH) = bM(H) \) for \( b \in A(U) \sim mA(U) \). If \( M \in M(H) \) then \( M + mA(U)[X] \subset M(H) \),
\begin{equation}
M + mA(U)[X]_{m(H)} + (\tau A f, \Theta - U \Pi)_{m(H)} \subset M(H)/mA(U)[X]_{m(H)}
\end{equation}
and \( M + mA(U)[X]_{m(H)} \in (\tau f, \Theta - U \Pi)/i \). \( M(H) \) will be considered as a subset of \( (\tau f, \Theta - U \Pi)/(\tau A f, \Theta - U \Pi) \).

Let \( H_1, \ldots, H_{\beta - \delta} \) be forms in \( mA(U)[X] \) such that their residues modulo \( mA(U)[X] \) are in \( (k(U) \cdot \tau A f : \Theta - U \Pi) \), such that their residues form a \( k(U) \)-basis for \( (k(U) \cdot \tau A f : \Theta - U \Pi)/i \), \( m(H_i) \leq m(H_{i+1}) \) and such that if \( m(H_i) = m(H_{i+1}) \) then \( \deg M_i = \deg M_{i+1} \).
Choose \( m_{i+1} \in M(H_i) \). Suppose \( \mu_1, \ldots, \mu_{t-1} \) are linearly independent over \( k(U) \) and \( \mu_t = \alpha_1 \mu_1 + \cdots + \alpha_{t-1} \mu_{t-1} \) where \( \alpha_i \in A(U) \). \( \alpha_i \) is nonzero only if \( m(H_i) = m(H_t) \), \( m(H_i) = m(H_t) \) for \( i \neq 0 \), and let \( H_i, \ldots, H_{t-1} \) be those \( H_i \) with \( i < t \), \( m(H_i) = m(H_t) \) and \( \deg H_i = \deg H_t \). Then \( m(H_i - \alpha_i H_{t-1} < \cdots < \alpha_{t-1} H_{t-1}) > m(H_t) \). Replace \( H_i \) by \( H_i - \alpha_i H_{t-1} \), choose \( \mu_t \) anew, reorder \( H_1, \ldots, H_{t-1}, \) with a finite number of repetitions the injection is defined, and \( \alpha - \gamma > \beta - \delta \).

Thus \( \alpha - \gamma = \beta - \delta \). The final goal in the proof of \( \alpha = \beta \) is to show that \( \gamma \) and \( \delta \) are equal.

Let \( \mathfrak{A} \subset \mathfrak{B} \) be two ideals of \( A(U) \). As either \( k(U) \) or \( A(U) \)-modules, \( \mathfrak{B} / \mathfrak{A} \cong \sigma \mathfrak{B} / \sigma \mathfrak{A} \). Now

\[
\sigma \mathfrak{B} / \sigma \mathfrak{A} \cong \sum_{n \geq 0} \bigoplus \left( \frac{m^n \cap \mathfrak{B} + m^{n+1}/m^{n+1}}{(m^n \cap \mathfrak{A} + m^{n+1}/m^{n+1})} \right)
\]

and

\[
\gamma = l_{A(U)}(P, f)(y - xU, f),
\]

and

\[
\delta = l_{A(U)}(A(U)f : y - xU) / A(U)f.
\]

Let \( \psi \in (A(U)f : y - xU) \). \( \psi / f \)(y - xU) \in A(U), \( f(\psi / f)(y - xU) \in P \), \( f \not\in P \), so \( (\psi / f)(y - xU) \in P \). Let \( \xi_1(\psi) = (\psi / f)(y - xU) \). If \( \psi \in A(U)f \) then \( \xi_1(\psi) \in A(U)(y - xU) \). Hence

\[
(\psi / f)(y - xU) \in A(U)f \rightarrow (P, f)(y - xU, f)
\]

is a homomorphism. Let \( \psi \in \text{Ker} \xi_1 \), that is, \( (\psi / f)(y - xU) = af + b(y - xU) \) for some \( a \) and \( b \) in \( A(U) \). Then \( (\psi - bf)(y - xU) = af^2 \), and \( \psi \in ((A(U)f)^2 : y - xU, f) \). If \( \phi \in (A(U)f)^2 : y - xU \), then \( \phi(y - xU) = af^2 \) for some \( a \) in \( A(U) \), \( \xi_1(\phi) = (\phi / f)(y - xU) = af \), and \( \phi \in \text{Ker} \xi_1 \). So

\[
\text{Ker} \xi_1 = (A(U)f^2 : y - xU, f) / A(U)f.
\]

