FAVARD'S SOLUTION IS THE LIMIT OF $W^{k,p}$-SPLINES

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Dedicated to Professor Michael Golomb
on the Occasion of His 65th Birthday

ABSTRACT. The purpose of this paper is to affirm a conjecture of C. de Boor, namely: The $W^{k,p}$-splines converge in $W^{k,r}[a, b]$ for all $r$, $1 < r < \infty$, to the Favard solution as $p$ tends to infinity.

1. Introduction. Let $f$ be an element of the Sobolev space $W^{k,p}[a, b]$ and let $\pi = \{t_i\}_{i=1}^{n+k}$, where $a \leq t_1 \leq t_2 \leq \cdots \leq t_{n+k} \leq b$ and $t_{i+k} - t_i > 0$, $i = 1, \ldots, n$. We define flats $G_p$, $1 < p < \infty$, in $W^{k,p}[a, b]$ by

$$G_p = \{g \in W^{k,p}[a, b] : g^{(j)}(t_i) = f^{(j)}(t_i) \text{ for } i = 1, \ldots, n + k \text{ where } j = j(i) = \max\{i - m : t_m = t_i\}\}.$$ (1.1)

Many authors, including Favard [4], Golomb [6], Jerome and Schumaker [8], Smith [9], Fisher and Jerome [5], Chui and Smith [1], and de Boor [3] have studied problems related to the minimum seminorm problem

$$\inf_{g \in G_p} \|D^k g\|_p.$$ (1.2)

We will call solutions to (1.2) $W^{k,p}$-splines (with respect to $G_p$). There are many equivalent formulations of this problem. Perhaps the most fruitful view was the formulation put forth by Favard [4] and then resurrected by de Boor [3]. A brief account of this method follows. If we denote by $[t_i, \ldots, t_{i+k}]$ the $k$th divided difference operator, then the set

$$G^k_p = \{g \in L^p[a, b] : g = D^k \hat{g} \text{ for some } \hat{g} \in G_p\}$$ (1.3)

has the equivalent description

$$G^k_p = \{g \in L^p[a, b] : \int_a^b M_{i, k} \hat{g} = (k!) [t_i, \ldots, t_{i+k}] \hat{g}, \quad i = 1, \ldots, n \text{ for some } \hat{g} \in G_p\}.$$ (1.4)
Here, $M_{i,k}(t)/k! = [t_i, \ldots, t_{i+k}](\cdot -t)_{+}^{k-1}/(k-1)!$ is a $B$-spline of order $k$ normalized to have unit integral. It is clear that finding the $W^{k,p}$-spline is equivalent to solving

$$\inf_{g \in G_p} \|g\|_p.$$  

Thus, viewing the $\{M_{i,k}\}_{i=1}^n$ as a sequence in $L^q$, $1/p + 1/q = 1$, we can in a standard fashion regard (1.5) as a Hahn-Banach extension problem, guaranteeing existence of solution for $1 < p < \infty$ and uniqueness of solution for $1 < p < \infty$.

Let $S_p$ for $1 < p < \infty$ denote the $W^{k,p}$-spline (hence $S_p$ solves (1.5)). Smith [9] proved that every sequence $p_n \rightarrow \infty$ has a subsequence $p_n' \rightarrow \infty$ so that $\{S_{p_n'}\}$ converges to a $W^{k,\infty}$-spline. de Boor [3], following Favard, produced a $W^{k,\infty}$-spline which is the unique solution of a certain intrinsic sequence of $L^\infty$ minimization problems. This solution is called the Favard solution. We will prove the following theorem which affirms a conjecture of de Boor [3].

**Theorem 1.1.** For $1 < p < \infty$, let $S_p$ be the element of minimal $L^p$ seminorm in $G_p$ and let $S_\infty$ be the Favard solution. Then as $p \rightarrow \infty$, $S_p \rightarrow S_\infty$ in $W^{k,r}$ for all $r, 1 \leq r < \infty$.

This will be proved by a sequence of lemmas in the next section which are of interest in their own right.

2. **Proof of the main result.** Suppose that we have a finite-dimensional subspace $M$ of $L^\infty(N_1)$, where $N_1 = [a, b]$ is a compact interval, and that $\{f_n\} \subset M$ is a sequence which is bounded and bounded away from zero. Let $\{\epsilon_n\}$ be a sequence of positive numbers which converges monotonically to zero. We wish to construct a "limit point," in some reasonable sense, for some subsequence of $\{h_n\}$, where $h_n \equiv |f_n|\epsilon_n \text{sgn}(f_n)$. As will be made clear below, this is a major step needed to prove Theorem 1.1.

