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FAVARD'S SOLUTION IS THE LIMIT OF $W^{k,p}$-SPLINES

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Dedicated to Professor Michael Golomb
on the Occasion of His 65th Birthday

ABSTRACT. The purpose of this paper is to affirm a conjecture of C. de Boor, namely: The $W^{k,p}$-splines converge in $W^{k,r}[a, b]$ for all r, $1 \leq r < \infty$, to the Favard solution as p tends to infinity.

1. **Introduction.** Let f be an element of the Sobolev space $W^{k,p}[a, b]$ and let $\pi = \{t_i\}_{i=1}^{n+k}$, where $a \leq t_1 \leq t_2 \leq \cdots \leq t_{n+k} \leq b$ and $t_{i+k} - t_i > 0$, $i = 1, \ldots, n$. We define flats G_p, $1 < p \leq \infty$, in $W^{k,p}[a, b]$ by

$$(1.1)\quad \begin{aligned} G_p = \{g \in W^{k,p}[a, b] : g^{(j)}(t_i) = f^{(j)}(t_i) \text{ for} \\ i = 1, \ldots, n + k \text{ where } j = j(i) = \max[i - m : t_m = t_i]\}. \end{aligned}$$

Many authors, including Favard [4], Golomb [6], Jerome and Schumaker [8], Smith [9], Fisher and Jerome [5], Chui and Smith [1], and de Boor [3] have studied problems related to the minimum seminorm problem

$$(1.2)\quad \inf_{g \in G_p} \|D^k g\|_p.$$

We will call solutions to (1.2) $W^{k,p}$-splines (with respect to G_p). There are many equivalent formulations of this problem. Perhaps the most fruitful view was the formulation put forth by Favard [4] and then resurrected by de Boor [3]. A brief account of this method follows. If we denote by $[t_i, \ldots, t_{i+k}]$ the kth divided difference operator, then the set

$$(1.3)\quad G_p^k = \{g \in L^p[a, b] : g = D^k \hat{g} \text{ for some } \hat{g} \in G_p\}$$

has the equivalent description

$$(1.4)\quad G_p^k = \left\{ g \in L^p[a, b] : \int_a^b M_{i,k} g = (k!)[t_i, \ldots, t_{i+k}]\hat{g}, \ i = 1, \ldots, n \text{ for some } \hat{g} \in G_p \right\}.$$

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Here, $M_{i,k}(t)/k! = [t_i, \ldots, t_{i+k}](\cdot - t)_+^{k-1}/(k-1)!$ is a B-spline of order k normalized to have unit integral. It is clear that finding the $W^{k,p}$-spline is equivalent to solving

$$\inf_{g \in G_p^k} \|g\|_p. \tag{1.5}$$

Thus, viewing the $\{M_{i,k}\}_{i=1}^n$ as a sequence in L^q, $1/p + 1/q = 1$, we can in a standard fashion regard (1.5) as a Hahn-Banach extension problem, guaranteeing existence of solution for $1 < p \leqslant \infty$ and uniqueness of solution for $1 < p < \infty$.

Let S_p for $1 < p < \infty$ denote the $W^{k,p}$-spline (hence S_p solves (1.5)). Smith [9] proved that every sequence $p_n \longrightarrow \infty$ has a subsequence $p_{n'} \longrightarrow \infty$ so that $\{S_{p_{n'}}\}$ converges to a $W^{k,\infty}$-spline. de Boor [3], following Favard, produced a $W^{k,\infty}$-spline which is the unique solution of a certain intrinsic sequence of L^∞ minimization problems. This solution is called the Favard solution. We will prove the following theorem which affirms a conjecture of de Boor [3].

THEOREM 1.1. *For $1 < p < \infty$, let S_p be the element of minimal L^p seminorm in G_p and let S_∞ be the Favard solution. Then as $p \longrightarrow \infty$, $S_p \longrightarrow S_\infty$ in $W^{k,r}$ for all r, $1 \leqslant r < \infty$.*

This will be proved by a sequence of lemmas in the next section which are of interest in their own right.

2. **Proof of the main result.** Suppose that we have a finite-dimensional subspace M of $L^\infty(N_1)$, where $N_1 = [a, b]$ is a compact interval, and that $\{f_n\} \subset M$ is a sequence which is bounded and bounded away from zero. Let $\{\epsilon_n\}$ be a sequence of positive numbers which converges monotonically to zero. We wish to construct a "limit point," in some reasonable sense, for some subsequence of $\{h_n\}$, where $h_n \equiv |f_n|^{\epsilon_n} \operatorname{sgn}(f_n)$. As will be made clear below, this is a major step needed to prove Theorem 1.1.