Now,

\[
(A(U)f^4 : y - xU) / (A(U)f^4 : y - xU) \cap A(U)f
\]

and a homomorphism
THE MULTIPLICITY FUNCTION OF A LOCAL RING

\[ \xi_i: (A(U)f^i: y - xU)/(A(U)f^i: y - xU) \cap A(U)f \]

\[ \rightarrow \{ \cdots ((P, f)/(y - xU, f))/\text{Im} \, \xi_1)/\cdots )/\text{Im} \, \xi_{i-1} \]

with

\[ \text{Ker} \, \xi_i = ((A(U)f^i-1: y - xU), f)/A(U)f \]

is to be defined inductively.

If \( \psi \in (A(U)f^i: y - xU) \), let \( \xi_0(\psi) = (\psi/f^i)(y - xU) \in P \). If \( \psi \in (A(U)f^i: y - xU) \cap A(U)f \), then \( \psi/f \in (A(U)f^{i-1}: y - xU) \), \( \xi_{i-1}(\psi/f) = (\psi/f^i)(y - xU) = \xi_i(\psi) \), and \( \xi_i(\psi) \in \text{Im} \, \xi_{i-1} \). Let \( \psi \in \text{Ker} \, \xi_i \). Then

\[ (\psi/f^i)(y - xU) = a^i + b(y - xU) \]

\[ + (\psi_1/f)(y - xU) + \cdots +(\psi_{i-1}/f^{i-1})(y - xU) \]

where \( \psi_j \in (A(U)f^j: y - xU) \) for \( j = 1, \ldots , i - 1 \), and

\[ \psi - b^i - f^{i-1}\psi_1 - \cdots - f\psi_{i-1})(y - xU) = a^i + 1 , \]

so \( \text{Ker} \, \xi_i \subset ((A(U)f^{i+1}: y - xU), f)/A(U)f \). If \( \phi \in (A(U)f^{i+1}: y - xU) \) then

\[ \xi_i(\phi) = (\phi/f^i)(y - xU), f, f \in A(U)f \), and \( \phi \in \text{Ker} \, \xi_i \) Thus

\[ \text{Ker} \, \xi_i = ((A(U)f^{i+1}: y - xU), f)/A(U)f \].

\[ \bigcap_i A(U)f^i = (0) \], so \( \bigcap_i (A(U)f^{i+1}: y - xU) = (0) \), and by [3, Theorem 30.1, p. 103], \( \bigcap_i \text{Ker} \, \xi_i \subset \bigcap_i (A(U)f^i + m^i) = A(U)f \). By [5, Theorem 1, p. 365], because \( y - xU \) is superficial of degree 1, \( (m_1+1 A(U): y - xU) = m^i \)

for all sufficiently large \( i \), so \( \bigcap_i \text{Ker} \, \xi_i \subset \bigcap_i (A(U)f^i + m^i) = A(U)f \). If \( \phi \in P \) there is an integer \( s \) such that \( f^s\phi \in A(U)(y - xU) \), for there is an integer \( s \) such that \( P \cap m^s = A(U)(y - xU) \cap m^s \). Then \( \xi_s(f^s\phi)(y - xU) = \phi \).

Let

\[ \mathcal{U}_i = ((A(U)f^i: y - xU), f) , \]

and let

\[ \mathcal{B}_i = (((\psi/f^i)(y - xU)\psi \in (A(U)f^i: y - xU)), f). \]

Then \( \bigcap_i \mathcal{B}_i = A(U)f \) and \( \mathcal{U}_i = A(U)f \) for some \( t \geq 1 \), for \( (A(U)f^i: y - xU)/A(U)f \)

is of finite length. Hence

\[ \mathcal{U}_0 = (A(U)f: y - xU) \cup \mathcal{U}_1 \cup \cdots \cup \mathcal{U}_i = A(U)f , \]

and

\[ (y - xU, f) = \mathcal{B}_0 \subset \mathcal{B}_1 \subset \cdots \subset \mathcal{B}_s = (P, f) \]

where \( \mathcal{U}_i/\mathcal{U}_{i+1} \cong \mathcal{B}_{i+1}/\mathcal{B}_i \) as \( A(U) \)-modules. Thus \( \gamma = \delta \).