We will actually show that a subsequence of $\{h_n\}$ converges almost uniformly, a.u. (i.e. in $L^p(N_1)$ for all $p, 1 \leq p < \infty$). Note that $h_n$ can be written as

$$h_n = (|f_n|/\|f_n\|_{N_1})\epsilon_n \|f_n\|_{N_1} \text{sgn}(f_n)/\|f_n\|_{N_1}$$

where $\|g\|_A$ denotes the $L^\infty$-norm of $g$ restricted to the set $A$. We may assume, by going to a subsequence, if necessary, that $f_n/\|f_n\|_{N_1} \rightarrow f \in M$ uniformly (since dim $M < \infty$) and that $\|f_n\|_{N_1} \rightarrow m \leq 1$. Setting $N_2 = \{t \in N_1: f(t) = 0\}$, we have

$$\{t \in N_1 \setminus N_2: |f(t)| < \epsilon\} \rightarrow 0 \text{ as } \epsilon \rightarrow 0$$
and hence

\[
\text{sgn}(f_n / \| f_n \|_{N_1}) \rightarrow \text{sgn } f
\]

(2.3) \left\{ \begin{array}{l}
\| f_n / \| f_n \|_{N_1} \|_{N_1}^\epsilon \rightarrow 1 \\
\text{a.u. on } N_1 \setminus N_2.
\end{array} \right.

(In the above and throughout, |A| denotes the Lebesgue measure of the set A.) Consequently, \( h_n \) converges a.u. on \( N_1 \setminus N_2 \) to \( m \text{ sgn } f \). If \( N_2 \) has measure zero, we are done; otherwise, we restrict our attention to \( N_2 \) and repeat the process.

The precise details are in the following algorithm, that constructs a function \( h \), which can easily be seen to be the a.u. limit of a subsequence of the sequence \( \{h_n\} \).

**The \( L^p \) Algorithm.** Set \( M_1 = M, N_1 = [a, b] \) and \( i = 1 \).

Step 1. Passing to a subsequence if necessary, set \( m_1 = \lim_{n \to \infty} \| f_n \|_{N_1}^\epsilon \).

Step 2. If \( m_1 = 0 \), set \( h |_{N_1} = 0 \) and stop.

Otherwise, we use the fact that \( M_1 \) is finite dimensional and, passing to a subsequence if necessary, set \( \psi_1 = \lim_{n \to \infty} (f_n |_{N_1}) / \| f_n \|_{N_1} \).

Step 3. Set \( N_{i+1} = \{ t \in N_i : \psi_1(t) = 0 \} \) and set \( h |_{N_i} = m_i \text{ sgn } \psi_1 \).

Step 4. Set \( M_{i+1} = \{ \phi |_{N_{i+1}} : \phi \in M_i \} \).

If \( \dim(M_{i+1}) = 0 \), stop. Otherwise increase \( i \) by one and go to Step 1.

It is easy to see that the algorithm terminates since \( \dim(M) < \infty \). We have just proved the following

**Lemma 2.1.** Let \( M, \{f_n\}, \) and \( e_n \) be as above. Then, for some subsequence \( i_n \) of positive integers, the sequence

\[
h_{i_n} = \| f_{i_n} \|_{N_1}^\epsilon \text{ sgn } f_{i_n}
\]

(2.4) converges almost uniformly (i.e. in \( L^r \) for all \( r, 1 < r < \infty \)) to a function \( h \) in \( L^\infty(N_1) \) with the following properties: For some strictly decreasing sequence \( N_1 \supset N_2 \supset \cdots \supset N_q+1 \) with \( q \leq \dim M \),

\[
h = m_i \text{ sgn } \psi_i \quad \text{on } N_i \setminus N_{i+1}, \quad i = 1, \ldots, q,
\]

(2.5) with \( m_i \in [0, 1] \) and \( \psi_i \in M_i = \{ \phi |_{N_i} : \phi \in M \} \).

Actually, with a little more work it could be shown that a subsequence could be obtained which converges in measure. We need one more lemma in order to prove Theorem 1.1. Let \( M = \text{span}[\phi_1, \ldots, \phi_n] \) be an \( n \)-dimensional subspace of \( L^\infty(a, b) \), \((-\infty < a < b < \infty)\). For \( g \in L^\infty(a, b) \) set

\[
\widetilde{G}_p \equiv \widetilde{G}_p(g) \equiv \left\{ h \in L^p[a, b] : \int_a^b h \phi = \int_a^b g \phi \quad \text{for all } \phi \in M \right\}.
\]

(2.6)
By a standard duality argument [7, p. 76] for $1 < p < \infty$ the element of smallest $L^p$-norm in $G_p$ is given by
\begin{equation}
(2.7) \quad s_p = R_p |\psi_q|^{q-1} \text{sgn } \psi_q, \quad 1/p + 1/q = 1,
\end{equation}
where $\psi_q \in M$ satisfies $\|\psi_q\|_q = 1$ and
\begin{equation}
(2.8) \quad R_p = \max_{\phi \in M; \|\phi\|_q = 1} \int_a^b g\phi = \int_a^b g\psi_q.
\end{equation}
Clearly, $R_p \to R_\infty$ as $p \to \infty$. We will show that $s_p$ converges to $s_\infty$ (the Favard solution to be defined below) a.u.

