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DEFORMATIONS OF FORMAL EMBEDDINGS OF SCHEMES([1])

BY

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ABSTRACT. A family of isolated singularities of k-varieties will be here called *equisingular* if it can be simultaneously resolved to a family of hypersurfaces embedded in nonsingular spaces which induce only locally trivial deformations of pairs of schemes over local artin k-algebras. The functor of locally trivial deformations of the formal embedding of an exceptional set has a versal object in the sense of Schlessinger. When the exceptional set X_0 is a collection of nonsingular curves meeting normally in a nonsingular surface X, the moduli correspond to Laufer's moduli of thick curves. When X is a nonsingular scheme of finite type over an algebraically closed field k and X_0 is a reduced closed subscheme of X, every deformation of (X, X_0) to $k[\epsilon]$ such that the deformation of X_0 is locally trivial, is in fact a locally trivial deformation of pairs.

1. **Introduction.** Much progress has been made recently in the classification of normal singularities of complex analytic surfaces by considering their resolutions (see [3], [7], [11], [12], [13]). The present paper investigates deformations of formally embedded schemes with the aim of eventually using these objects in the algebraic category to classify singularities of dimension two and higher. The present work suggests the following notion of equisingularity: A family of (isolated) singularities is *equisingular* if it can be resolved simultaneously to a family of embeddings in nonsingular spaces which induce only locally trivial deformations of pairs of schemes over any local artin k-algebra.

This paper looks at infinitesimal deformations from the functorial point of view of Rim [9] and Schlessinger [10]. Specifically, fix a pair $(\mathfrak{X}, X_0)$ consisting of a locally noetherian formal scheme $\mathfrak{X}$ and a locally noetherian ordinary scheme X_0, both defined over a field k, such that X_0 is a subscheme of $\mathfrak{X}$ given by an $\mathfrak{O}_{\mathfrak{X}}$-ideal of definition $\mathfrak{J}$. More generally, we set $X_n = (\mathfrak{X}, \mathfrak{O}_{\mathfrak{X}}/\mathfrak{J}^{n+1})$ for any nonnegative integer n. §2 defines the functor $[M_{\mathfrak{X},X_0}]$ of isomorphism classes of deformations of the pair $(\mathfrak{X}, X_0)$ to artin local k-algebras with the same residue field k and establishes that it has a "good" deformation theory (Proposition 2.7). We also consider the subfunctor $[L_{\mathfrak{X},X_0}]$ of isomorphism classes of locally

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trivial deformations of pairs (Definition 2.6).

For certain nice pairs, in particular those corresponding to the completion of a nonsingular scheme X of finite type over k along a contractable subscheme, $[L_{\mathfrak{X},X_0}]$ has a pro-representable hull in the sense of Schlessinger [10, Theorem 4.4 and corollaries]. In these cases, the tangent space of the locally trivial functor is precisely the space $H^1(X_n, \mathfrak{Der}^{\mathfrak{J}}(X_n))$, for large enough n, where $\mathfrak{Der}^{\mathfrak{J}}(X_n)$ is the sheaf of derivations of $\mathfrak{O}_{X_n}$ into itself which induce derivations of $\mathfrak{O}_{X_0}$. When X_0 is locally a complete intersection in X, and k has characteristic zero, then $\mathfrak{Der}^{\mathfrak{J}}(X_n)$ is the full tangent sheaf $\mathfrak{Der}(X_n)$, corresponding to Laufer's sheaf ${}_{X_n}\theta$ in the analytic case (Corollary 4.10). Thus, our tangent space is the analog of Laufer's moduli for deformations of thick curves [7, Theorems 2.2, 2.3]. Our proof is very straightforward, allowing explicit calculations at the end.

In some cases we may compare the functors $[M_{X,X_0}]$, $[M_{\mathfrak{X},X_0}]$ and $[L_{\mathfrak{X},X_0}]$ (§§5 and 7). §6 gives a geometric interpretation of our results and definitions and suggests that $[L_{\mathfrak{X},X_0}]$ is the proper functor for preserving equisingularity. The final section contains several examples, most notably the case of exceptionally embedded curves in a nonsingular surface.

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2. **Deformation functors of the pair $(\mathfrak{X}, X_0)$.** Let k be an arbitrary field and let $\mathfrak{C}_k = \mathfrak{C}$ be the category of local artinian k-algebras with residue field k. Each ring in $\mathfrak{C}$ is given the discrete topology. For an object R in $\mathfrak{C}$, the formal spectrum of R, Spf(R), is exactly Spec(R) as ringed topological spaces. We mimic the constructions and arguments in the ordinary case found in Schlessinger [10] and Rim [9], using the notation of §1. Let R be an object of $\mathfrak{C}$.

DEFINITION 2.1. *An (infinitesimal) deformation of* $(\mathfrak{X}, X_0)$ *to* R will be a pair $(\mathfrak{Y}, Y_0)$, where $\mathfrak{Y}$ is a formal adic prescheme over R and Y_0 is a closed subscheme of $\mathfrak{Y}$ given by an ideal of definition $\mathfrak{I}$ of $\mathfrak{Y}$, with the following properties:

(1) $\mathfrak{O}_{\mathfrak{Y}}$ and $\mathfrak{O}_{Y_0}$ are flat sheaves of R-modules.

(2) There is a commutative diagram of formal schemes

$$\begin{array}{cccc} i_Y: & \mathfrak{X} & \longrightarrow & \mathfrak{Y} \\ & \downarrow & & \downarrow \\ & \mathrm{Spf}(k) & \longrightarrow & \mathrm{Spf}(R) \end{array}$$

identifying $\mathfrak{X}$ with $\mathfrak{Y} \times_R k$ and X_0 with $Y_0 \times_R k$.

Note that we write $\mathfrak{Y} \times_R S$ for $\mathfrak{Y} \times_{\mathrm{Spf}(R)} \mathrm{Spf}(S)$ for a morphism $R \longrightarrow S$ in $\mathfrak{C}$.

Choose any open affine formal subscheme U of $\mathfrak{Y}$. If $B = \Gamma(U, \mathfrak{O}_{\mathfrak{Y}})$ and $J = \Gamma(U, \mathfrak{I})$, then by (2) we know that $i_{\mathfrak{Y}}^{-1}(U) = \mathrm{Spf}(A)$, where $A = B \hat{\otimes}_R k = \mathrm{proj\,lim}(B/J^{n+1} \otimes_R k)$ is an open formal affine subscheme of $\mathfrak{X}$, and $B/J \otimes_R k = A_0$ is a quotient of A by an ideal of definition of A. Since A is a complete noetherian adic ring, so is B. Indeed, it is not difficult to prove the following fact.

PROPOSITION 2.2. *Let B be an adic R-algebra, for R an artin local k-algebra with residue field k. Suppose that $B \hat{\otimes}_R k = A$ is a noetherian k-algebra. Then B is noetherian.*

In particular, $B \hat{\otimes}_R k = B \otimes_R k$.

We may note that $i_{\mathfrak{Y}}^*(\mathfrak{I}^n) = \mathfrak{J}^n$, for all positive integers n. Since R is an artinian ring, $i_{\mathfrak{Y}}\colon \mathfrak{X} \longrightarrow \mathfrak{Y}$ is a closed immersion which is a homeomorphism on underlying topological spaces.

The *trivial* deformation of $(\mathfrak{X}, X_0)$ to R is the pair $(\mathfrak{X} \times_k R, X_0 \times_k R)$, with the obvious map $i_{\mathfrak{Y}}$.

Let $(\mathfrak{Y}, Y_0)$ and $(\mathfrak{Z}, Z_0)$ be deformations of $(\mathfrak{X}, X_0)$ to R and S respectively. A *morphism* $(\mathfrak{Z}, Z_0) \longrightarrow (\mathfrak{Y}, Y_0)$ is a pair (u, F) where $u\colon S \longrightarrow R$ is a morphism in $\mathfrak{C}$ and $F\colon \mathfrak{Y} \longrightarrow \mathfrak{Z}$ is a morphism of formal schemes such that if $\mathfrak{I}$ is the $\mathfrak{O}_{\mathfrak{Y}}$-ideal defining Y_0 and $\mathfrak{K}$ the $\mathfrak{O}_{\mathfrak{Z}}$-ideal defining Z_0, the following conditions are satisfied:

(1) $F^*(\mathfrak{K}) \cdot \mathfrak{O}_{\mathfrak{Y}} = \mathfrak{I}$.

(2) Diagram (2.3) is commutative.