The above construction is inductive to dimension one. Let \( B_d = A \) and
$B_{d-1} = B$ where $d$ is again the dimension of $A$, let $\Theta_{d-1} = \Theta, y_{d-1} = y, u_{d-1} = u$ and $L_{d-1} = \Theta - U$. $\Pi$ and $x = \Pi(x_i)$ remain fixed throughout the induction.

Suppose $B_{j+1}$ has been defined with the required properties. Let $\Theta_j$ be a form of degree one in $A[X]$ such that $y_j = \Theta_j(x_j)$ is a superficial element of $B_{j+1}$ and of $B_{j+1} \mid f, \Theta_j$ is not contained in any associated prime ideal of $(\rho_j, L_{d-1}, \ldots, L_{j+1})$ other than possibly $I$ nor contained in any isolated prime ideal of $(\rho_j, L_{d-1}, \ldots, L_{j+1}, \Pi)$ for any isolated prime ideal $\rho_j$ of $\tau A_0$, and such that $y_j$ is contained in no associated prime ideal of $B_{j+1}x$ except possibly $mB_{j+1}$.

The above arguments hold when $A$ is replaced by $B_{j+1}$ and $B$ is replaced by $B_j = S^{-1}B_{j+1}[u_j]$ where $u_j = y_j/x$ and $S = B_{j+1}[u_j] \sim mB_{j+1}[u_j]$.

Let $B = B_1$. $B$ is one dimensional, $B$ is local with maximal ideal $mB$, and $\mu_A(f) = \mu_B(f)$.

Let $\mathfrak{R}_1$ be $\mathfrak{T}^{-1}\mathfrak{R}$ where $T = \mathfrak{T} \sim (\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_r)$ and where $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ are the height one prime ideals of $\mathfrak{T}$. For every $i = 1, \ldots, r$,

$$\mathfrak{R}_1 \mathfrak{p}_i \cap A[u_{d-1}, \ldots, u_1] = m[u_{d-1}, \ldots, u_1].$$

For let $z \in A[u_{d-1}, \ldots, U_1] \cap \mathfrak{R}_1 \mathfrak{p}$ where $\mathfrak{p}$ denotes one of the $\mathfrak{p}_i$. Then $z \in A[u_{d-1}, \ldots, u_1] \cap \mathfrak{p}$. Let $\mathfrak{p}$ be the prime ideal corresponding to $\mathfrak{p}$ which is associated to $\tau A_0$, and let $F(\Theta_{d-1}, \ldots, \Theta_1, \Pi)$ be a form in $\Theta_{d-1}, \ldots, \Theta_1$ and $\Pi$ with coefficients in $A$ such that

$$F(\Theta_{d-1}(x_j/x), \ldots, \Theta_1(x_j/x), \Pi(x_j/x)) = z.$$

$A[u_{d-1}, \ldots, u_1] \subset \mathfrak{R}$, so $z \in \mathfrak{p}$ and by the correspondence between $\mathfrak{p}$ and $\mathfrak{p}$, $F(\Theta_{d-1}, \ldots, \Theta_1, \Pi) + m[X] \subset \mathfrak{p}$. Suppose $F$ modulo $m, \bar{F}$, is nonzero. If $\bar{F}$ were a power of $\Pi$, then $\Pi \in \mathfrak{p}$ which is a contradiction. So there is an integer $j$ such that $d - 1 \geq j > 1$, $\bar{F} \in k[\Theta_{d-1}, \ldots, \Theta_j, \Pi]$ and $\bar{F} \notin k[\Theta_{d-1}, \ldots, \Theta_{j+1}, \Pi]$.

Then

$$\bar{F} = \bar{G} \Pi^e \mod (\Theta_{d-1} - \Pi, \ldots, \Theta_{j+1} - \Pi) \subset (\rho_j, L_{d-1}, \ldots, L_{j+1}, \Pi)$$

for some form $\bar{G} \in k[\Theta_j, \Pi]$ which is not divisible by $\Pi$. Letting $s > 1$ be the degree of $\bar{G}, \Theta_j \in (\rho_j, L_{d-1}, \ldots, L_{j+1}, \Pi)$ which is a contradiction to the choice of $\Theta_j$. Hence $\bar{F} = 0$, and $z \in m[u_{d-1}, \ldots, u_1]$.