We will actually show that a subsequence of $\{h_n\}$ converges almost uniformly, a.u. (i.e. in $L^p(N_1)$ for all p, $1 \leqslant p < \infty$). Note that h_n can be written as

$$h_n = (|f_n|/\|f_n\|_{N_1})^{\epsilon_n} \|f_n\|_{N_1}^{\epsilon_n} \operatorname{sgn}(f_n/\|f_n\|_{N_1}) \tag{2.1}$$

where $\|g\|_A$ denotes the L^∞-norm of g restricted to the set A. We may assume, by going to a subsequence, if necessary, that $f_n/\|f_n\|_{N_1} \longrightarrow f \in M$ uniformly (since $\dim M < \infty$) and that $\|f_n\|_{N_1}^{\epsilon_n} \longrightarrow m \leqslant 1$. Setting $N_2 = \{t \in N_1 : f(t) = 0\}$, we have

$$|\{t \in N_1 \backslash N_2 : |f(t)| < \epsilon\}| \longrightarrow 0 \quad \text{as } \epsilon \longrightarrow 0 \tag{2.2}$$

and hence

$$\left.\begin{aligned} \operatorname{sgn}(f_n/\|f_n\|_{N_1}) &\longrightarrow \operatorname{sgn} f \\ (|f_n|/\|f_n\|_{N_1})^{\epsilon_n} &\longrightarrow 1 \end{aligned}\right\} \text{a.u. on } N_1\backslash N_2. \tag{2.3}$$

(In the above and throughout, $|A|$ denotes the Lebesgue measure of the set A.) Consequently, h_n converges a.u. on $N_1\backslash N_2$ to $m \operatorname{sgn} f$. If N_2 has measure zero, we are done; otherwise, we restrict our attention to N_2 and repeat the process. The precise details are in the following algorithm, that constructs a function h, which can easily be seen to be the a.u. limit of a subsequence of the sequence $\{h_n\}$.

THE L^p ALGORITHM. Set $M_1 = M$, $N_1 = [a, b]$ and $i = 1$.

Step 1. Passing to a subsequence if necessary, set $m_1 = \lim_{n\to\infty} \|f_n\|_{N_i}^{\epsilon_n}$.

Step 2. If $m_1 = 0$, set $h|_{N_i} = 0$ and stop.

Otherwise, we use the fact that M_i is finite dimensional and, passing to a subsequence if necessary, set $\psi_i = \lim_{n\to\infty} (f_n|_{N_i})/\|f_n\|_{N_i}$.

Step 3. Set $N_{i+1} = \{t \in N_i\colon \psi_i(t) = 0\}$ and set $h|_{N_i} = m_i \operatorname{sgn} \psi_i$.

Step 4. Set $M_{i+1} = \{\phi|_{N_{i+1}}\colon \phi \in M_i\}$.

If $\dim(M_{i+1}) = 0$, stop. Otherwise increase i by one and go to Step 1.

It is easy to see that the algorithm terminates since $\dim(M) < \infty$. We have just proved the following

LEMMA 2.1. *Let M, $\{f_n\}$, and ϵ_n be as above. Then, for some subsequence i_n of positive integers, the sequence*

$$h_{i_n} \equiv |f_{i_n}|^{\epsilon_{i_n}} \operatorname{sgn} f_{i_n} \tag{2.4}$$

converges almost uniformly (i.e. in L^r for all r, $1 \leqslant r < \infty$) to a function h in $L^\infty(N_1)$ with the following properties: For some strictly decreasing sequence $N_1 \supset N_2 \supset \cdots \supset N_{q+1}$ with $q \leqslant \dim M$,

$$h = M_i \operatorname{sgn} \psi_i \quad \text{on } N_i\backslash N_{i+1}, \qquad i = 1, \ldots, q, \tag{2.5}$$

with $m_i \in [0, 1]$ and $\psi_i \in M_i = \{\phi|_{N_i}\colon \phi \in M\}$.

Actually, with a little more work it could be shown that a subsequence could be obtained which converges in measure. We need one more lemma in order to prove Theorem 1.1. Let $M = \operatorname{span}[\phi_1, \ldots, \phi_n]$ be an n-dimensional subspace of $L^\infty[a, b]$, $(-\infty < a < b < \infty)$. For $g \in L^\infty[a, b]$ set

$$\widetilde{G}_p \equiv \widetilde{G}_p(g) \equiv \left\{h \in L^p[a, b]\colon \int_a^b h\phi = \int_a^b g\phi \text{ for all } \phi \in M\right\}. \tag{2.6}$$

By a standard duality argument [7, p. 76] for $1 < p < \infty$ the element of smallest L^p-norm in $\widetilde{G}_p$ is given by

$$s_p = R_p |\psi_q|^{q-1} \operatorname{sgn} \psi_q, \qquad 1/p + 1/q = 1, \tag{2.7}$$

where $\psi_q \in M$ satisfies $\|\psi_q\|_q = 1$ and

$$R_p \equiv \max_{\phi \in M; \|\phi\|_q = 1} \int_a^b g\phi = \int_a^b g\psi_q. \tag{2.8}$$

Clearly, $R_p \longrightarrow R_\infty$ as $p \longrightarrow \infty$. We will show that s_p converges to s_∞ (the Favard solution to be defined below) a.u.