(2.3)
$$\begin{tikzcd}
\mathfrak{Y} \arrow[rrrr, "F"] \arrow[dddd] & & & & \mathfrak{Z} \arrow[dddd] \\
& & \mathfrak{X} \arrow[ull, "i_{\mathfrak{Y}}"'] \arrow[urr, "i_{\mathfrak{Z}}"] \arrow[d] & & \\
& & \mathrm{Spf}(k) \arrow[ddll] \arrow[ddrr] & & \\
& & & & \\
\mathrm{Spf}(R) \arrow[rrrr, "\mathrm{Spf}(u)"] & & & & \mathrm{Spf}(S)
\end{tikzcd}$$

In particular, if $(\mathfrak{Y}, Y_0)$ and $(\mathfrak{Z}, Z_0)$ are both deformations of $(\mathfrak{X}, X_0)$ to R, we say that $(\mathfrak{Y}, Y_0)$ and $(\mathfrak{Z}, Z_0)$ are *isomorphic* if there is a morphism (id_R, F):

$(\mathfrak{Z}, Z_0) \longrightarrow (\mathfrak{Y}, Y_0)$. In this case, $\mathfrak{Y}$ and $\mathfrak{Z}$ are isomorphic as formal schemes in such a way that Y_0 and Z_0 are identified.

DEFINITION 2.4. $M_{\mathfrak{X},X_0}(R) \equiv$ the collection of all deformations of $(\mathfrak{X}, X_0)$ to R, for R an object of $\mathfrak{C}$. $[M_{\mathfrak{X},X_0}(R)] \equiv$ {isomorphism classes of deformations of $(\mathfrak{X}, X_0)$ to R}.

The latter defines a covariant functor from $\mathfrak{C}$ to $\mathfrak{Sets}$ such that $[M_{\mathfrak{X},X_0}(k)]$ consists of the single class $[(\mathfrak{X}, X_0)]$. For u: $R \longrightarrow S$ any $\mathfrak{C}$-morphism and $(\mathfrak{Y}, Y_0)$ a deformation of $(\mathfrak{X}, X_0)$ to R, we find that $[M_{\mathfrak{X},X_0}](u) \cdot [\mathfrak{Y}, Y_0] = [\mathfrak{Z}, Z_0]$, where $\mathfrak{Z} = \mathfrak{Y} \times_R S$ and Z_0 is defined by the $\mathfrak{O}_{\mathfrak{Z}}$-ideal of definition $\mathfrak{K} = \mathfrak{I} \otimes_R S$, $\mathfrak{I}$ the $\mathfrak{O}_{\mathfrak{Y}}$-ideal defining Z_0.

Let $U \subset \mathfrak{X}$ be an open formal subscheme of $\mathfrak{X}$. We have the natural restriction functors over $\mathfrak{C}$, $M_{\mathfrak{X},X_0} \longrightarrow M_{\mathfrak{X}|_U, X_0|_U}$ and $[M_{\mathfrak{X},X_0}] \longrightarrow [M_{\mathfrak{X}|_U, X_0|_U}]$.

DEFINITION 2.6. A *locally trivial* deformation of $(\mathfrak{X}, X_0)$ to R is a deformation $(\mathfrak{Y}, Y_0)$ of $(\mathfrak{X}, X_0)$ to R with the following property: There is a covering of $\mathfrak{X}$ by affine open formal subschemes U_α such that $(\mathfrak{Y}|_{U_\alpha}, Y_0|_{U_\alpha})$ is isomorphic to the trivial deformation of $(\mathfrak{X}|_{U_\alpha}, X_0|_{U_\alpha})$ to R.

We define a subfunctor of $[M_{\mathfrak{X},X_0}]$.

$$[L_{\mathfrak{X},X_0}(R)] \equiv \{\text{isomorphism classes of locally trivial deformations of } (\mathfrak{X}, X_0) \text{ to } R\}.$$

PROPOSITION 2.7. *Suppose $\mathfrak{X}$ is a locally noetherian formal prescheme over k, $\mathfrak{J}$ an ideal of definition for $\mathfrak{X}$, and $X_0 = (\mathfrak{X}, \mathfrak{O}_{\mathfrak{X}}/\mathfrak{J})$. Then the functors $[M_{\mathfrak{X},X_0}]$ and $[L_{\mathfrak{X},X_0}]$ both satisfy conditions* H_1 *and* H_2 *of Schlessinger's Theorem* 2.11 [10].

PROOF. Suppose we have a diagram of morphisms

$$(2.8)\qquad \begin{array}{ccc} (\mathfrak{Y}', Y_0') & & (\mathfrak{Y}'', Y_0'') \\ & {\scriptstyle (u', F')}\searrow \quad \swarrow{\scriptstyle (u'', F'')} & \\ & (\mathfrak{Y}, Y_0) & \end{array}$$

of deformations $(\mathfrak{Y}', Y_0')$, $(\mathfrak{Y}'', Y_0'')$ and $(\mathfrak{Y}, Y_0)$ to R', R'', and R, respectively. We may set $\mathfrak{O}_{\mathfrak{Z}} = \mathfrak{O}_{\mathfrak{Y}'} \times_{\mathfrak{O}_{\mathfrak{Y}}} \mathfrak{O}_{\mathfrak{Y}''}$, a sheaf of rings on the underlying topological space of $\mathfrak{X}$. Let the ideal $\mathfrak{I}'$ (resp. $\mathfrak{I}''$, resp. $\mathfrak{I}$) define the embedding of Y_0' (resp. Y_0'', resp. Y_0) in $\mathfrak{Y}'$ (resp. $\mathfrak{Y}''$, resp. $\mathfrak{Y}$). Then the sheaf of ideals $\mathfrak{K} = \mathfrak{I}' \times_{\mathfrak{I}} \mathfrak{J}''$ defines an (adic) topology on $\mathfrak{O}_{\mathfrak{Z}}$ in such a way that whenever u'' is surjective, the ring of sections of $\mathfrak{O}_{\mathfrak{Z}}$ over an affine open U is complete in its topology, and the projection maps $\mathfrak{P}_1: \mathfrak{O}_{\mathfrak{Z}} \longrightarrow \mathfrak{O}_{\mathfrak{Y}}$ and $\mathfrak{P}_2: \mathfrak{O}_{\mathfrak{Z}} \longrightarrow \mathfrak{O}_{\mathfrak{Y}''}$ are continuous. Furthermore, setting $Z_0 = (\mathfrak{X}, \mathfrak{O}_{\mathfrak{Z}}/\mathfrak{K})$ we see that

$$\mathfrak{O}_{Z_0} = \mathfrak{O}_{Y_0'} \times_{\mathfrak{O}_{Y_0}} \mathfrak{O}_{Y_0''}.$$

Now the proof that H_1 and H_2 hold for $[M_{\mathfrak{X},X_0}]$ follows exactly as in the scheme case [10, §3]. If the deformations of diagram (2.8) are all locally trivial, then so is the deformation $(\mathfrak{Z}, Z_0)$ defined above. Therefore $[L_{\mathfrak{X},X_0}]$ satisfies H_1 and H_2 as well.

The existence of versal elements for $[M_{\mathfrak{X},X_0}]$ and $[L_{\mathfrak{X},X_0}]$ now rests entirely on the dimension of the tangent space to these functors. The obvious statement corresponding to [10, Lemma 3.8] also holds, giving a condition for H_4.

3. **Special automorphisms of $A \otimes k[\epsilon]$ and the sheaves $\mathfrak{Der}^{\mathfrak{I}}(X_n)$.** Let A be any k-algebra. The group of $k[\epsilon]$-automorphisms ν of the ring $A \otimes_k k[\epsilon]$ which induce the identity when tensored with k is isomorphic to the A-module $\mathrm{Der}_k(A) = \mathrm{Hom}_A(\Omega_A, A)$. Here $\Omega_A = \Omega_{A/k}$ is the A-module of 1-differentials with respect to k. Specifically, for a and b in A, we can write uniquely $\nu(a + b\epsilon) = a + b\epsilon + D(a)\epsilon$. If we write $\nu \sim D$, then $\nu^{-1} \sim -D$.

Let I be an ideal of A. The group of $k[\epsilon]$-automorphisms ν of the ring $A \otimes_k k[\epsilon]$ which induce the identity when tensored with k and such that $\nu(I \otimes_k k[\epsilon]) = I \otimes_k k[\epsilon]$ will be denoted $\mathrm{Autid}(A, I)$. The derivation D associated to such a ν has the property that $D(I) \subset I$. Conversely, if $D(I) \subset I$, then the $k[\epsilon]$-automorphism derived from D lies in $\mathrm{Autid}(A, I)$.

Definition 3.1. $\mathrm{Der}_k^I(A) = \mathrm{Ker}\, p$, where p is the natural map $p\colon \mathrm{Der}_k(A) \longrightarrow \mathrm{Hom}_{A/I}(I/I^2, A/I)$.