$B$ is a ring of fractions of $A[u_{d-1}, \ldots, u_1]$ with $m[u_{d-1}, \ldots, u_1] \subset mB \cap A[u_{d-1}, \ldots, u_1]$. $mB$ is a prime ideal of height one of $B$, so $mB \cap A[u_{d-1}, \ldots, u_1]$ must be of height one also, and

$$mB \cap A[u_{d-1}, \ldots, u_1] = m[u_{d-1}, \ldots, u_1].$$

It follows that

$$B = A[u_{d-1}, \ldots, u_1] m[u_{d-1}, \ldots, u_1],$$

and therefore $B \subset \mathfrak{R}_1$. 

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The multiplicity function of a local ring

\$R_1 = R_1 \cap \cdots \cap R_r \) is a finite integral extension of \( B = B_1 \). The proof is an adaptation of the proof of Theorem 10 [5, p. 371]. Let \( \mathfrak{p}_1, \ldots, \mathfrak{p}_r \) also denote the proper prime ideals \( R_1 \mathfrak{p}_1, \ldots, R_1 \mathfrak{p}_r \) of \( R_1 \), let \( m_j \) be integers such that \( \mathfrak{p}_1^{m_1} \cdots \mathfrak{p}_r^{m_r} \subset R_1 m \), and let \( n = \mathfrak{p}_1^{m_1} \cdots \mathfrak{p}_r^{m_r} \). Then \( m^2 \subset (R_1 m)^t \) and \( (R_1 m)^{tt} \subset m^2 \) where \( t = \max \{ m_1, \ldots, m_r \} \). Let \( \hat{B} \) be the m-B-adic completion of \( B \), and let \( \hat{R} \) be the \( R_1 \)-adic completion of \( R_1 \). \( \hat{R} \) is a \( \hat{B} \)-module, \( \hat{R} \) is the m-adic completion of \( R_1 \cap \bigcap_{n>0} m^n = (0) \), and by [7, Corollary 2, p. 273], the \( mB \)-adic topology of \( B \) is induced by the m-adic topology of \( R_1 \). It is clear that \( \hat{R}/\hat{R} m = R_1/R_1 m \).

\( B[x_1/x, \ldots, x_n/x] \) is of dimension one [3, Theorem 33.2, p. 115], and \( R_1 \) is a ring of quotients of \( B[x_1/x, \ldots, x_n/x] \). \( \mathfrak{p}_j \cap B[x_1/x, \ldots, x_n/x] \) for \( j = 1, \ldots, r \) are distinct proper prime ideals of \( B[x_1/x, \ldots, x_n/x] \). Let \( p \) be a proper prime ideal of \( B[x_1/x, \ldots, x_n/x] \). \( B[x_1/x, \ldots, x_n/x] \) is a ring of fractions of \( A[x_1/x, \ldots, x_n/x] \), so \( p \cap A[x_1/x, \ldots, x_n/x] \) is a prime ideal of height one, therefore there is a prime ideal \( \mathfrak{p}_j \) of \( R_1 \) such that \( \mathfrak{p}_j \cap A[x_1/x, \ldots, x_n/x] = p \cap A[x_1/x, \ldots, x_n/x] \), and \( \mathfrak{p} \cap B[x_1/x, \ldots, x_n/x] = p \). From the above assertions it is immediate that \( A[x_1/x, \ldots, x_n/x] = B[x_1/x, \ldots, x_n/x] \).

Let \( \theta_j \) be the residue of \( x_j/x \) modulo \( \mathfrak{p}_j \). \( R_1/\mathfrak{p}_j = k(\vec{u}_1, \ldots, \vec{u}_{d-1}) \) \([\theta_{j_1}, \ldots, \theta_{jn}] \) is a field, and \( \theta_j \) are algebraic over \( k(\vec{u}) = k(\vec{u}_1, \ldots, \vec{u}_{d-1}) \). By multiplying together the \( m_j \)-th power of a polynomial which modulo \( \mathfrak{p}_j \) is the algebraic relation of \( \theta_j \) over \( k(\vec{u}) \) for \( j = 1, \ldots, r \), there is a relation

