ON REPRESENTATIONS OF THE GROUP $SU(n, 1)^{(1)}$

BY

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ABSTRACT. A natural bijection is established between the set of equivalence classes of irreducible unitary representations of the group $G = SU(n, 1)$, which are not induced from a proper parabolic subgroup, and the set of equivalence classes of irreducible representations of a maximal compact subgroup.

1. Introduction. In [4] all irreducible subquotients of all elementary (= nonunitary principal series) representations of the group $G = SU(n, 1)$ and of its universal covering group were determined. We have also found all infinitesimal equivalences among these irreducible representations and in this way, owing to the subquotient theorem [1], [6], [9], we have described the set $\hat{G}$ of all infinitesimal equivalence classes of irreducible quasi-simple representations of $G$. Furthermore, we have found a necessary and sufficient condition for a class $\pi \in \hat{G}$ to contain a unitary representation and so the set $\hat{G}$ of all equivalence classes of unitary irreducible representations of $G$ was completely described.

The aim of this paper is to give another description of the results of [4]. This description will use only very general terms and will not depend on almost any particular property of the group $SU(n, 1)$. Furthermore, it will provide us with very natural parametrizations of the sets $\hat{G}$ and $\hat{G}$.

We shall also describe all elementary representations in which a given unitary irreducible nonelementary representation occurs as a subquotient. Using Blattner's conjecture (which is proven to be true for linear groups acting on hermitian symmetric spaces [8]) it will be very easy to identify the discrete series representations. This will, especially, give us all possible imbeddings of a discrete series representation as a subquotient of an elementary representation.

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2. Statements of the results. In the following $G$ will denote the group $SU(n, 1)$, $\mathfrak{g}$ its Lie algebra, $\mathfrak{g}_c (= \mathfrak{s}(n + 1, \mathbb{C}))$ the complexification of $\mathfrak{g}$, $\mathfrak{d}$ the universal enveloping algebra of $\mathfrak{g}$, $\mathfrak{z}$ the center of $\mathfrak{d}$. For any Cartan subalgebra
$S(b)$M' denote the algebra of Weyl group invariants in the symmetric algebra $S(b)$ over $\mathfrak{h}$, and let $\varphi_\mathfrak{h}$ be the canonical isomorphism of $S(b)$ onto $S(b)$ [9]. $S(b)$ being identified with the polynomials on the dual space $\mathfrak{h}^*$ of $\mathfrak{h}$, let, for $\lambda \in \mathfrak{h}^*$, $\chi_\lambda$ denote the element of Hom($\mathfrak{h}$, $\mathbb{C}$) obtained by composing the evaluation at $\lambda$ with $\varphi_\mathfrak{h}$: $\chi_\lambda(z) = \varphi_\mathfrak{h}(z)(\lambda)$, $z \in \mathfrak{h}$ (the notation here differs from that in [4, p. 26]). Then $\lambda \mapsto \chi_\lambda$ defines a bijection from the set of Weyl group orbits in $\mathfrak{h}^*$ onto Hom($\mathfrak{h}$, $\mathbb{C}$). For any $\lambda \in \mathfrak{h}^*$ let $[\lambda]$ denote the Weyl group orbit containing $\lambda$. Furthermore, if $\pi$ is a quasi-simple representation of $G$, denote by $[\pi]$ the Weyl group orbit in $\mathfrak{h}^*$ which corresponds to the infinitesimal character of $\pi$.

Let $(K, A, N)$ be an Iwasawa decomposition of $G$ and $(\mathfrak{f}, \mathfrak{a}, \mathfrak{n})$ the corresponding decomposition of $\mathfrak{g}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{f}$. Denote by $\mathfrak{p}$ the orthogonal complement of $\mathfrak{f}$ in $\mathfrak{g}$ with respect to the Killing form. Let $R$ be the root system of $(\mathfrak{g}, \mathfrak{h})$ and let $W$ be its Weyl group. For $\alpha \in R$, $\mathfrak{g}_\alpha$ will denote the corresponding root subspace of $\mathfrak{g}_c$. Let $R_K$ and $R_P$ denote the sets of compact and noncompact roots, respectively; $R_K = \{\alpha \in R; \mathfrak{g}_\alpha \subset \mathfrak{f}_c\}$, $R_P = \{\alpha \in R; \mathfrak{g}_\alpha \subset \mathfrak{p}_c\}$. Then $R_K$ is the root system of $(\mathfrak{f}_c, \mathfrak{h}_c)$. Fix an $R_K$-Weyl chamber $C$ in $i\mathfrak{h}^*$ and denote by $R_K^C$ the corresponding positive roots in $R_K$. Let $C$ be the set of all $R$-Weyl chambers in $i\mathfrak{h}^*$ contained in $C$. For any $D \in C$ denote by $R_D$ the corresponding positive roots in $R$. Furthermore, put

$$R_D^D = R_D \setminus R_K^C = R_D \cap R_P,$$

$$\rho_K = \frac{1}{2} \sum_{\alpha \in R_K^C} \alpha, \quad \rho_P^D = \frac{1}{2} \sum_{\alpha \in R_P^D} \alpha.$$

$\hat{K}$, the set of equivalence classes of finite-dimensional irreducible representations of $K$, will be regarded as a subset of the closure of $C$ by identifying any $q \in \hat{K}$ with its maximal weight.