The association of automorphisms and derivations described above induces a group isomorphism between $\mathrm{Autid}(A, I)$ and $\mathrm{Der}_k^I(A)$. In particular, $\mathrm{Autid}(A, I)$ is an abelian group which is in fact an A-module. When there is no danger of confusion we will drop the subscript k.

Assume that A is noetherian and give A the I-adic topology. If $\hat{A}$ is the completion of A, then $\hat{I} = I\hat{A}$ is the completion of I in the induced topology. The ideal $\hat{I} \otimes_k k[\epsilon]$ is an ideal of definition for the tensor product $\hat{A} \otimes_k k[\epsilon] = \hat{A} \hat{\otimes}_k k[\epsilon]$. For all nonnegative integers n, set $A_n = A/I^{n+1}$ and $I_n = I/I^{n+1}$.

Lemma 3.2. $\mathrm{Autid}(\hat{A}, \hat{I}) = \mathrm{proj\,lim}\, \mathrm{Autid}(A_n, I_n)$.

The proof of this is easy and will be left to the reader.

Since $\mathrm{Autid}(A_n, I_n) = \mathrm{Der}^{I_n}(A_n)$, we also have the identity $\mathrm{Autid}(A_n, I_n) = \mathrm{proj\,lim}\, \mathrm{Der}^{I_n}(A_n)$. We investigate the projective sequence $\{\mathrm{Der}^{I_n}(A_n)\}$ a little more closely. Whenever $0 < m < n$, we have a diagram of canonical homomorphisms with exact rows

$$
\begin{array}{ll}
 & 0 \longrightarrow \mathrm{Der}(A_n) \xrightarrow{i_n} \mathrm{Hom}_{A_n}(\Omega_A \otimes_A A_n, A_n) \xrightarrow{j_n} \mathrm{Hom}_{A_n}(I^{n+1}/(I^{n+1})^2, A_n) \\
 & \qquad\qquad\qquad\qquad\qquad\qquad \Big\downarrow g_{mn} \\
(3.3) & 0 \longrightarrow \mathrm{Der}(A_m) \xrightarrow{i_m} \mathrm{Hom}_{A_m}(\Omega_A \otimes_A A_m, A_m) \xrightarrow{j_m} \mathrm{Hom}_{A_m}(I^{m+1}/(I^{m+1})^2, A_m) \\
 & \qquad\qquad\qquad\qquad\qquad\qquad \Big\downarrow g_{0m} \\
 & 0 \longrightarrow \mathrm{Der}(A_0) \xrightarrow{i_0} \mathrm{Hom}_{A_0}(\Omega_A \otimes_A A_0, A_0) \xrightarrow{j_0} \mathrm{Hom}_{A_0}(I/I^2, A_0)
\end{array}
$$

Now we claim that for all m, $\mathrm{Ker}\, j_0 \circ g_{0m} \subset \mathrm{Ker}\, j_m$. In fact, suppose f is an element of $\mathrm{Hom}_{A_m}(\Omega_A \times_A A_m, A_m) = \mathrm{Hom}_A(\Omega_A, A_m)$ and that f lies in $\mathrm{Ker}\, j_0 \circ g_{0m}$. This means that $(j_0 \circ g_{0m})(f)[i^*] = 0$, for all elements i in I, where i^* stands for residue class modulo I^2. In turn, this means that $g_{0m}(f)[di] = 0$ in A_0, for all i in I, which is equivalent to $f[di] \in I_m$ in A_m. But then certainly $f[dI^{m+1}] \subset (m+1)I_m^{m+1} = 0$ in A_m, so f lies in $\mathrm{Ker}\, j_m$.

However, for all $m \geqslant 1$, the canonical isomorphism $\Omega_A \otimes_A A_0 \cong \Omega_{A_m} \otimes_A A_0$ [EGA 0, 20.5.12] allows us to write $j_0 \circ g_{0m} \circ i_m = p_m$, where p_m is the natural map $\mathrm{Der}(A_m) \longrightarrow \mathrm{Hom}_{A_0}(I/I^2, A_0)$ of Definition 3.1. Thus, since i_m is injective, we have $\mathrm{Der}^{I_m}(A_m) = \mathrm{Ker}\, j_0 \circ g_{0m}$, for all $m \geqslant 1$. Since $g_{0n} = g_{0m} \circ g_{mn}$ whenever $0 < m < n$, we see that the maps making $\{\mathrm{Der}^{I_n}(A_n)\}$ into a projective sequence are just the restrictions g^*_{mn} of the maps g_{mn} to the submodules $\mathrm{Der}^{I_n}(A_n)$. Notice that g^*_{mn} is surjective if g_{mn} is.

We shall leave out the subscript n in I_n when there is no danger of confusion. Thus, $\mathrm{Der}^I(A_n) = \mathrm{Der}^{I_n}(A_n)$.

Let X be a scheme of finite type over k, $\mathfrak{J}$ a coherent $\mathfrak{O}_X$-ideal and $X_0 = (X, \mathfrak{O}_X/\mathfrak{J})$. All the groups and morphisms above may be sheafified. We will use German type and the scheme letters X, X_0, etc. for the sheaves and sheaf maps whose sections over an affine open U of X are given by the groups and homomorphisms above, with the corresponding Roman names. In particular, there is a natural isomorphism $\mathfrak{Der}^{\mathfrak{J}}(X) \approx \mathfrak{Autid}(X, X_0)$ where the right-hand side is the sheaf of sections of $k[\epsilon]$-automorphisms of $X \times k[\epsilon]$ which induce automorphisms of $X_0 \times k[\epsilon]$, and the left-hand side is the coherent sheaf of $\mathfrak{O}_X$-modules which is the kernel of the map

$$
(3.4) \qquad \mathfrak{p}\colon \mathfrak{Der}(X) = \mathfrak{Hom}_{\mathfrak{O}_X}(\Omega_X, \mathfrak{O}_X) \longrightarrow \mathfrak{Hom}_{\mathfrak{O}/\mathfrak{J}}(\mathfrak{J}/\mathfrak{J}^2, \mathfrak{O}/\mathfrak{J}).
$$

Now let $\mathfrak{X}$ be a locally noetherian formal scheme over k, $\mathfrak{J}$ an ideal of definition for $\mathfrak{X}$. With the notation of §1, each sheaf $\mathfrak{Der}^{\mathfrak{J}_n}(X_n) = \mathfrak{Der}^{\mathfrak{J}}(X_n)$

is a coherent $\mathfrak{O}_{X_n}$-module. Define $\mathfrak{F} = \text{proj lim } \mathfrak{Der}^{\mathfrak{J}}(X_n)$ in the category of sheaves of abelian groups on the underlying topological space of $\mathfrak{X}$, where the maps $\mathfrak{g}_{mn}$: $\mathfrak{Der}^{\mathfrak{J}}(X_n) \longrightarrow \mathfrak{Der}^{\mathfrak{J}}(X_m)$ for $n > m$ correspond to the restrictions of the maps g_{mn} of (3.3) to the kernels of $j_0 \circ g_{0n}$. It follows that for U an open affine subscheme of $\mathfrak{X}$, $A = \Gamma(U, \mathfrak{O}_{\mathfrak{X}})$ is complete in the I-adic topology when $I = \Gamma(U, \mathfrak{J})$, and

$$\begin{aligned}\mathfrak{F}(U) &= \text{proj lim } \Gamma(U, \mathfrak{Der}^{\mathfrak{J}}(X_n)) = \text{proj lim } \mathrm{Der}^I(A_n)\\ &= \mathrm{Autid}(A, I) = \mathrm{Der}^I(A).\end{aligned}$$

We will denote the sheaf $\mathfrak{F}$ thus defined by $\mathfrak{Autid}(\mathfrak{X}, X_0)$ or $\mathfrak{Der}^{\mathfrak{J}}(\mathfrak{X})$.

When $\mathfrak{X}$ is locally the completion of a scheme X of finite type over k along a closed subscheme X_0, the association $\mathfrak{G} \longrightarrow \hat{\mathfrak{G}}$ attaching to a coherent sheaf of $\mathfrak{O}_X$-modules its completion along X_0 is an exact functor. In particular, if we write $\Omega_{\mathfrak{X}} = \hat{\Omega}_X$ and set $\mathfrak{Der}\ \mathfrak{X} = \widehat{\mathfrak{Der}(X)}$, then $\mathfrak{Der}\ \mathfrak{X} = \mathfrak{Hom}_{\mathfrak{O}_{\mathfrak{X}}}(\Omega_{\mathfrak{X}}, \mathfrak{O}_{\mathfrak{X}})$ and $\mathfrak{Der}^{\mathfrak{J}}(\mathfrak{X}) = \mathfrak{Autid}(\mathfrak{X}, X_0) = \mathfrak{Ker}\ \mathfrak{p}$, where $\mathfrak{p}$ is the natural morphism of coherent sheaves of $\mathfrak{O}_{\mathfrak{X}}$-modules corresponding to (3.1).