\[
(x_j/x)^{t_j} + \alpha_{t_j-1}(x_j/x)^{t_j-1} + \cdots + \alpha_0 \in R_1 m
\]

where \( \alpha_0, \ldots, \alpha_{t_j-1} \in B \). Therefore \( R_1/R_1 m \) is a finite \( B/mB \) module, and \( \hat{R} \) is a finite \( \hat{B} \) module [7, Corollary 2, p. 259]. So for every positive integer \( s \) there is a relation

\[
(x_j/x)^s \in [\hat{B}(x_j/x)^{t_j-1} + \cdots + \hat{B}(x_j/x) + \hat{B}] \cap B
\]

for the latter module is finitely generated over the Zariski ring \( B \) and is therefore closed. \( R_1 \) is thus finite integral over \( B \).

It is to be shown that \( [R_1/\mathfrak{p}_j : B/mB] = e_j(k[X]/\mathfrak{p}_j) \). From the choice of \( \Theta_j \) it follows that \( L_j \) is a superficial element of

\[
k(\vec{u}_{d-1}, \ldots, \vec{u}_j)[X]/(\mathfrak{p}_j, L_{d-1}, \ldots, L_{j+1}),
\]

where \( \vec{u}_j \) is transcendental over \( k(\vec{u}_{d-1}, \ldots, \vec{u}_{j+1}) \). The dimensions are greater than one, so

\[
e_j(k[X]/\mathfrak{p}_j) = e_j(k(\vec{u})[X]/(\mathfrak{p}_j, L_{d-1}, \ldots, L_1)),
\]
where \( k(\bar{u}) \) now denotes \( k(\bar{u}_{d-1}, \ldots, \bar{u}_1) \). Let \( M_k(X) \in A[X] \) for \( k = 1, \ldots, t \) be forms of degree \( d_k \) such that the residues of \( M_1(x_i/x), \ldots, M_t(x_i/x) \) modulo \( \mathfrak{p}_x \) form a basis of \( \mathcal{R}^1/\mathfrak{p}_x \) over \( k(\bar{u}) = B/mB \). If \( G \) is a form in \( A[X] \) of degree \( g \geq \) max \( \{d_1, \ldots, d_t\} \), then

\[
G(\theta_{si}) = \sum_{k=1}^{t} \alpha_k(\Pi(\theta_{si}))^{d_k-1}M_k(\theta_{si})
\]

for some \( \alpha_1, \ldots, \alpha_t \in k(\bar{u}) \), for \( \Pi(\theta_{si}) = 1 \). Letting

\[
0 \rightarrow K \rightarrow k(\bar{u})[X_1, \ldots, X_n] \rightarrow k(\bar{u})[\theta_{s1}, \ldots, \theta_{sn}] \rightarrow 0
\]

be the exact where \( X_i \rightarrow \theta_{si}, k(\bar{u})[X]_g/K_g \) is of dimension \( t \) over \( k(\bar{u}) \) for \( g \geq \) max \( \{d_1, \ldots, d_t\} \). \( K \supset (\mathfrak{p}_x, L_{d-1}, \ldots, L_1) \) by the correspondence between \( \mathfrak{p}_x \) and \( \mathfrak{p}_s \). Let \( G \in K_g \). There is a unit \( \beta \) in \( k(\bar{u}) \) such that \( \beta G \in k[\bar{u}][X]_g \), and there are \( F_j \in k[\bar{u}][X] \) for \( j = 1, \ldots, d - 1 \) such that

\[
E' = \Pi^c \beta G = \sum_{j=1}^{d-1} (\Theta_j - \bar{u}_j \Pi) F_j \in k[X]_{g+c}
\]

where \( c \) is the degree of \( \bar{u} \) in \( \beta G \). Let \( E \in A[X]_{g+c} \) be a representative of \( E' \). \( E(x_i/x) \in \mathfrak{p}_s \), so \( E' \in \mathfrak{p}_s \). Thus \( \Pi^c G \in (\mathfrak{p}_s, L_{d-1}, \ldots, L_1) \). Inductively \( \Pi \) is contained in no minimal prime ideal of \( (\mathfrak{p}_s, L_{d-1}, \ldots, L_1) \). For let \( P \) be such a minimal prime ideal and suppose \( \Pi \in P \). Then \( \Theta_j \in P \), and inductively by dimension, \( P \) is a minimal prime ideal of \( (\mathfrak{p}_s, L_{d-1}, \ldots, L_1) \) except perhaps the primary component belonging to \( I, K_g = (\mathfrak{p}_s, L_d, \ldots, L_1)_g \) for all large enough values of \( g \), and by comparison of the Hilbert polynomials, \( t = e_f(k[X]/\mathfrak{p}). \)