If $\pi$ is an admissible quasi-simple representation of $G$ and $q \in \hat{K}$, let $(\pi : q)$ denote the multiplicity of $q$ in $\pi|_K$; furthermore, put, as in [4], $\Gamma(\pi) = \{q \in \hat{K}; (\pi : q) > 0\}$.

**Definition.** Let $\pi$ be an admissible quasi-simple representation of $G$, $q \in \hat{K}$, $D \in C$.

(i) $q$ is called a $D$-fundamental weight of $\pi$ if $q \in \Gamma(\pi)$ and $[\pi] = [q + \rho_K - \rho_P^D]$.

(ii) $q$ is called a $D$-corner of $\pi$ if $q \in \Gamma(\pi)$ and $q - \alpha \notin \Gamma(\pi) \forall \alpha \in R_P^D$.

(iii) $q$ is called a $D$-fundamental corner of $\pi$ if it is a $D$-fundamental weight and a $D$-corner of $\pi$.

(iv) $q$ is called a fundamental corner of $\pi$ if it is a $D$-fundamental corner of $\pi$ for some $D \in C$. 

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THE GROUP SU(n, 1)

Remark. This definition is, of course, inspired by Blattner’s conjecture for the K-multiplicities of a discrete series representation (see [7]).

Let \( M \) be the centralizer of \( A \) in \( K \). For \( p \in \hat{M} \) and \( \lambda \in \mathfrak{a}_e^* \) let \( \pi_{p, \lambda} \) denote the corresponding elementary representation of \( G \) (\( \pi_{p, \lambda} \) is induced by the representation \( \text{man} \mapsto \exp((\lambda + p)(\log a))p(m) \) of \( MAN, \rho(H) = \frac{1}{2} \text{tr}(\text{ad} H | n), H \in \mathfrak{a} \)).

Let \( \widehat{G}_e \) be the set of all elements in \( \widehat{G} \) containing some elementary representation. Put

\[
\widehat{G}^0 = \widehat{G} \cap \widehat{G}_e, \quad \widehat{G}^e = \widehat{G}_e \cap \widehat{G}, \quad \widehat{G}_0 = \widehat{G} \cap \widehat{G}_0.
\]

The following theorems give the description and parametrizations of these sets.

**Theorem 1.** \( \pi_{p, \lambda} \) is reducible if and only if there exist \( q \in \hat{K} \) and \( D \in \mathbb{C} \) such that \( q \) is a \( D \)-fundamental weight of \( \pi_{p, \lambda} \) but is not a \( D \)-corner of \( \pi_{p, \lambda} \).

This theorem, together with the fact that two irreducible elementary representations \( \pi_{p, \lambda} \) and \( \pi_{p', \lambda'} \) are infinitesimally equivalent if and only if \( (p, \lambda) \) and \( (p', \lambda') \) are conjugated by the action of the Weyl group of \( (g, a) \), completely describes \( \widehat{G}_e \).

For any \( \pi \in \widehat{G}_0 \) let \( F(\pi) \) be the set of all fundamental corners of \( \pi \).

**Theorem 2.** (i) For every \( \pi \in \widehat{G}_0 \), \( F(\pi) \) has either one or two elements.

(ii) Let \( \Omega \) be the set of all nonordered pairs \( (q, q') \), \( q, q' \in \hat{K} \). \( \pi \mapsto F(\pi) \) defines an injection of \( \widehat{G}_0 \) into \( \Omega \) (the image of \( \pi \in \widehat{G}_0 \) with only one fundamental corner \( q \) being \( (q, q) \)).

Let \( \Omega_0 \) be the image of this injection.

(iii) \( \Omega_0 \) is the set of all pairs \( (q, q') \in \Omega \) with the property that there exist \( D, D' \in \mathbb{C} \) such that \( \{q + \rho_K - \rho_D, q' + \rho_K - \rho_{D'}\} = \{q + \rho_K - \rho_D, q + \rho_K - \rho_{D'}\} \). Here \( D \) denotes the closure of \( D \) in \( \mathbb{C} \) and \( D' \) is the closure of \( D \) in \( \mathbb{C}^* \). Furthermore, for any \( (q, q') \in \Omega_0 \), \( D \) and \( D' \) are well determined.