4. Versal objects for $[L_{\mathfrak{X}, X_0}]$.

PROPOSITION 4.1. *Suppose $\mathfrak{X}$ is a locally noetherian formal scheme over k, $\mathfrak{J}$ an ideal of definition for X such that $X_n = (\mathfrak{X}, \mathfrak{O}_{\mathfrak{X}}/\mathfrak{J}^{n+1})$ is a scheme of finite type over k for $n = 0, 1, 2, \ldots$. Then there is an isomorphism of k-vector spaces $t_L \cong H^1(\mathfrak{X}, \mathfrak{Autid}(\mathfrak{X}, X_0))$ where L is the functor of isomorphism classes of locally trivial deformations of $(\mathfrak{X}, X_0)$ to objects of the category $\mathfrak{C}$, and $t_L = [L(\mathfrak{X}, X_0)](k[\epsilon])$.*

PROOF. When $(\mathfrak{Y}, Y_0)$ is a locally trivial deformation of $(\mathfrak{X}, X_0)$ to $k[\epsilon]$, there is a covering of $\mathfrak{X}$ by open affine formal schemes $\{U_\alpha\}$ and $k[\epsilon]$-sheaf isomorphisms:

$$i_\alpha\colon (\mathfrak{O}_{\mathfrak{Y}}, \mathfrak{O}_{Y_0})|_{U_\alpha} \xrightarrow{\sim} (\mathfrak{O}_{\mathfrak{X}} \otimes k[\epsilon], \mathfrak{O}_{X_0} \otimes k[\epsilon])|_{U_\alpha}$$

inducing the identity when tensored with k. The collection $\{\nu_{\alpha\beta}\}$, where $\nu_{\alpha\beta} = i_\alpha \circ i_\beta^{-1}$, forms a cocycle in $Z'(\{U_{\alpha\beta}\}, \mathfrak{Autid}(\mathfrak{X}, X_0))$. The rest of the details are as usual.

In order to calculate the dimension of t_L as a k-vector space, we will want to know when taking first cohomology commutes with taking projective limits of sheaves. From [EGA 0, 13.1 and 13.3] we get the following definition and proposition.

DEFINITION 4.2. A projective system $(A_\alpha, f_{\alpha\beta})$ satisfies *condition ML* (the Mittag-Leffler condition) if for all α, there is a $\beta \geqslant \alpha$ such that $\nu \geqslant \beta$ implies that $f_{\alpha\nu}(A_\nu) = f_{\alpha\beta}(A_\beta)$.

DEFINITION 4.3. Let X be a topological space, $(\mathfrak{F}_k)_{k\in N}$ a projective system of sheaves of abelian groups on X, $\mathfrak{F} = \text{proj lim } \mathfrak{F}_k$. Let $\mathfrak{B}$ be a basis of open sets for the topological space X and suppose the following conditions are satisfied:

(a) $H^i(U, \mathfrak{F}_k) = 0$ for all $k \geqslant 0$, $i > 0$, and $U \in \mathfrak{B}$.

(b) For all $U \in \mathfrak{B}$, $\Gamma(U, \mathfrak{F}_k) \longrightarrow \Gamma(U, \mathfrak{F}_h)$ is surjective whenever $h \leqslant k$.

Then for $i > 0$, the natural map h_i: $H^i(X, \mathfrak{F}) \longrightarrow \text{proj lim}_k H^i(X, \mathfrak{F}_k)$ is surjective.

If in addition, for some i, $(H^{i-1}(X, \mathfrak{F}_k))_{k\in N}$ satisfies condition ML, then h_i is a bijection, while h_0 is always an isomorphism without any hypotheses.

THEOREM 4.4. *Let $\mathfrak{X}$ be a formal scheme which is locally the completion of a nonsingular scheme of finite type over k along a closed subscheme. Let $\mathfrak{I}$ be an ideal of definition for $\mathfrak{X}$ and set $X_n = (\mathfrak{X}, \mathfrak{O}_{\mathfrak{X}}/\mathfrak{I}^{n+1})$. Then there is a surjective map $t_L \longrightarrow$ proj lim $H^1(\mathfrak{X}, \mathfrak{Der}^{\mathfrak{I}}(X_n))$ when L is the functor of isomorphism classes of locally trivial infinitesimal deformations of $(\mathfrak{X}, X_0)$.*

If, in addition, X_0 is proper over k, then this map is a bijection of k-vector spaces.

PROOF. The basis $\mathfrak{B}$ will consist of all the affine open subschemes U such that on U, $\mathfrak{X}$ is the completion of a nonsingular scheme X along a closed subscheme, which we may assume to be X_0. Consider the projective system of sheaves $(\mathfrak{Der}^{\mathfrak{I}}(X_n))_{n\in N}$. Let $A = \Gamma(U, \mathfrak{O}_X)$ and $I = \Gamma(U, \mathfrak{I})$ where $\mathfrak{I}$ is the $\mathfrak{O}_X$-sheaf of ideals defining the embedding of X_0 in X. The hypotheses imply that the finitely generated A-module Ω_A is projective, so that the maps g_{mn} of (3.3) are surjective for $m < n$. Then their restrictions g^*_{mn} to $\Gamma(U, \mathfrak{Der}^{\mathfrak{I}}(X_n))$ are surjective for all U in $\mathfrak{B}$. Hence condition 4.3(b) holds. Condition 4.3(a) is satisfied for the basis $\mathfrak{B}$ since the sheaves are all coherent. Propositions 4.1 and 4.3 imply the first statement of the theorem.

Now assume that X_0 is proper over k and fix $m \geqslant 0$. The scheme X_m is also proper and of finite type over k. Fix i and set for $n \geqslant m$

$$C_n = \text{Im } \mathfrak{g}_{mn}: H^i(\mathfrak{X}, \mathfrak{Der}^{\mathfrak{I}}(X_n)) \longrightarrow H^i(\mathfrak{X}, \mathfrak{Der}^{\mathfrak{I}}(X_m)).$$

The collection $\{C_n\}$ forms a descending chain of k-vector spaces of the finite dimensional space $H^i(\mathfrak{X}, \mathfrak{Der}^{\mathfrak{I}}(X_m))$. This means there is an integer $n' \geqslant m$ such that $C_n = C_{n'}$, whenever $n \geqslant n'$. In other words, the ML condition holds for the sequence $(H^i(\mathfrak{X}, \mathfrak{Der}^{\mathfrak{I}}(X_n)))$, for all i. The second part of Proposition 4.3 now gives us the final statement of our theorem.

Note that what was really needed for 4.3(b) to be satisfied was the formal projectivity of $\Omega_{\mathfrak{X}}$ or Ω_X in the $\mathfrak{I}$-adic topology.

In the following, $\mathfrak{O}_n$ stands for $\mathfrak{O}_{X_n}$.

COROLLARY 4.5. *If in the situation of Theorem* 4.4, X_0 *is proper over* k *and* $H^1(\mathfrak{X}, \mathfrak{Der}(\mathfrak{O}_n, \mathfrak{I}^n/\mathfrak{I}^{n+1})) = 0$ *for all large* n, *then* t_L *is finite dimensional and the functor* $[L_{\mathfrak{X},X_0}]$ *has a pro-representable hull.*

PROOF. $\mathfrak{Der}(\mathfrak{O}_n, \mathfrak{I}^n/\mathfrak{I}^{n+1})$ is the sheaf of k-derivations of $\mathfrak{O}_n$ into the ideal $\mathfrak{I}^n/\mathfrak{I}^{n+1}$. Under the hypotheses given, there is an exact sequence of sheaves over the underlying topological space of $\mathfrak{X}$.

$$(4.6)\quad 0 \longrightarrow \mathfrak{Der}(\mathfrak{O}_n, \mathfrak{I}^n/\mathfrak{I}^{n+1}) \longrightarrow \mathfrak{Der}^{\mathfrak{I}}(X_n) \xrightarrow{\mathfrak{g}} \mathfrak{Der}^{\mathfrak{I}}(X_{n-1}) \longrightarrow 0$$

whenever $n > 0$. Part of the induced long exact sequence of cohomology groups is

$$(4.7)\quad \begin{aligned} H^1(\mathfrak{X}, \mathfrak{Der}(\mathfrak{O}_n, \mathfrak{I}^n/\mathfrak{I}^{n+1})) &\longrightarrow H^1(\mathfrak{X}, \mathfrak{Der}^{\mathfrak{I}}(X_n)) \\ &\longrightarrow H^1(\mathfrak{X}, \mathfrak{Der}^{\mathfrak{I}}(X_{n-1})). \end{aligned}$$

Since these cohomology groups are finite dimensional in any case, the corollary follows.