*Apply the first part of the proof of Lemma 1 to \( \mathcal{R}^1 \) over \( B = B_1 \), and obtain

\[
\mu_A(f) = \mu_B(f) = \sum_{i=1}^{r} e_f(k[X]/\mathfrak{p}_i)\mu_{\mathcal{R}_1}(f).
\]

4. The valuation formula. Let \( A \) be a local ring with maximal ideal \( m \). For a definition of a valuation of \( A \), finite on \( A \) and centered at a prime ideal of \( A \), see \( [2, \S 1] \). By the additivity formula \( \mu_A(f) = \sum_{p} \lambda_p(f)e_{m}(A/p) \) where the sum ranges over all prime ideals \( p \) of \( A \) which are of depth equal to the dimension of \( A \). Assume that \( A \) is nonimbedded. Then the prime ideals \( p \) are all of height one, but they do not necessarily include all the prime ideals of height one. Then also \( \lambda_p(Af) \) is a finite sum of finite rank one discrete valuations centered at \( p \).

As an example, let \( A \) be an entire factorial ring of dimension greater than
one. Let \( \{v_i\}_{i \in I} \) be the set of prime divisors of type one of \( A \), and let \( p_i \) be a prime element of \( A \) with \( v_i(p_i) = 1 \). Let \( w_1 \) and \( w_2 \) be two distinct prime divisors of \( A \) centered at \( m \), let \( a_i = w_1(p_i) \) and \( b_i = w_2(p_i) \), and then \( w_1 = \Sigma_i a_i v_i \) and \( w_2 = \Sigma_i b_i v_i \). Let \( c_i = \min \{a_i, b_i\} \). Then \( \Sigma_i c_i v_i \geq w_1, \Sigma_i c_i v_i \neq w_1, \) and \( \Sigma_i c_i v_i \) is not a sum of valuations centered at \( m \).

**Theorem.** Let \( A \) be a local ring with maximal ideal \( m \). There are integral valued valuations \( v_1, \ldots, v_s \) finite on \( A \) centered at \( m \), and there are positive integers \( n_1, \ldots, n_s \) such that for every regular element \( f \) of \( A \),

\[
\mu_A(f) = n_1 v_1(f) + \cdots + n_s v_s(f).
\]

If \( A \) is nonimbedded if \( \mu_A(f) = n_1 v_1(f) + \cdots + n_s v_s(f) \) for all regular elements \( f \) of \( A \), if the valuations \( v_1, \ldots, v_s \) are independent, and if the ideal generated by each \( v_i(A) \) is all of the integers, then the valuations \( v_1, \ldots, v_s \) and the integers \( n_1, \ldots, n_s \) are unique. (If \( A \) is of dimension zero, \( \mu_A \) is the trivial valuation: \( \mu_A(f) = 0 \) if \( f \notin m \).)

The proof of the formula is now straightforward. By Lemma 2, \( A \) can be assumed to be entire. It may also be assumed that the residue field of \( A \) is infinite. In fact let \( A[x] \) be the polynomial ring in one variable over \( A \), let \( S = A[x] \sim mA[x], \) and let \( A(x) = S^{-1}A[x], \) a local ring with maximal ideal \( m \cdot A(x) \) and residue field \( A(x)/mA(x) = k(x) \) a simple transcendental extension of \( k = A/m. \) Then \( \mu_A = \mu_{A(x)}, \) for \( A(x)/A(x)f = (A/Af)(x) \) and letting \( B = A/Af \)

\[
G_{mB(x)}B(x) = \sum_{n>0} \frac{m^n B(x)}{m^{n+1}B(x)} \approx \sum_{n>0} \frac{m^n}{m^{n+1}} \otimes_A B(x)
\]

\[
\approx \sum_{n>0} \frac{m^n + Af}{m^{n+1} + Af} \otimes_k k(x) \approx (G_{mB}) \otimes_k k(x),
\]

so the multiplicities of \( A/Af \) and of \( A(x)/A(x)f \) are equal. A valuation of \( A(x) \) restricted to \( A \) remains a valuation. By Lemma 4, \( A \) can be assumed to be one dimensional, by Lemma 3, \( A \) can be assumed to be normal, and apply the Corollary of Proposition 2 to obtain the formula.