Hence, if we denote by \( \pi(q, q') \) the unique element of \( \widehat{G}_0 \) with the property \( F(\pi(q, q')) = \{q, q'\} \) \( ((q, q') \in \Omega_0) \) then we get

\[
\widehat{G}_0 = \{\pi(q, q'); (q, q') \in \Omega_0\}.
\]

We give also another description of \( \widehat{G}_0 \) which is in a sense more convenient because of the nontransparent characterization of the set \( \Omega_0 \).

**Theorem 3.** For \( D \in \mathbb{C} \) let \( \hat{K}^D \) denote the set of all \( q \in \hat{K} \) such that there is \( \pi \in \widehat{G}_0 \) which has \( q \) as a \( D \)-fundamental corner. Put \( \Omega_1 = \{(q, D); q \in \hat{K}^D, D \in \mathbb{C}\} \).

(i) For any \( (q, D) \in \Omega_1 \) there is a unique element \( \pi(q, D) \) in \( \widehat{G}_0 \) which has \( q \) as a \( D \)-fundamental corner.
(ii) $\hat{G}^0 = \{ \pi(q, D); (q, D) \in \Omega_1 \}$.

(iii) If $(q, D) \in \Omega_1$, then there exists at most one $(q', D') \in \Omega_1$ different from $(q, D)$ such that $\pi(q, D) = \pi(q', D')$. Then $F(\pi(q, D)) = \{ q, q' \}$.

(iv) If $D$ has a wall in common with $C$ then $\hat{K}^D = \hat{K}$. If $D$ has not a wall in common with $C$, then there are two roots $\alpha_D, \beta_D$ in $R^D_\rho$ perpendicular to the walls of $D$ intersecting $C$ and $\hat{K}^D$ is the set of all $q \in \hat{K}$ such that either $(\alpha_D + \beta_D|q) \neq 0$ or $(\alpha|q + \rho_K - \rho^D) \neq 0 \forall \alpha \in R^D_\rho$.

By Proposition 11.4 in [4] we have the following description of $\hat{G}^e$:

**Theorem 4.** Identify $\alpha^* \subset C$ so that $p = n$. For any $p \in \hat{M}$ put

$$\lambda_p = \min \{ \lambda \geq 0; \pi^{p, \lambda} \text{ reducible} \}.$$  

Then

$$\hat{G}^e = \{ \pi^{p, \lambda}; p \in \hat{M}, \lambda \in i(0, +\infty) \cup [0, \lambda_p) \}.$$  

We shall give also the explicit value of $\lambda_p$ for any $p \in \hat{M}$ (it is determined in [3] for "generic" $p$'s).

**Theorem 5.** (i) $\pi \in \hat{G}^0$ is unitary if and only if it has only one fundamental corner, i.e., Card $F(\pi) = 1$.

(ii) For every $q \in \hat{K}$ there is a unique $\pi_q \in \hat{G}^0$ with $F(\pi_q) = \{ q \}$. $q \mapsto \pi_q$ is a bijection of $\hat{K}$ onto $\hat{G}^0$.

Furthermore, we shall prove

**Theorem 6.** (i) $\pi_q$ is a discrete series representation if and only if $q + \rho_K - \rho^D \in D$ for some $D \subset C$; this $D$ is then uniquely determined.

(ii) Every $\pi \in \hat{G}^0$ which is not in the discrete series occurs as a subquotient of some $\pi^{p, \lambda}_p$, $p \in \hat{M}$.

**Remark.** Statement (ii) can also be deduced from the results of [3] and [5].

3. Irreducibility of elementary representations. $G$ is the group $SU(n, 1)$ of all complex square matrices $g$ of order $n + 1$ with $\det g = 1$ and $gGg^* = \Gamma = \text{diag}(1, \ldots , 1, -1)$. We choose $K$ to be the subgroup of all unitary matrices in $G$. $G_c$ is naturally identified with $\mathfrak{sl}(n + 1, \mathbb{C})$. Let $\mathfrak{h}$ be the Cartan subalgebra of $g$ (and $\mathfrak{f}$) of all diagonal matrices in $g$. Let $e_{ij}$ be the matrix with $1$ on the place $(i, j)$ and $0$ elsewhere $(1 \leq i, j \leq n + 1)$. Put $h_i = e_{ii} - e_{i+1,i+1} (1 \leq i \leq n)$. Then $h_1, \ldots , h_n$ is a basis of $\mathfrak{h}_c$. We identify $\mathfrak{h}^*_c \subset \mathfrak{h}^*_c$ with the set

$$C_0^{n+1} = \left\{ s \in C^{n+1}; \sum_{j=1}^{n+1} s_j = 0 \right\}$$

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in such a way that \( s(h_j) = s_j - s_{j+1} \), \( 1 \leq j \leq n \). Then \( i\mathfrak{h}^* = C_0^{n+1} \cap \mathbb{R}^{n+1} = R_0^{n+1} \).