DEFINITION 4.8. Let $\mathfrak{I}$ be an $\mathfrak{O}_X$-ideal for a locally noetherian scheme X. The closed subscheme $X_0 = (X, \mathfrak{O}_X/\mathfrak{I})$ is *locally a complete intersection in* X if $\mathfrak{I}$ is a regular ideal; that is, for each point x of X, $\mathfrak{I}_x$ is generated by an $\mathfrak{O}_{X,x}$-regular sequence. When this is the case, $\mathfrak{I}^n/\mathfrak{I}^{n+1}$ is a locally free $\mathfrak{O}_X/\mathfrak{I}$-module, for all positive integers n [SGA, Exposé II, p. 18].

We may replace X by $\mathfrak{X}$ and $\mathfrak{O}_X$ by $\mathfrak{O}_{\mathfrak{X}}$ in the definition above to get the notion of a locally complete intersection in a formal scheme $\mathfrak{X}$, requiring in addition that $\mathfrak{I}$ be an $\mathfrak{O}_{\mathfrak{X}}$-ideal of definition.

LEMMA 6.8. *Let* A *be a local algebra over a field* k *of characteristic zero and let* $f_1, \ldots, f_k$ *be an* A*-regular sequence contained in the maximal ideal of* A. *Then if* I *is the ideal* $(f_1, \ldots, f_k)$, *it is true that, whenever* $n > 1$, $\mathrm{Der}_k^{I^n}(A) = \mathrm{Der}_k^I(A)$.

The proof of this lemma is left to the reader.

COROLLARY 4.10. *Let* $\mathfrak{X}$ *be the completion of a nonsingular scheme* X *of finite type over a field* k *of characteristic zero along a proper closed subscheme* X_0. *If* X_0 *is locally a complete intersection, then* $t_L = \operatorname{proj\,lim} H^1(X, \mathfrak{Der}(X_n))$. *If furthermore* $\mathfrak{I}/\mathfrak{I}^2$ *is an ample* $\mathfrak{O}_{X_0}$*-vector bundle in the sense of Hartshorne* [5], *then* $\dim_k t_L < \infty$ *and the functor* $[L_{\mathfrak{X},X_0}]$ *has a pro-representable hull.*

PROOF. The first statement is obtained by applying the lemma to the stalks of the sheaves $\mathfrak{Der}^{\mathfrak{I}^n}(X)$ and $\mathfrak{Der}^{\mathfrak{I}}(X)$, and from the fact that $\mathfrak{Der}(X_n)$ (resp. $\mathfrak{Der}^{\mathfrak{I}}(X_n)$) is the image of $\mathfrak{Der}^{\mathfrak{I}^n}(X)$ (resp. $\mathfrak{Der}^{\mathfrak{I}}(X)$) under the map

$$\mathfrak{g}_n \colon \mathfrak{Hom}_{\mathfrak{O}_X}(\Omega_X, \mathfrak{O}_X) \longrightarrow \mathfrak{Hom}_{\mathfrak{O}_X}(\Omega_X, \mathfrak{O}_X/\mathfrak{I}^{n+1}).$$

For the second statement, notice that we have for all $n > 0$ the isomorphisms

$$\begin{aligned}\mathfrak{Der}(\mathfrak{O}_n, \mathfrak{J}^n/\mathfrak{J}^{n+1}) &= \mathfrak{Hom}_{\mathfrak{O}_n}(\Omega_n, \mathfrak{J}^n/\mathfrak{J}^{n+1})\\ &= \mathfrak{Hom}_{\mathfrak{O}_0}(\Omega_X \otimes_{\mathfrak{O}_X} \mathfrak{O}_0, \mathfrak{J}^n/\mathfrak{J}^{n+1})\\ &= \mathfrak{Hom}_{\mathfrak{O}_X}(\Omega_X, \mathfrak{J}^n/\mathfrak{J}^{n+1})\\ &= \mathfrak{Hom}_{\mathfrak{O}_X}(\Omega_X, \mathfrak{O}_X) \otimes_{\mathfrak{O}_X} \mathfrak{J}^n/\mathfrak{J}^{n+1}.\end{aligned}$$

The ampleness of $\mathfrak{J}/\mathfrak{J}^2$ now implies that $H^1(\mathfrak{X}, \mathfrak{Der}(\mathfrak{O}_n, \mathfrak{J}^n/\mathfrak{J}^{n+1})) = H^1(X, \mathfrak{Der}(X) \otimes \mathfrak{J}^n/\mathfrak{J}^{n+1})$ vanishes for large n. The corollary follows from the previous one.

5. **Relation between $[L_{\mathfrak{X},X_0}]$ and $[M_{\mathfrak{X},X_0}]$.** The fact is that when $\mathfrak{X}$ and X_0 are sufficiently nice, then $[L_{\mathfrak{X},X_0}]$ has a pro-representable hull if and only if $[M_{\mathfrak{X},X_0}]$ does. This results stems from the sequence of Theorem 5.2 below. First recall [9, Lemma 4.4] that there is a canonical exact sequence of vector spaces, for any scheme X_0 of finite type over k,

$$(5.1)\quad 0 \longrightarrow H^1(X_0, \mathfrak{Der}(X_0)) \longrightarrow [M_{X_0}(k[\epsilon])] \longrightarrow H^0(X_0, \mathfrak{D}^1_{X_0})$$

where $\mathfrak{D}^1_{X_0}$ is the coherent sheaf of $\mathfrak{O}_{X_0}$-modules which attaches to each open affine subscheme Spec(A) of X_0 the A-module Exalcomm$_k(A, A)$ of commutative k-algebra extensions of A by A. The sheaf $\mathfrak{D}^1_{X_0}$ is supported on the singular set of X_0. The space $H^1(X_0, \mathfrak{Der}(X_0))$ then corresponds under (5.1) to the set of isomorphism classes of locally trivial deformations of X_0 to $k[\epsilon]$, that is, to $[L_{X_0}(k[\epsilon])]$, where L_{X_0} is the obvious functor.

THEOREM 5.2. *Let $\mathfrak{X}$ be a noetherian formal k-scheme, X_0 a subscheme of finite type over k, locally a complete intersection in $\mathfrak{X}$, defined by the ideal $\mathfrak{J}$. Suppose that $\mathfrak{X}$ is formally smooth over k. Then there is an exact sequence of k-vector spaces*

$$0 \longrightarrow [L_{\mathfrak{X},X_0}(k[\epsilon])] \longrightarrow [M_{\mathfrak{X},X_0}(k[\epsilon])] \longrightarrow H^0(X_0, \mathfrak{D}^1_{X_0}).$$

That is, any deformation of $(\mathfrak{X}, X_0)$ to $k[\epsilon]$ such that the deformation of X_0 is locally trivial, is in fact a locally trivial deformation of pairs.

PROOF. The theorem is local in nature and we may assume that $(\mathfrak{Y}, Y_0)$ is a deformation of an affine pair $(\mathfrak{X}, X_0)$ to $k[\epsilon]$ such that Y_0 is the trivial deformation of X_0. We will show that then the pair $(\mathfrak{Y}, Y_0)$ is the trivial deformation of $(\mathfrak{X}, X_0)$ under the hypotheses stated. Setting $B = \Gamma(\mathfrak{Y}, \mathfrak{O}_{\mathfrak{Y}})$ and

$J = \Gamma(\mathfrak{Y}, \mathfrak{I})$, A and I as usual, we have a commuting diagram

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & I \arrow[r, "\epsilon"] \arrow[d] & J \arrow[r] \arrow[d] & I \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r, "\epsilon"] \arrow[d] & B \arrow[r, "p"] \arrow[d] & A \arrow[r] \arrow[d, "j_0"] & 0 \\
0 \arrow[r] & A_0 \arrow[r] \arrow[d] & A_0 \otimes_k k[\epsilon] \arrow[r, "p_0"'] \arrow[d] & A_0 \arrow[r] \arrow[d] \arrow[l, dashed, bend right, "s_0"'] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
\tag{5.3}
$$

where $A_0 = A/I$ and s_0 is a k-algebra map such that $p_0 \circ s_0 = \mathrm{id}_A$.

We first demonstrate that J is a regular ideal of B. Namely, we have

LEMMA 5.4. *Let A be a noetherian k-algebra, I a regular ideal of A. Suppose B is a k-algebra extension of A by A, with augmentation map p: $B \longrightarrow A$, J is an ideal of B such that $p(J) = I$, and B/J is an extension of A/I by A/I under the induced map p_0: $B/J \longrightarrow A/I$. Then J is a regular ideal of B.*

PROOF. The situation is a generalization of diagram (5.3), with $B_0 = B/J$ replacing $A_0 \otimes k[\epsilon]$ and no section s_0 assumed.