The proof of the unicity uses a slight generalization of the approximation theorem. Define two valuations of \( A \) to be equivalent if there is an order isomorphism and the usual commutative diagram, and to be independent if they are not equivalent.

**Lemma.** Let \( Q \) be a noetherian nonimbedded ring which is its own total quotient ring. Let \( v_1, \ldots, v_s \) be independent rank one valuations of \( Q \), let \( u_1, \ldots, u_s \in Q \) and let \( \alpha_i \in v_i(A) \) be finite for \( i = 1, \ldots, s \). There is an element \( u \) of \( Q \) such that \( v_i(u - u_i) = \alpha_i \) for \( i = 1, \ldots, s \).
Proof. \( Q = Q_1 \oplus \cdots \oplus Q_n \) where \( Q_j \) is a local ring of dimension zero, and let

\[
\mathfrak{R}_j = Q_1 \oplus \cdots \oplus Q_{j-1} \oplus Q_j \oplus Q_{j+1} \oplus \cdots \oplus Q_n
\]

where \( \mathfrak{R}_j \) is the nil radical of \( Q_j \). Let \( v_1, \ldots, v_t \) be all of the valuations \( v_1, \ldots, v_s \) which have \( N_{v_j} = N_j \). Then \( v_1, \ldots, v_s \) are naturally independent valuations of \( Q/N_j = k_j \). By the approximation theorem for a field [7, Theorem 18, p. 45], there is an element \( u'_1 \) of \( Q_1 \) with \( v_i(u'_1 - \text{proj}_1 u_i) = \alpha_i \) for \( i = 1, \ldots, t \). Repeat this for each \( N_j \), obtaining \( u'_j \in Q_j \) for \( 2 \leq j \leq n \).

Let \( u = u'_1 \oplus \cdots \oplus u'_n \), and the proof of lemma is complete.

A is assumed to be nonimbedded. Suppose \( n_1 v_1 + \cdots + n_s v_s > 0 \) where \( v_1, \ldots, v_s \) are independent nontrivial rank one valuations finite on \( A \).

It is to be seen that \( n_1 > 0, \ldots, n_{s-1} > 0 \) and \( n_s > 0 \). Let \( u = f/g \in QA \) where \( f \) and \( g \) are elements of \( A \), such that for some \( i \), \( v_i(u) > 0 \) and \( v_i(u) = 0 \) for \( j \neq i \). Then \( v_i(f) > v_i(g), v_j(f) = v_j(g) \) for \( j \neq i \), \( n_i(v_i(f) - v_i(g)) > 0 \) and \( n_i > 0 \).

Example. Let

\[
A = \mathbb{C}[x, y, z]/(x^4+y^2+3z^3) = \mathbb{C}[x, y, Z]/(xy-z^3)
\]

which is normal, analytically irreducible and Cohen-Macaulay. By direct computation \( \mu_A(x) = \mu_A(y) = 3, \mu_A(x+y) = 2 \), and \( \mu_A \) is not a valuation. In fact, \( \mu_A = v_x + v_y \) where \( \mathbb{C}(y/z)[z] \) and \( \mathbb{C}(x/z)[z] \) are the valuation rings of \( v_x \) and \( v_y \) respectively. Note that neither \( x \) nor \( y \) are superficial elements of \( A \).

Example. Let

\[
A = k[w, x, y, z]/(w, x, y, z) = k[s^4, t^4, s^3t^3, t^4] \subset k[s, t]
\]

the polynomial ring in two variables over a field \( k \). \( IA = k[s^4, s^3t, s^2t^2, st^3, t^4] \), \( DA = \{s^4, s^3t, st^3, t^4\} \) and \( A \) is not Cohen-Macaulay. \( A \) is the localization of a projective (graded) ring, and by Proposition 2, \( \mu_A = e_m(A)v_A \) where \( v_A \) is the order valuation of \( A \). By direct computation \( \mu_A(x) = 4 \), so \( e_m(A) = 4 \). Also \( R = k(s/t)[t^4] \) which verifies the formula of the theorem for this example.

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