In [3], de Boor describes a procedure first suggested by Favard [4], for producing an "extremal" element of minimal norm in $G_\infty$. In our notation this procedure is described below.

**Favard's Procedure.** Let $N_1 = [a, b], M_1 = M$ be as above, and $g_0 \in G_\infty$.

**Step 1.** Set $i = 1$.

**Step 2.** Set $r_i = \inf \{\|g\|_{N_i}: g \in G_{\infty,i}\}$ with
\begin{equation}
(2.9) \quad g_i \in G_{\infty,i} \text{ satisfying } \|g_i\|_{N_i} = r_i.
\end{equation}
and pick $g_i \in G_{\infty,i}$ satisfying $\|g_i\|_{N_i} = r_i$.

**Step 3.** If $M_i = \{\psi_i: \phi \in M\}$ has dimension zero, set $N_{i+1} = \emptyset$. Otherwise, pick $\psi_i \in M_i$ so that $\|\psi_i\|_{1,N_i} = 1$ and $\|g_i\|_{N_i} = \int_{N_i} g_i \psi_i$ and set $N_{i+1} = \{t \in N_i: \psi_i(t) = 0\}$.

**Step 4.** Redefine $g_0$ to equal $g_i$ on $N_i$.

**Step 5.** If $m(N_{i+1}) > 0$, increase $i$ by 1 and go to Step 2. Otherwise stop.

A remarkable fact proved in [3] is contained in the theorem below.

**Theorem (de Boor).** Favard's procedure produces a unique function which depends only on $N_1, M_1$, and the flat $G_\infty$, and does not depend on the initial function $g_0 \in G_\infty$.

This result is the key to proving that the $L^p$ solutions $s_p$ converge almost uniformly, as $p \to \infty$, to the solution produced by Favard's procedure (which we call "Favard's solution").

**Lemma 2.2.** Let $s_p$ be the element of minimal $L^p$-norm in $G_p$ where $1 < p < \infty$, and let $s_\infty$ be the Favard solution. Then
\begin{equation}
(2.9) \quad s_p \to s_\infty \text{ almost uniformly, as } p \to \infty.
\end{equation}
Proof. We will show that the function $h$ produced in Lemma 2.1 as a limit of $S_{pn} \equiv h_{pn}$ is $s_{\infty}$, independent of the initial sequence $p_n \to \infty$. This shows via Lemma 2.1 that every sequence has a subsequence with a unique limit which by standard arguments implies that $s_p \to s_{\infty}$ almost uniformly.

It remains to verify that $h = s_{\infty}$. Let $\psi_{i}, m_{i}$, and $N_{i}'$ be as in Lemma 2.1 for a given subsequence of $p_n \to \infty$. If we begin Favard's procedure with $g_{0} = h$, then, by choosing

$$r_{1} = m_{1}', \psi_{1} = \psi_{1}', N_{2} = N_{2}'$$

we can insure that $g_{0}$ is not changed during the first step. Suppose we have arrived at the $i$th step with $h$ unchanged on $N_{i} \setminus N_{i+1}'$. Then it is easy to see that $h$ is an element of minimal norm in $G_{\infty,i+1}$, since there is a $\psi_{i+1} \equiv \psi_{i+1}$ $\in M_{i+1}$ satisfying

$$\|h\|_{N_{i+1}} = \left(\int_{N_{i+1}} h\psi_{i+1}'\right)/\|\psi_{i+1}'\|_{1,N_{i+1}}$$

$$= \left(\int_{N_{i+1}} g\psi_{i+1}'\right)/\|\psi_{i+1}'\|_{1,N_{i+1}} \leq \|g\|_{N_{i+1}}$$

for any $g \in G_{\infty,i+1}$. Thus, at each step $h$ can be left unchanged and hence $h \equiv s_{\infty}$. This completes the proof.