Since the underlying topological spaces of Spec(B) and Spec(A) are the same, the ideal I is regular if and only if for all maximal ideals M containing J, there is an element $g \in B - M$ such that I_f is generated by a regular A-sequence, where $f = p(g)$ [SGA I, Exposé II, pp. 16–18]. By localization, therefore, we may assume that there is a regular A-sequence $z_1, \ldots, z_p$ generating the ideal I. We claim that if $y_1, \ldots, y_p$ in J are such that $p(y_i) = z_i$, $i = 1, \ldots, p$, then $y_1, \ldots, y_p$ is a regular B-sequence generating J. Since such y_i can always be chosen, this will establish the proposition.

Note that if we abuse notation by writing $\epsilon = \epsilon(1_A)$ and if $b \in B$, then the product $b \cdot \epsilon$ in B is just $\epsilon(p(b))$. Using this fact, it is easy to check that $\{y_1, \ldots, y_p\}$ generates J. The reader may also use induction to show that for $0 \leqslant i \leqslant p$, $(y_1, \ldots, y_i) \cap \epsilon(A) = \epsilon(z_1, \ldots, z_i)$. Now it follows by induction that $y_1, \ldots, y_i$ is a regular B-sequence for $1 \leqslant i \leqslant p$, completing the demonstration of the lemma.

To return to the proof of Theorem 5.2, we may conclude that J^n/J^{n+1} is a locally free, hence flat, B_0-module for all positive integers n. Since B_0 is flat over $k[\epsilon]$, we may use induction on n to prove that B_n is a flat $k[\epsilon]$-module, for all n. Thus, tensoring the middle row of (5.3) with $B_n = B/J^{n+1}$, we get a commuting diagram of $k[\epsilon]$-algebra extensions, where $A_n = A/I^{n+1}$,

$$(5.5)\qquad \begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A_n \arrow[r] \arrow[d, equal] & E_n \arrow[r, "q_n"] \arrow[d] & A \arrow[r] \arrow[d, "j_n"] & 0 \\
0 \arrow[r] & A_n \arrow[r] & B_n \arrow[r, "p_n"] & A_n \arrow[r] & 0
\end{tikzcd}$$

Here, the k-algebra extension E_n is the common image of B and B_n under the maps

$$\mathrm{Exalcomm}_k(A, A) \longrightarrow \mathrm{Exalcomm}_k(A, A_n),$$

$$\mathrm{Exalcomm}_k(A_n, A_n) \longrightarrow \mathrm{Exalcomm}_k(A, A_n).$$

We give $E_n = A \times_{A_n} B_n$ the J_n-adic topology, where $J_n = \mathrm{Im}(J) = I \times_{A_n} B_n$. We claim there is a continuous section c_n of the continuous map q_n. To see this, just note that since A is formally smooth in the I-adic topology, and p_n is a surjection of discrete k-algebras with nilpotent kernel, the map $A \longrightarrow A_n$ splits into a composition of continuous maps $A \xrightarrow{f} B_n \longrightarrow A_n$. The map c_n is then given by $c_n(a) = (a, f(a))$, $a \in A$.

Now consider a commuting diagram of solid arrows with exact rows and fixed sections, $n < m$,

$$\begin{tikzcd}
0 \arrow[r] & A_m \arrow[r, "\epsilon_m"] \arrow[d] & E_m \arrow[r, "q_m"'] \arrow[d, "h_{nm}"'] & A \arrow[r] \arrow[d, equal] \arrow[l, dashed, bend right, "c_m"'] & 0 \\
0 \arrow[r] & A_n \arrow[r, "\epsilon_m"] & E_n \arrow[r, "q_n"'] & A \arrow[r] \arrow[l, dashed, bend right, "c_n"'] & 0
\end{tikzcd}$$

Define a map $D \in \mathrm{Hom}_k(A, E_n)$ by $D = h_{nm} \circ c_m - c_n$. Then $D(A) \subset \epsilon_n(A_n)$ and it can easily be checked that $\bar{D}$ is actually a k-derivation of A into A_n, which we will henceforth denote by D_n. The formal smoothness of A implies the formal projectivity of $\Omega_{A/k}$, and so the surjectivity of the maps g_{mn} of (3.3). Let D_m be an element of $\mathrm{Der}_k(A, A_m) = \mathrm{Hom}_A(\Omega_A, A_m)$ such that $g_{nm}(D_m) = D_n$. Then the continuous map $c'_m = c_m + \epsilon \circ D_m$ is a new section of p_m and $h_{nm} \circ c'_m = c_n$.

This means that any fixed section c_0 of the extension E_0 induces a sequence of sections c_n of the extensions E_n such that $h_{nm} \circ c_m = c_n$. Let s be the projective limit of the maps c_n. Noting that $B = \text{proj lim } E_n$ in the category of topological k-algebras (use diagram (5.5)), it is clear that s is a k-algebra section of the extension B of A by A, and that s is continuous. If now c_0 is the section given by $c_0(a) = (a, s_0 \circ j_0(a))$ then clearly s induces s_0 and we may conclude that (B, B_0) is isomorphic to the trivial deformation of the k-algebra pair (A, A_0) to $k[\epsilon]$. This completes the proof of the theorem.

COROLLARY 5.6. *Let $\mathfrak{X}$ be the completion of a nonsingular scheme of finite type over k along a proper closed subscheme X_0 which is locally a complete intersection in X. Then $[M_{\dot{\mathfrak{X}},X_0}]$ has a pro-representable hull if and only if $[L_{\mathfrak{X},X_0}]$ does.*

6. **Geometric interpretation.** The reason for using the functor $[L_{\mathfrak{X},X_0}]$ as a basis for the notion of equisingularity, rather than all of $[M_{\mathfrak{X},X_0}]$, is based on the result of Proposition 6.3 below. Essentially in the complex case, if $(\mathfrak{X}, X_0)$ is the completion of a resolution of a normal singularity on a surface along the exceptional curve, with X_0 having only normal crossings, then deformations of $(\mathfrak{X}, X_0)$ to an arbitrary scheme which preserve the graph of X_0 are precisely those which induce only locally trivial tangent deformations. First, however, we need a simple calculation.

LEMMA 6.1. *Suppose X_0 is a finite collection of complete nonsingular divisors meeting normally in a nonsingular surface X defined over an algebraically closed field. Then the stalk of $\mathfrak{D}^1_{X_0}$ at the intersection point x of any pair of components of X_0 is just k, and any deformation of the local ring $\mathfrak{O}_{X_0,x}$ to $k[\epsilon]$ is of the form*

$$(6.2) \qquad \mathfrak{O}_{Y_0,x} = (\mathfrak{O}_{X,x} \otimes_k k[\epsilon])/(f + \alpha\epsilon), \quad \alpha \in k,$$

where f is any fixed generator of the $\mathfrak{O}_X$-ideal $\mathfrak{J}$ at x.

PROOF. Let x be the intersection of two components. The completion of the local ring $\mathfrak{O}_{X_0}$ at x is isomorphic to the quotient of the formal power series ring in two variables, $k[[X, Y]]/(XY)$. Hence, we need only calculate $\mathfrak{D}^1_{X_0}$ in the case that A is the polynomial ring $k[X, Y]$ and I is the ideal (XY), by [9, Corollary 3.14].

In this case, [9, Theorem 3.12(8)] asserts that

$$\mathfrak{D}^1_{X_0,x} = D^1_{A_0} = \text{Coker: } \mathrm{Der}_k(A, A_0) \longrightarrow \mathrm{Hom}_{A_0}(I/I^2, A_0).$$

We know that the association $\{f \longrightarrow$ the value f takes on the residue of XY modulo $(XY)^2\}$ defines an isomorphism of $\mathrm{Hom}_{A_0}(I/I^2, A_0)$ with A_0. Now if D is an element of $\mathrm{Der}_k(A, A_0)$, then $D(XY) = X^*D(Y) + Y^*D(X)$, where "$*$" means residue modulo (XY). Therefore, f comes from such a D if and only if the value of f on the residue of XY lies in the maximal ideal (X^*, Y^*) of A_0 at x. Hence the cokernel of the map is isomorphic to $A_0/(X^*, Y^*) = k$.

Furthermore, the deformations of $\mathfrak{O}_{X_0,x}$ of the form (6.2) are certainly nonisomorphic as $k[\epsilon]$-algebras, and so span $\mathfrak{D}^1_{X_0,x}$.