The root system of \((\mathfrak{g}_c, \mathfrak{h}_c)\) is

\[
R = \{ \alpha_{jk}; 1 \leq j, k \leq n + 1, j \neq k \}
\]

where \( \alpha_{jk}(\text{diag}(t_1, \ldots, t_{n+1})) = t_j - t_k \). Furthermore,

\[
R_K = \{ \alpha_{jk}; 1 \leq j, k \leq n, j \neq k \}, \quad R_P = \{ \alpha_{j,n+1}, \alpha_{n+1,j}; 1 \leq j \leq n \}.
\]

The Weyl group \( W \) of \( R \) is the group \( S_{n+1} \) of permutations of coordinates.

We choose the \( R_K \)-Weyl chamber in \( R_0^{n+1} \) to be

\[
C = \{ s \in R_0^{n+1}; s_j > s_{j+1}, 1 \leq j \leq n - 1 \}.
\]

Then

\[
R_K^C = \{ \alpha_{jk}; 1 \leq j < k \leq n \} \quad \text{and} \quad C = \{ D_0, D_1, \ldots, D_n \},
\]

where

\[
D_0 = \{ s \in C; s_{n+1} > s_1 \},
\]

\[
D_j = \{ s \in C; s_j > s_{n+1} > s_{j+1} \}, \quad 1 \leq j \leq n - 1,
\]

\[
D_n = \{ s \in C; s_n > s_{n+1} \}.
\]

Put \( R'_K \) and \( \rho'_P \) instead of \( R_K^D \) and \( \rho_P^D \), respectively. Then we have

\[
R'_P = \{ \alpha_{k,n+1}; 1 < k < j \} \cup \{ \alpha_{n+1,k}; j + 1 < k \leq n \}
\]

and

\[
(\rho_K - \rho'_P)_i = \begin{cases} 
    n/2 - i, & 1 \leq i < j, \\
    n/2 - i + 1, & j + 1 \leq i \leq n, \\
    j - n/2, & i = n + 1.
\end{cases}
\]

\( \hat{K} \) was in [4] identified with \( ((n + 1)^{-1}Z)^n_+ = \{ q \in ((n + 1)^{-1}Z)^n; q_j - q_{n+1} \in Z_+, 1 \leq j \leq n - 1 \} \). In the new parametrization of \( \mathfrak{h}_c^n \) and identifying \( \hat{K} \) with a subset of the closure of \( C \) we have \( q = (q_1, \ldots, q_n, -\Sigma_{j=1}^n q_j) \).

As in [4] we choose \( a \) to be spanned over \( R \) by \( e_{n,n+1} + e_{n+1,n} \). \( \hat{M} \) was identified with \( ((n + 1)^{-1}Z)^n_{-1} \) and \( a_c^n = C \) with \( \rho = n \). Let \( \nu: \hat{M} \times C \rightarrow C_0^{n+1} \) be defined by

\[
\nu(\rho, \lambda)_i = \begin{cases} 
    \frac{1}{2}(\lambda - \Sigma_{k=1}^{i-1} p_k), & i = 1, \\
    p_{i-1} + n/2 - i + 1, & 2 \leq i \leq n, \\
    -\frac{1}{2}(\lambda + \Sigma_{k=1}^{n+1} p_k), & i = n + 1.
\end{cases}
\]
Then $\nu$ is an injection and its image is

$$E = \left\{ s \in \mathbb{C}_0^{n+1}; s_j \in \frac{1}{n+1} \mathbb{Z}, s_k - s_{k+1} \in \mathbb{N}, 2 \leq j < n, 2 \leq k < n - 1 \right\}.$$ 

Denote the elementary representation $\pi^{\rho, \lambda}$ by $\pi^s$, $s = \nu(p, \lambda)$. Then $[\pi^s] = [s]$, i.e., the infinitesimal character of $\pi^s$ is $\chi_s$ (see [4, p. 45]).

The result about reducibility of elementary representations from [4] (Theorems 7.5 and 8.7) now gets the form:

**Proposition 1.** $\pi^s$ is reducible if and only if either $s_1 - s_j \in \mathbb{Z}\{0\} \forall j \in \{2, \ldots, n\}$ or $s_{n+1} - s_j \in \mathbb{Z}\{0\} \forall j \in \{2, \ldots, n\}$.