As was observed in the introduction, the minimization problem

$$(1.2) \inf_{g \in G_{p}} \|D^{k}g\|_{p}$$

where $G_{p} = \{g \in W^{k,p}[a,b] : g^{(n)}(t_{i}) = f^{(n)}(t_{i})\}$ is equivalent to

$$(1.5) \inf_{g \in G_{p}} \|g\|_{p}$$

where

$$G_{p}^{k} = \left\{ g \in L^{p}[a,b] : \int_{a}^{b} M_{i,k}g = (k!)\left[t_{i}, \ldots, t_{i+k}\right]\hat{g}, \right. \left. i = 1, \ldots, n \text{ for some fixed } \hat{g} \in G_{p} \right\}$$

or

$$G_{p}^{k} = \left\{ g \in L^{p}[a,b] : \int_{a}^{b} M_{i,k}g = \int_{a}^{b} M_{i,k}\hat{g}, \right. \left. i = 1, \ldots, n \text{ for some fixed } \hat{g} \in G_{p} \right\}.$$
Lemma 2.2 the solutions $s_p$ of (1.5) converge almost uniformly to $s_\infty$. Let $S_p$ and $S_\infty$ denote the corresponding elements in $G_1$ having $s_p$ and $s_\infty$ for their $k$th derivatives. Now $S_p$ is obtained from $s_p$ by integrating $s_p$ back $k$ times and matching boundary conditions. Since $S_p$ and $S_\infty$ are in $G_p$, they agree at $k$ points (counting multiplicity) and due to the fact that $s_p$ converges to $s_\infty$ in the $L^r$-norm ($1 \leq r < \infty$) our proof is completed.

3. General interpolation constraints. Let $\{\lambda_i\}_{i=1}^{k+n} \subset (W_\infty^{k+n}[a, b])^*$ be linearly independent and have the form

\begin{equation}
\lambda_i(f) = \sum_{i=1}^{m_k} \sum_{j=0}^{k-1} \alpha_{ij}^{(j)} f^{(j)}(t_{i,j})
\end{equation}

where $t_{i,j} \in [a, b]$. For simplicity we will assume that

\begin{equation}
\lambda_i(f) = f(t_i), \quad i = 0, \ldots, k.
\end{equation}

As before we set, for $1 < p \leq \infty$, and $g \in W_\infty^{k,n}[a, b]$,

\begin{equation}
G_p = \{ f \in W_\infty^{k,p}[a, b] : \lambda_i(f) = \lambda_i(g) \text{ for } i = 1, \ldots, n + k \}.
\end{equation}

For $1 < p \leq \infty$ a solution to problem (1.2) will be called a $W_\infty^{k,p}$-spline and if $1 < p < \infty$ the solution is unique (by (3.2) and uniform convexity of $L^p$) and will be denoted by $T_p$. We now prove

**Theorem 3.1.** Let $G_p$ and $T_p$ be as above. Then

\begin{equation}
T_p \rightarrow T_\infty \text{ in } W_\infty^{k,r}[a, b], \quad 1 \leq r < \infty,
\end{equation}

as $p \rightarrow \infty$. Here, the $k$th derivative of $T_\infty$ is the solution produced by Favard's procedure.

The proof will just be sketched here since it follows the same pattern as the proof of Theorem 1.1. Every $f \in G_p$ for $1 < p \leq \infty$ has the representation

\begin{equation}
f(x) = P_k^p(x) + \int_a^b k(x, t) D^k f(t) dt
\end{equation}

where $P_k^p$ is a polynomial of degree $(k - 1)$ or less and hence is uniquely determined by (see (3.2))

\begin{equation}
\lambda_i(P_k^p) = \lambda_i(g), \quad i = 1, \ldots, k,
\end{equation}

and where (letting $l_j$ be the Lagrange polynomial of degree $(k - 1)$ determined by $\lambda_j$, $j = 1, \ldots, k$)

\begin{equation}
(k - 1)! k(x, t) = \sum_{j=1}^{k} [(x - t)_j^{k-1} - (t_j - t)^{k-1}] l_j(x).
\end{equation}

The kernel $k(x, t)$ is called Kowalewski's exact remainder for polynomial interpolation (see Davis [2, p. 71]).
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Thus finding an element of smallest $L^p$–seminorm in $G_p$ (i.e. $L^p$-norm of $k$th derivative), $1 < p \leq \infty$, is equivalent to finding an element of smallest $L^p$-norm in

$$G_p^k = \left\{ f \in L^p[a, b] : \int_a^b \phi_i f = \lambda_{i+k}(\xi) - \lambda_{i+k}(\xi_k^*) \text{ for } i = 1, \ldots, n \right\}.$$ 

Here, $\phi_i(x) = \lambda_{i+k}(x, i)$, $i = 1, \ldots, n$. Since the $\phi_i \in L^\infty [a, b]$ for $i = 1, \ldots, n$, Lemma 2.1, de Boor’s theorem, Favard’s procedure, and Theorem 2.1 apply.

Of course, much more generality is possible and the authors are currently investigating extensions of these results. The fact that there are, in general, so many best $L^\infty$ interpolants coupled with the fact that the best $L^p$ interpolants converge to exactly one best $L^\infty$ interpolant, leads one to believe that there may be some very sharp theorems involving the structure of the Favard solution. For the interpolation problem discussed in §1, de Boor [3] has derived a partial characterization.

REFERENCES


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