PROPOSITION 6.3. *Suppose $k = \mathbf{C}$, the field of complex numbers, and let π: $(Z, Z_0) \longrightarrow \mathrm{Spec}(\mathbf{C}[t])$ be a flat family of pairs of schemes, together with an isomorphism of (X, X_0) with the fiber over $t = 0$. Assume X is a nonsingular surface and X_0 is a curve with only normal crossings in X. Let $\mathfrak{X}$ (resp. $\mathfrak{Z}$) be the completion of X (resp. Z) along X_0 (resp. Z_0). For any algebraic $\mathbf{C}$-space W, we write W_{an} for the corresponding complex analytic space. Then the following are equivalent*:

(i) *The deformation $(\mathfrak{Y}, Y_0)$ of $(\mathfrak{X}, X_0)$ to $\mathbf{C}[\epsilon]$ induced by the map $t \longmapsto \epsilon$ is locally trivial.*

(ii) *There is an open disc U about 0 in $\mathbf{C}$, and an open neighborhood V of $X_{0,an}$ in Z_{an} such that the family π: $(V, Z_{0,an} \cap V) \longrightarrow U$ is locally on $X_{0,an}$ a trivial family of pairs of analytic spaces.*

PROOF. The deformation $(\mathfrak{Y}, Y_0)$ induced by the map $t \longmapsto \epsilon$ has the property that the local ring at a point x, which is the intersection point of two components of Z_0, is of the form (6.2). The deformation is trivial in some neighborhood of x if and only if $\alpha = 0$.

Let $\mathfrak{H}_{Z,x}$ (resp. $\mathfrak{H}_{Z_0,x}$) be the local ring of Z_{an} (resp. $Z_{0,an}$) at x. Then (6.2) implies that by a suitable analytic change of coordinates,

$$\mathfrak{H}_{Z,x} \approx \mathbf{C}\{X, Y, t\} \quad \text{and} \quad \mathfrak{H}_{Z_0,x}/(t^2) \approx \mathbf{C}\{X, Y, t\}/(XY + \alpha t, t^2).$$

This means that the ideal of Z_0 at x is generated by an element of the form $XY + \alpha t(1 + tP(X, Y, t))$. In some small open polydisc $D_x = \{|x| < \delta, |Y| < \delta, |t| < \delta\}$ about x in Z, we may use the coordinate change $X' = X/(1 + tP(X, Y, t))$, $Y' = Y$, $t' = t$, and obtain $\mathfrak{H}_{Z_0,x} = \mathbf{C}\{X', Y, t\}/(X'Y + \alpha t)$. Now $(D_x, D_x \cap Z_0)$ is a trivial deformation of pairs of analytic spaces to $|t| < \delta$ precisely when $\alpha = 0$, for otherwise the fiber of Z_0 over any $t \neq 0$ is nonsingular.

7. **Relation between deformations of pairs of schemes and deformations of formal embeddings.** The formal pairs we generally look at are, of course, pairs $(\mathfrak{X}, X_0)$ where $\mathfrak{X}$ is the completion of a smooth scheme of finite type X along a closed proper hypersurface. Therefore, it will be useful to look at the functors $[M_{X,X_0}]$ and $[L_{X,X_0}]$ corresponding to isomorphism classes of deformations of the pair of schemes (X, X_0). The definition of $[M_{X,X_0}]$ is exactly analogous to our functor $[M_{\mathfrak{X},X_0}]$ (see [9]). The functor of isomorphism classes of deformations of the single scheme X to objects in $\mathfrak{C}$ is $[M_X] = [M_{X,\phi}]$. We have (note the correction to Lemma 4.7 in [9])

LEMMA 7.1. *Suppose that $\mathfrak{J}$ is the ideal sheaf of $\mathfrak{O}_X$ defining $\mathfrak{O}_{X_0} = \mathfrak{O}_0$. Then there is an exact sequence of k-vector spaces*

$$H^0(X, \mathfrak{Der}(X)) \xrightarrow{p^*} H^0(X_0, \mathfrak{Hom}_{\mathfrak{O}_0}(\mathfrak{J}/\mathfrak{J}^2, \mathfrak{O}_0))$$
$$\longrightarrow [M_{X,X_0}(k[\epsilon])] \xrightarrow{h} [M_X(k[\epsilon])].$$

The map p^* is derived from the natural map of sheaves $\mathfrak{p}$ *of* (3.4).

PROOF. The sequence is derived as follows. The kernel of h is the collection of isomorphism classes of deformations of (X, X_0) to $k[\epsilon]$ "sitting inside $X \times k[\epsilon]$." That is, if the class of (Y, Y_0) lies in Ker h, then there is an isomorphism of Y with $X \times k[\epsilon]$ from which we may deduce the commutative diagram of sheaves on the underlying topological space of X, with exact rows and columns

$$(7.2)\qquad \begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & \mathfrak{J} \arrow[r] \arrow[d] & \mathfrak{J} \otimes k[\epsilon] \arrow[r] \arrow[d] & \mathfrak{J} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathfrak{O}_X \arrow[r] \arrow[d] & \mathfrak{O}_X \otimes k[\epsilon] \arrow[r, "\mathfrak{q}"'] \arrow[d, "\mathfrak{i}"] & \mathfrak{O}_X \arrow[r] \arrow[d] \arrow[l, bend right, dashed, "\mathfrak{s}"'] & 0 \\ 0 \arrow[r] & \mathfrak{O}_{X_0} \arrow[r] \arrow[d] & \mathfrak{O}_{Y_0} \arrow[r, "\mathfrak{q}_0"'] \arrow[d] & \mathfrak{O}_{X_0} \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

Fixing the section $\mathfrak{s}$ of $\mathfrak{q}$, we let $\mathfrak{f} = \mathfrak{i} \circ \mathfrak{s}$ and

$$f^* \in H^0(X, \mathfrak{Hom}(\mathfrak{J}/\mathfrak{J}^2, \mathfrak{O}_{X_0}))$$

be the morphism induced by $\mathfrak{f}$ by the commutativity of the diagram.

Now $\mathfrak{I}$ completely determines $\mathfrak{O}_{Y_0}$ and hence the isomorphism class of the pair (Y, Y_0). In turn, the sections of $\mathfrak{I}$ over an open subscheme U of X are all the sections of the form $i \otimes 1 - h_i \otimes \epsilon$, where i is a section of $\mathfrak{J}$ and h_i is a section of $\mathfrak{O}_X$ whose residue modulo $\mathfrak{J}$ if $f^*(i)$.

Let D be an element of $H^0(X, \mathfrak{Der}(X))$. Under the canonical isomorphism of $H^0(X, \mathfrak{Der}(X))$ with the group of $k[\epsilon]$-automorphisms of $X \times k[\epsilon]$ inducing the identity modulo ϵ, D corresponds to the automorphism ν^* of $\mathfrak{O}_X \otimes k[\epsilon]$ which takes sections $a \otimes 1 + b \otimes \epsilon$ to $a \otimes 1 + (b + D(a)) \otimes \epsilon$, a and b sections of $\mathfrak{O}_X$. Hence, the sections of $\nu^*(\mathfrak{J} \otimes k[\epsilon])$ are all sections of the form $i \otimes 1 + (D(i) + i')\epsilon$, where i and i' are sections of $\mathfrak{J}$. If $\mathfrak{I} = \nu^*(\mathfrak{J} \otimes k[\epsilon])$ for such a ν, then (Y, Y_0) is isomorphic as a deformation of the pair (X, X_0) to the trivial deformation $(X \times k[\epsilon], X_0 \times k[\epsilon])$. This will happen if and only if $-D$ induces f^* via the map p^*.

THEOREM 7.3. *Let X be a nonsingular scheme of finite type over an algebraically closed field k. Suppose X_0 is a reduced closed subscheme of X. Then there is an exact sequence of k-vector spaces*

$$0 \longrightarrow [L_{X,X_0}(k[\epsilon])] \longrightarrow [M_{X,X_0}(k[\epsilon])] \longrightarrow H^0(X_0, \mathfrak{D}^1_{X_0}).$$

That is, every deformation of (X, X_0) to $k[\epsilon]$ such that the deformation of X_0 is locally trivial is a locally trivial deformation of pairs.

PROOF. Since X is nonsingular, any deformation Y of X to $k[\epsilon]$ is locally trivial. Hence, as in the proof of Theorem 5.2, we may assume that (X, X_0) is an affine pair and that (Y, Y_0) is a deformation of (X, X_0) to $k[\epsilon]$ such that the deformations Y and Y_0 of X and X_0 are separately trivial. In particular, if $A = \Gamma(X, \mathfrak{O}_X)$ and $I = \Gamma(X, \mathfrak{I})$, the isomorphism class of (Y, Y_0) lies in the image of $\mathrm{Hom}_{A_0}(I/I^2, A_0)$ in $[M_{X,X_0}(k[\epsilon])]$ under the sequence of Lemma 7.1. If $B = \Gamma(Y, \mathfrak{O}_Y)$ and J is the ideal defining the embedding of Y_0 in Y, there are isomorphisms of B with $A \otimes k[\epsilon]$ and B_0 with $A_0 \otimes k[\epsilon]$, so that we get diagram (5.3) again, together with a section s of p.