In [3], de Boor describes a procedure first suggested by Favard [4], for producing an "extremal" element of minimal norm in $\widetilde{G}_\infty$. In our notation this procedure is described below.

FAVARD'S PROCEDURE. Let $N_1 = [a, b]$, $M_1 = M$ be as above, and $g_0 \in \widetilde{G}_\infty$.

Step 1. Set $i = 1$.

Step 2. Set $r_i = \inf\{\|g\|_{N_i}: g \in G_{\infty,i}\}$ with

$$G_{\infty,i} = \left\{g \in L^\infty(N_i): \int_{N_i} g\phi = \int_{N_i} g_0\phi \text{ for all } \phi \in M_i\right\}$$

and pick $g_i \in G_{\infty,i}$ satisfying $\|g_i\|_{N_i} = r_i$.

Step 3. If $M_i \equiv \{\phi|_{N_i}: \phi \in M\}$ has dimension zero, set $N_{i+1} \equiv \emptyset$. Otherwise, pick $\psi_i \in M_i$ so that $\|\psi_i\|_{1,N_i} = 1$ and $\|g_i\|_{N_i} = \int_{N_i} g_i\psi_i$ and set $N_{i+1} \equiv \{t \in N_i: \psi_i(t) = 0\}$.

Step 4. Redefine g_0 to equal g_i on N_i.

Step 5. If $m(N_{i+1}) > 0$, increase i by 1 and go to Step 2. Otherwise stop.

A remarkable fact proved in [3] is contained in the theorem below.

THEOREM (DE BOOR). *Favard's procedure produces a unique function which depends only on N_1, M_1, and the flat $\widetilde{G}_\infty$, and does not depend on the initial function $g_0 \in G_\infty$.*

This result is the key to proving that the L^p solutions s_p converge almost uniformly, as $p \longrightarrow \infty$, to the solution produced by Favard's procedure (which we call "Favard's solution").

LEMMA 2.2. *Let s_p be the element of minimal L^p-norm in $\widetilde{G}_p$ where $1 < p < \infty$, and let s_∞ be the Favard solution. Then*

$$s_p \longrightarrow s_\infty \text{ almost uniformly, as } p \longrightarrow \infty. \tag{2.9}$$

PROOF. We will show that the function h produced in Lemma 2.1 as a limit of $S_{p_n} \equiv h_{p_n}$ is s_∞, independent of the initial sequence $p_n \longrightarrow \infty$. This shows via Lemma 2.1 that every sequence has a subsequence with a unique limit which by standard arguments implies that $s_p \longrightarrow s_\infty$ almost uniformly.

It remains to verify that $h = s_\infty$. Let ψ_i', m_i', and N_i' be as in Lemma 2.1 for a given subsequence of $p_n \longrightarrow \infty$. If we begin Favard's procedure with $g_0 = h$, then, by choosing

$$r_1 = m_1', \quad \psi_1 = \psi_1', \quad N_2 = N_2', \tag{2.10}$$

we can insure that g_0 is not changed during the first step. Suppose we have arrived at the ith step with h unchanged on $N_1 \backslash N_{i+1}'$. Then it is easy to see that h is an element of minimal norm in $G_{\infty,i+1}$, since there is a $\psi_{i+1} \equiv \psi_{i+1}' \in M_{i+1}$ satisfying

$$\begin{aligned} \|h\|_{N_{i+1}} &= \left(\int_{N_{i+1}} h\psi_{i+1}'\right) \Big/ \|\psi_{1+1}'\|_{1,N_{i+1}} \\ &= \left(\int_{N_{i+1}} g\psi_{i+1}'\right) \Big/ \|\psi_{i+1}'\|_{1,N_{i+1}} \leqslant \|g\|_{N_{i+1}} \end{aligned} \tag{2.11}$$

for any $g \in G_{\infty,i+1}$. Thus, at each step h can be left unchanged and hence $h \equiv s_\infty$. This completes the proof.