**Proof of Theorem 1.** Suppose that $\pi^s$ is reducible. Interchanging $s_1$ and $s_{n+1}$ if necessary (this corresponds to the action of the nontrivial element of the little Weyl group), we can suppose that $s_1 - s_j \in \mathbb{Z}\{0\} \forall j \in \{2, \ldots, n\}$. We have $\Gamma(\pi^s) = \{ q \in \hat{\mathbb{K}}; p < q \}$ ($\nu(p, \lambda) = s$), where $p < q$ means $q_i - p_i \in \mathbb{Z}_+$ and $p_i - q_{i+1} \in \mathbb{Z}_+$, $1 \leq i < n - 1$. We have to show that there are $q \in \hat{\mathbb{K}}$ and $j \in \{0, \ldots, n\}$ such that $p < q$, $[q + \rho_K - \rho_P] = [s]$ and $p < q - \alpha$ for some $\alpha \in R_P$. Let $j$ be the smallest element of $\{1, \ldots, n-1\}$ such that $s_1 > s_{j+1}$ and put $j = n$ if $s_n > s_1$. Let $q \in \hat{\mathbb{K}}$ be defined by

$$q_i = \begin{cases} p_i, & 1 \leq i \leq j-1, \\ s_1 + j - n/2, & i = j, \\ p_i - 1, & j + 1 \leq i \leq n. \end{cases}$$

Then it is easily seen that $q \in \hat{\mathbb{K}}$, $p < q$ and $\alpha_{j,n+1} \in R_P^\prime$, $q - \alpha_{j,n+1} \in \hat{\mathbb{K}}$, $p < q - \alpha_{j,n+1}$. Furthermore, by (1)

$$q + \rho_K - \rho_P = (s_2, \ldots, s_j, s_1, s_{j+1}, \ldots, s_{n+1});$$

hence $[q + \rho_K - \rho_P] = [s]$.

Suppose now that there are $j \in \{0, \ldots, n\}$, $q \in \hat{\mathbb{K}}$ and $\alpha \in R_P^\prime$ such that $q - \alpha \in \hat{\mathbb{K}}$, $p < q$, $p < q - \alpha$ and $[q + \rho_K - \rho_P] = [s]$. Put $t = q + \rho_K - \rho_P$ and let $w \in \mathcal{W}$ be such that $t = ws$. We have $s_2 > \cdots > s_n$ and $t_1 > \cdots > t_j > t_{j+1} > \cdots > t_n$. Now, $p < q$ and the formulas relating $p$ and $q$ with $s$ and $t$ give

$$t_1 > s_2 > t_2 > s_3 > \cdots > s_j > t_j > s_{j+1} > t_{j+1} > s_{j+2} > \cdots > s_n > t_n.$$ 

These inequalities, together with the fact that $w$ is a permutation, imply $\{s_1, s_{n+1}\} \subset \{t_j, t_{j+1}, t_{n+1}\}$. Interchanging $s_1$ and $s_{n+1}$ if necessary, we have to consider the following possibilities:

(a) $s_1 = t_{n+1}, s_{n+1} = t_j$. Then $s_i = t_{i-1}$, $2 \leq i \leq j$, and $s_j = t_j$. 

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j + 1 < i < n. Therefore \( q_i = p_i, 1 < i < j - 1, q_i = p_{i-1}, j + 1 < i < n, q_j = s_{n+1} - n/2 + j \). Hence \( p < q - \alpha \) for all \( \alpha \in \mathbb{R}^n \) except possibly \( \alpha = \alpha_{j,n+1} \). So \( \alpha \) in the assumption must be \( \alpha_{j,n+1} \). This means \( q_j - p_j \in \mathbb{N} \), or \( s_{n+1} - s_j + 1 \in \mathbb{N} \). Furthermore, \( p < q \) implies \( p_{j-1} - q_j \in \mathbb{Z}_+ \), or \( s_j - s_{n+1} \in \mathbb{N} \). Hence, \( s_{n+1} - s_k \in \mathbb{Z} \setminus \{0\} \) for all \( k \in \{2, \ldots, n\} \), and \( n^x \) is reducible by Proposition 1.

(b) \( s_1 = t_{n+1}, s_{n+1} = t_{j+1} \). Similarly as above we get \( q_i = p_i, 1 < i < j, \)
\( q_i = p_{i-1}, j + 2 < i < n, q_{j+1} = s_{n+1} - n/2 + j, \) and \( \alpha = \alpha_{n+1,j+1} \). Furthermore, \( p < q \) implies \( s_{n+1} - s_{j+1} \in \mathbb{N} \), and \( p < q - \alpha_{n+1,j+1} \) implies \( s_{j+1} - s_{n+1} \in \mathbb{N} \). Hence, \( \pi^x \) is again reducible.