Fix a closed point x and shrink X if necessary to an affine open neighborhood of x such that there are elements $z_1, \ldots, z_n$ in A, n the dimension of A, with the property that X is étale over $\mathrm{Spec}(C) = Z$ where $C = k[z_1, \ldots, z_n]$. We claim that in some open neighborhood of x and its image, $(\mathfrak{I} \cap \mathfrak{O}_Z) \cdot \mathfrak{O}_X = \mathfrak{I}$. This would mean that $\mathfrak{I}$ is generated near x by some rational functions $Q(z)$ with coefficients in k, where $z = (z_1, \ldots, z_n)$ and the elements $\{dz_1, \ldots, dz_n\}$ form a free basis of Ω_X. But the claim follows from the following lemma.

LEMMA 7.4. *Let $C \longrightarrow A$ be an étale map of local k-algebras such that each is the localization of a k-algebra of finite type, and assume that k is algebraically closed. Suppose I is a radical ideal of A (i.e. A/I is reduced). Then $(I \cap C) \cdot A = I$.*

PROOF. The function $n(z)$ which assigns to any point z of $Z = \mathrm{Spec}(C)$ the sum of the separable degrees of the residual extensions $k(x)$ over $k(z)$, for $x \in f^{-1}(z)$, is upper semicontinuous. At the closed point of Z, $n(z) = 1$, since k is algebraically closed, hence n is the constant 1. In particular, if Q is a prime ideal of C, there is one and only one prime P of A such that $P \cap C = Q$. On the other hand, A/QA is étale over C/Q, hence reduced since C/Q is [SGA I, Exposé I, Corollary 9.3]. Thus $Q \cdot A = P$ or $(P \cap C) \cdot A = P$, establishing the lemma for I prime. Note that $P \subseteq P'$ if and only if $Q \subset Q'$, where $Q' = P' \cap C$, P' another prime of A.

In the general case, we may write I as the intersection $\bigcap_{i=1}^r P_i$ of the distinct primes minimal among those containing I. Then if $J = I \cap C$, J is reduced and $J = \bigcap_{i=1}^r Q_i$, where $P_i \cap C = Q_i$. Each prime Q_i is minimal among those containing J. Since A/JA is étale over C/J, it is reduced by the reference already cited, and so JA is the intersection of the minimal primes containing it. These are in one-to-one correspondence with the minimal primes of C containing J, so that $JA = \bigcap_{i=1}^r P_i = I$.

Getting back to the proof of Theorem 7.3, denote the residue class modulo I of an element $a \in A$ by a^* and set $f = j \circ s$ as in Lemma 7.1. Using the section s_0, we can find elements $a_i \in A$ such that $f(z_i) = z_i^* + a_i^* \cdot \epsilon$, for $i = 1, \ldots, n$. Then we can define an element D in $\mathrm{Der}(A)$ by setting $D(dz_i) = a_i$, and extending it linearly to $\Omega_{A/k}$. We claim that the image of D in $\mathrm{Hom}_{A_0}(I/I^2, A_0)$ under p^* is precisely f^*, so that (Y, Y_0) is the trivial deformation of (X, X_0) to $k[\epsilon]$.

In fact, $D(dQ(z)) = \Sigma_{i=1}^n \partial Q(z)/\partial z_i \cdot a_i$, by the definition of D, while $f(Q(z)) = Q(z^*) + \Sigma_{i=1}^n \partial Q(z^*)/\partial z_i \cdot a_i^*\epsilon$, for any rational function Q of z with coefficients in k. This means that if $Q(z) \in I$, then f^* applied to the residue of $Q(z)$ in I/I^2 gives rise to precisely $D(dQ(z))^*$. Since the image of D and f in $\mathrm{Hom}_{A_0}(I/I^2, A_0)$ are completely determined by their actions on these generators of I/I^2 as an A_0-module, the claim is established. This proves the theorem.

8. **Examples.** All our examples will involve pairs $(\mathfrak{X}, X_0)$ where $\mathfrak{X}$ is the completion of a nonsingular scheme X, of finite type over an algebraically closed field of characteristic zero, along a proper hypersurface X_0. We will write t_L for $[L_{\mathfrak{X}, X_0}(k[\epsilon])]$. If $\mathfrak{J}$ is the ideal defining the embedding of X_0 in X, then $\mathfrak{J}$ is invertible and so Corollary 4.10 applies.

(1) NONSINGULAR CURVES. Let X_0 be a complete nonsingular curve of genus g lying in a nonsingular surface X. Suppose the normal bundle $\mathfrak{N}_{X_0}$ of the embedding is negative of degree $-n$. Then $\mathfrak{J}/\mathfrak{J}^2 = \mathfrak{N}_{X_0}^*$ and is ample with degree n. Thus, $\dim t_L < \infty$.

When $g > 1$ and $n > 4g - 4$, the reader may check easily that

$$t_L = \text{proj lim } H^1(\mathfrak{X}, \mathfrak{Der}(X_n)) = H^1(\mathfrak{X}, \mathfrak{Der}(X_1)).$$

Furthermore, we have the exact sequence

$$0 \longrightarrow H^1(X_0, \mathfrak{Der}(X) \otimes \mathfrak{J}/\mathfrak{J}^2) \longrightarrow H^1(\mathfrak{X}, \mathfrak{Der}(X_1)) \longrightarrow H^1(X_0, \mathfrak{Der}(X_0)) \longrightarrow 0$$

and a natural isomorphism $H^1(X_0, \mathfrak{Der}(X) \otimes I/I^2) = H^1(X_0, \mathfrak{O}_{X_0})$. Roughly speaking, all the tangent deformations of $(\mathfrak{X}, X_0)$ which induce trivial deformations of X_0 are obtained by deforming the normal bundle of the embedding, since $H^1(X_0, \mathfrak{O}_{X_0})$ is the tangent space to the Picard functor at $(X_0, \mathfrak{N}_{X_0})$.

Similar computations show that when $g = 0$ and $-n$ is any nonpositive integer, then $t_L = 0$. On the other hand, if X is the nth power of the hyperplane bundle over the projective line, n a positive integer, then t_L is infinite dimensional.

(2) We make the following definition.

DEFINITION 8.1. Let X_0 be a closed subscheme of a locally noetherian scheme X. Let $\mathfrak{X}$ be the completion of X along X_0. We will call the pair $(\mathfrak{X}, X_0)$ *rigid* (resp. *L-rigid*) or say that the embedding of X_0 in $\mathfrak{X}$ is *formally rigid* (resp. *formally L-rigid*) if $[M_{\mathfrak{X},X_0}(k[\epsilon])] = 0$ (resp. $[L_{\mathfrak{X},X_0}(k[\epsilon])] = 0$).

When $k = \mathbf{C}$, the field of complex numbers, and X_0 is a finite collection of complete curves meeting normally in a surface X with negative definite intersection matrix, then X_0 is formally L-rigid if and only if the singularity obtained by blowing down X_0 to a point is analytically taut. Recall

DEFINITION 8.2. Let p be a normal two-dimensional singularity in a complex analytic surface. Then p is *taut* if any normal two-dimensional singularities having the same (dual) weighted graph as p for its minimal resolution is analytically isomorphic to p.

If Z_{an} is the unique analytic structure associated to a given complex algebraic variety Z and $\mathfrak{F}_{an}$ the coherent analytic sheaf associated to an algebraic sheaf $\mathfrak{F}$ on Z [GAGA], then the sheaf ${}_{X_n}\theta$ of Laufer [7] is precisely the sheaf $(\mathfrak{Der}(X_n))_{an}$ and by Theorem 1 of [GAGA] we have canonical isomorphisms $H^i(X_n, \mathfrak{Der}(X_n)) \longrightarrow H^i(X_{n,an}, \theta_n)$, for all positive integers n.

The results of Laufer [7, Theorems 3.9 and 3.10] then show that analytic tautness of a normal surface singularity is equivalent to formal L-rigidity of the embedding of the exceptional set of the desingularization of p obtained by blowing up just enough points in the minimal resolution until a configuration of nonsingular curves meeting normally in a nonsingular surface is obtained. In particular, Tjurina's proof [11] that all rational double and triple points are taut implies that all rational double and triple embeddings of curves are formally L-rigid, while the nontaut examples of Brieskorn [3] and Wagreich [12] are non-L-rigid.

(3) We note only a simple example of formal rigidity of higher dimensional pairs. If k is any integer and $n > 2$, and if $\mathbf{P}^n$ is identified with the zero section of $\mathfrak{H}^k$, the kth power of the hyperplane bundle over $\mathbf{P}^n$, then the pair $(\mathfrak{H}^k, \mathbf{P}^n)$ is formally rigid. If $n = 2$, this is true as long as $k \leqslant 0$, while for $k > 0$, $t_L = t_M$ will at least be finite dimensional.

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