As was observed in the introduction, the minimization problem

$$\inf_{g \in G_p} \|D^k g\|_p \tag{1.2}$$

where $G_p = \{g \in W^{k,p}[a, b] : g^{(j)}(t_i) = f^{(j)}(t_i)\}$ is equivalent to

$$\inf_{g \in G_p^k} \|g\|_p \tag{1.5}$$

where

$$G_p^k = \left\{ g \in L^p[a, b] : \int_a^b M_{i,k} g = (k!)[t_i, \ldots, t_{i+k}]\hat{g}, \right.$$
$$\left. i = 1, \ldots, n \text{ for some fixed } \hat{g} \in G_p \right\}$$

or

$$G_p^k = \left\{ g \in L^p[a, b] : \int_a^b M_{i,k} g = \int_a^b M_{i,k}\hat{g}, \right.$$
$$\left. i = 1, \ldots, n \text{ for some fixed } \hat{g} \in G_p \right\}.$$

PROOF OF THEOREM 1.1. With $M = \operatorname{span}\{M_{i,k}\}_{i=1}^n$, the minimization problem (1.5) falls into the category of those considered in this section. By

Lemma 2.2 the solutions s_p of (1.5) converge almost uniformly to s_∞. Let S_p and S_∞ denote the corresponding elements in G_1 having s_p and s_∞ for their kth derivatives. Now S_p is obtained from s_p by integrating s_p back k times and matching boundary conditions. Since S_p and S_∞ are in G_p, they agree at k points (counting multiplicity) and due to the fact that s_p converges to s_∞ in the L^r-norm ($1 \leqslant r < \infty$) our proof is completed.

3. **General interpolation constraints.** Let $\{\lambda_i\}_{i=1}^{k+n} \subset (W^{k,\infty}[a, b])^*$ be linearly independent and have the form

$$\lambda_i(f) = \sum_{l=1}^{m_i} \sum_{j=0}^{k-1} \alpha_{ij}^{(l)} f^{(j)}(t_{l,i}) \tag{3.1}$$

where $t_{l,i} \in [a, b]$. For simplicity we will assume that

$$\lambda_i(f) = f(t_i), \qquad i = 0, \ldots, k. \tag{3.2}$$

As before we set, for $1 < p \leqslant \infty$, and $\hat{g} \in W^{k,\infty}[a, b]$,

$$G_p = \{f \in W^{k,p}[a, b] : \lambda_i(f) = \lambda_i(\hat{g}) \text{ for } i = 1, \ldots, n + k\}. \tag{3.3}$$

For $1 < p \leqslant \infty$ a solution to problem (1.2) will be called a $W^{k,p}$-spline and if $1 < p < \infty$ the solution is unique (by (3.2) and uniform convexity of L^p) and will be denoted by T_p. We now prove

THEOREM 3.1. *Let G_p and T_p be as above. Then*

$$T_p \longrightarrow T_\infty \quad \text{in } W^{k,r}[a, b], \qquad 1 \leqslant r < \infty, \tag{3.4}$$

as $p \longrightarrow \infty$. Here, the kth derivative of T_∞ is the solution produced by Favard's procedure.

The proof will just be sketched here since it follows the same pattern as the proof of Theorem 1.1. Every $f \in G_p$ for $1 < p \leqslant \infty$ has the representation

$$f(x) = P_k^*(x) + \int_a^b k(x, t) D^k f(t)\, dt \tag{3.5}$$

where P_k^* is a polynomial of degree $(k - 1)$ or less and hence is uniquely determined by (see (3.2))

$$\lambda_i(P_k^*) = \lambda_i(\hat{g}), \qquad i = 1, \ldots, k, \tag{3.6}$$

and where (letting l_j be the Lagrange polynomial of degree $(k - 1)$ determined by λ_j, $j = 1, \ldots, k$)

$$(k - 1)!\,k(x, t) = \sum_{j=1}^{k} [(x - t)_+^{k-1} - (t_j - t)_+^{k-1}]\, l_j(x). \tag{3.7}$$

The kernel $k(x, t)$ is called Kowalewski's exact remainder for polynomial interpolation (see Davis [2, p. 71]).

Thus finding an element of smallest L^p-seminorm in G_p (i.e. L^p-norm of kth derivative), $1 < p \leq \infty$, is equivalent to finding an element of smallest L^p-norm in

$$G_p^k \equiv \left\{ f \in L^p[a, b] : \int_a^b \phi_i f = \lambda_{i+k}(\hat{g}) - \lambda_{i+k}(P_k^*) \text{ for } i = 1, \ldots, n \right\}.$$

Here, $\phi_i(t) = \lambda_{i+k,x} k(x, t)$, $i = 1, \ldots, n$. Since the $\phi_i \in L^\infty[a, b]$ for $i = 1, \ldots, n$, Lemma 2.1, de Boor's theorem, Favard's procedure, and Theorem 2.1 apply.

Of course, much more generality is possible and the authors are currently investigating extensions of these results. The fact that there are, in general, so many best L^∞ interpolants coupled with the fact that the best L^p interpolants converge to exactly one best L^∞ interpolant, leads one to believe that there may be some very sharp theorems involving the structure of the Favard solution. For the interpolation problem discussed in §1, de Boor [3] has derived a partial characterization.

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