(c) \( s_1 = t_{j}, s_{n+1} = t_{j+1} \). Then \( q_i = p_i, 1 < i < j - 1, q_i = p_{i-1}, j + 2 < i < n, q_j = s_1 - n/2 + j, q_{j+1} = s_{n+1} - n/2 + j, -\Sigma_{k=1}^n q_k = p_j + n - 2j \). By
\( p < q \) we have \( s_j - s_n \in \mathbb{N} \) and \( s_{n+1} - s_{j+2} \in \mathbb{N} \). So we have to show that either \( s_1 \neq s_{j+1} \) or \( s_{n+1} \neq s_{j+1} \). Suppose that \( s_1 = s_{n+1} = s_{j+1} \). Then \( q = (p_1, \ldots, p_j, p_j, \ldots, p_{n+1}) \). But then \( p < q - \alpha \) \( \forall \alpha \in \mathbb{R}^n \) contradicting the assumptions. Q.E.D.

4. Irreducible nonelementary representations. \( \mathcal{G}^0 \) was parametrized in [4, §10] as follows.

\[ \mathcal{G}^0 = \{ \pi_{j,r}; r \in T_j, 0 < j < n \} \cup \{ \pi_{j,k,r}; r \in T_{j,k}, 0 \leq k < j < n \}. \]

Here \( T_j \) is the set of all \( r \in ((n + 1)^{-1} \mathbb{Z})^n \) such that \( r_j > r_{j+1} \) \( (r_0 = \infty, r_{n+1} = -\infty) \) and one of the following conditions is satisfied:

(a) \( r_k + \Sigma_{t=1}^n r_t = j + k - n \) for some \( k \in \{1, \ldots, j - 1\} \);

(b) \( r_{k+1} + \Sigma_{t=1}^n r_t = j + k - n \) for some \( k \in \{j + 1, \ldots, n - 1\} \);

(c) \( r_j > 2j - n - \Sigma_{t=0}^{n-1} r_t \geq r_{j+1} \).

Furthermore, \( T_{j,k} \) is the set of all \( r \in ((n + 1)^{-1} \mathbb{Z})^{n+1} \) such that \( r_j > r_{j+1}, r_{k+1} > r_{k+2} \) and \( \Sigma_{t=1}^n r_t = j + k - n \) \( (r_0 = +\infty, r_{n+2} = -\infty) \).

We simplify this in the following way. Let, for \( 0 \leq j < k < n, S_{jk} \) be the set of all \( r \in ((n + 1)^{-1} \mathbb{Z})^{n+1} \) such that \( \Sigma_{t=1}^{n+1} r_t = j + k - n \) and either \( r_j > r_{j+1} \) or \( r_{k+1} > r_{k+2} \). Then \( T_{j,k} \subset S_{jk} \). Let \( r \) be in \( T_j \) such that \( r_k + \Sigma_{t=1}^n r_t = j + k - n \) for some \( k \in \{1, \ldots, j - 1\} \). Put

\[ r' = (r_1, \ldots, r_k, r_{k+1}, \ldots, r_j, r_{j+1}, \ldots, r_n). \]

Then \( r' \in S_{kj} \). Let \( r \in T_j \) be such that \( r_{k+1} + \Sigma_{t=1}^n r_t = j + k - n \) for some \( k \in \{j + 1, \ldots, n - 1\} \). Now put

\[ r' = (r_1, \ldots, r_j, r_{j+1}, \ldots, r_{k+1}, r_{k+1}, \ldots, r_n). \]

Then \( r' \in S_{jk} \). Finally, if \( r \in T_j, r_j > 2j - n - \Sigma_{t=1}^{n-1} r_t \geq r_{j+1} \), put
With the obvious notation we get

\[ G^0 = \{ \pi_{j,k,r}; r \in S_{jk}, 0 \leq j \leq k \leq n \}. \]

Furthermore, with the notation from [4, §10]:

\[ \Gamma(\pi_{j,k,r}) = \{ q \in \hat{K}; (q_1, \ldots, q_j) < (\infty, r_1, \ldots, r_j), \]

\[ (q_{j+1}, \ldots, q_k) < (r_{j+1}, \ldots, r_{k+1}), \]

\[ (q_{k+1}, \ldots, q_n) < (r_{k+2}, \ldots, r_{n+1}, -\infty) \}. \]

Using the imbeddings of \( \pi_{j,k,r} \)'s as subquotients of elementary representations in the proof of Proposition 10.2 in [4] we find \( [\pi_{j,k,r}] = [s] \) where

\[ \begin{align*}
    r_i &= n/2 - i, & 1 \leq i \leq j, \\
    s_i &= \begin{cases} 
        r_i + n/2 - i, & j + 1 \leq i \leq k + 1, \\
        r_i + n/2 - i + 1, & k + 2 \leq i \leq n + 1.
    \end{cases}
\end{align*} \] (3)

From (2) and (3) we get easily

**PROPOSITION 2.** Let \( 0 \leq j \leq k \leq n \) and \( r \in S_{jk} \).

(i) \( \lambda^j(r) = (r_1, \ldots, r_k, r_{k+2}, \ldots, r_{n+1}) \) is the unique \( j \)-fundamental corner of \( \pi_{j,k,r} \).

(ii) \( \lambda^k(r) = (r_1, \ldots, r_j, r_{j+2}, \ldots, r_{n+1}) \) is the unique \( k \)-fundamental corner of \( \pi_{j,k,r} \).

(iii) For \( i \in \{0, \ldots, n\}, i \neq j, i \neq k, \pi_{j,k,r} \) has no \( i \)-fundamental corners.

**PROOF OF THEOREM 2.** From Proposition 2, we see that \( F(\pi_{j,k,r}) = \{ \lambda^j(r), \lambda^k(r) \} \) and (i) is proven.

(ii) Suppose that \( 0 \leq j \leq k \leq n \) and \( r \in S_{jk}, s \in S_{im} \) and \( \{ \lambda^j(r), \lambda^k(r) \} = \{ \lambda^l(s), \lambda^m(s) \} \). We can suppose \( j \leq l \). Then we have the following possibilities:

(a) \( j \leq k \leq l \leq m \). Then it follows \( r_i = s_i \) for \( 1 \leq i \leq j \) and \( m + 2 \leq i \leq n + 1, r_i = s_{i-1} \) for \( k + 2 \leq i \leq l + 1 \), and \( r_{j+1} = \cdots = r_{k+1} = s_{j+1} = \cdots = s_k = a \geq r_{l+2} = \cdots = r_{m+1} = s_{l+1} = \cdots = s_{m+1} = b \). Hence, \( m + l - j - k = \Sigma_{i=1}^{n+1} s_i - \Sigma_{i=1}^{n+1} r_i = b - a \). This yields \( a = b \) and \( j = k = l = m \). Hence,

\[ \pi_{j,k,r} = \pi_{l,m,s}. \]

(b) \( j \leq l \leq k < m \). Then \( r = s \); hence, \( 0 = \Sigma_{i=1}^{n+1} s_i - \Sigma_{i=1}^{n+1} r_i = (m - k) + (l - j) \), i.e., \( l = j, m = k \). Therefore \( \pi_{j,k,r} = \pi_{l,m,s} \).

(c) \( j \leq l < m < k \). It follows \( r = s \) and \( s_{j+1} = \cdots = s_{l+1}, s_{m+1} = \cdots = s_{k+1} \). Now, \( s_{m+1} = s_{m+2} \) and \( s \in S_{l,m} \) implies \( s_l > s_{l+1} \); hence \( l = j \).
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Furthermore, \(0 = \Sigma_{i=1}^{n+1}s_i - \Sigma_{i=1}^{n+1}t_i = l + m - j - k = m - k\); therefore \(m = k\) and again \(m = k\). Therefore \(\pi \mapsto F(\pi)\) defines an injection of \(\hat{G}_0\) into \(\Omega\) and (ii) is proven.

(iii) Let \(\Omega_0\) be the image of this injection and let \(\Omega_0'\) be the subset of \(\Omega\) defined by the properties in (iii). We have to show \(\Omega_0 = \Omega_0'\).

Let \(\pi = \pi_{i,k,r} \in \hat{G}_0\). Then \(F(\pi) = (\lambda^i(r), \lambda^k(r))\). Put \(t = \lambda^i(r) + \rho_K - \rho_P, s = \lambda^k(r) + \rho_K - \rho_P\). Obviously, \([t] = [s]\). We have either \(r_j > r_{j+1}\) or \(r_{k+1} > r_{k+2}\). It is easy to check that \(t \in \bar{D}_k, s \in \bar{D}_j\). Furthermore, \(r_j > r_{j+1}\) implies \(t \in \bar{D}_k\) and \(r_{k+1} > r_{k+2}\) implies \(s \in \bar{D}_j\). Hence, \(F(\pi) \in \Omega_0'\) and \(\Omega_0 \subset \Omega_0'\).

Take \((q, q') \in \Omega_0'\) and let \(j, k \in \{0, \ldots, n\}\) be such that \([q + \rho_K - \rho_P] = [q' + \rho_K - \rho_P], q + \rho_K - \rho_P \in \bar{D}_k, q' + \rho_K - \rho_P \in \bar{D}_j\). Put \(t = q + \rho_K - \rho_P, s = q' + \rho_K - \rho_P\). Suppose first \(j \leq k\). Then \(t \in \bar{D}_k, s \in \bar{D}_j\) and \([t] = [s]\) implies \(t = s\) for \(1 \leq i \leq j\) and \(k + 1 \leq i \leq n, t_i = s_{i-1}\) for \(j + 2 \leq i \leq k, t_{j+1} = s_{k+1}\). By (i), this yields \(q_i = q'_i\) for \(1 \leq i \leq j\) and \(k + 1 \leq i \leq n, q_i = q'_i\) for \(1 \leq i < k, q_{j+1} + \Sigma_{i=1}^{n} q'_i = j + k - n\) and \(q'_k + \Sigma_{j=1}^{n} q_i = j + k - n\) and \(t \in D_j\) implies \(t \leq j\) and \(t_{j+1} > t_{j+1}\) implies \(t > j\). Hence, \(t \in S_{j,k}\). Obviously, \(\lambda^i(r) = q, \lambda^k(r) = q';\) therefore, \(F(\pi_{i,k,r}) = (q, q')\) and \((q, q') \in \Omega_0\).

Suppose now \(k < j\). Then, similarly as above, we find that \((q_1, \ldots, q_k, q_{k+1}, \ldots, q_n) \in S_{k,j}, \lambda^k(r) = q', \lambda^i(r) = q;\) hence, \(F(\pi_{j,k,r}) = (q, q')\) and again \((q, q') \in \Omega_0\). Thus, \(\Omega_0 = \Omega_0'\). Finally, the injectivity in (ii) shows that \(j\) and \(k\) are well determined by \((q, q')\) and Theorem 2 is proven.

Proof of Theorem 3. Let \(\hat{K}'\) be the set of all \(q \in \hat{K}\) such that there is \(\pi \in \hat{G}_0\) which has \(q\) as a \(j\)-fundamental corner. We have \(\Omega_1 = \{(q, j); q \in \hat{K}', 0 < j < n\}\). Statement (iv) of Theorem 3 is equivalent to

**Lemma 1.** \(\hat{K}^0 = \hat{K}^n = \hat{K}\). For \(1 \leq j \leq n - 1, \hat{K}^j\) is the set of all \(q \in \hat{K}\) such that either \(q_j > q_{j+1}\) or \((\alpha q + \rho_K - \rho_P) \neq 0 \forall \alpha \in R_P\).

**Proof.** Obviously, \(\hat{K}^j = \{(\lambda^j(r); r \in S_{k,j}, 0 \leq k < j\} \cup \{\lambda^j(r); r \in S_{j,k}, j < k \leq n\}\).

(a) Suppose first \(j = 0\). Then \(\hat{K}^0 = \{\lambda^0(r); r \in S_{0,k}, 0 \leq k \leq n\}\). Take any \(q \in \hat{K}\). Then \(q_i + n - i > q_{i+1} + n - (i + 1), 1 \leq i \leq n - 1\). Hence, there is unique \(k \in \{0, \ldots, n\}\) such that \(q_k = n - k > -\Sigma_{i=1}^{n} q_i > q_{k+1} + n - (k + 1)\). \((q_0 = \infty, q_{n+1} = -\infty)\). Put \(r = (q_1, \ldots, q_k, -\Sigma_{i=1}^{n} q_i + k - n, q_{k+1}, \ldots, q_n)\). Then \(r \in S_{0,k}\) and \(\lambda^0(r) = q\). Hence, \(\hat{K}^0 = \hat{K}\).

(b) Suppose now \(j = n\). Then for \(q \in \hat{K}\) we see that there is unique \(k \in \{0, \ldots, n\}\) such that \(q_k - k > -\Sigma_{i=1}^{n} q_i > q_{k+1} - (k + 1)\). Now, \(r = (q_1, \ldots, q_k, -\Sigma_{i=1}^{n} q_i + k, q_{k+1}, \ldots, q_n)\) is in \(S_{k,n}\) and \(\lambda^n(r) = q\). Hence, \(\hat{K}^n = \hat{K}\).

(c) Finally, suppose \(1 < j < n - 1\). Put \(\hat{K}_j = \{q \in \hat{K}; q_j > q_{j+1}\) or \((\alpha q + \rho_K - \rho_P) \neq 0 \forall \alpha \in R_P\)\).
Let $0 \leq k < j$, $r \in S_{kj}$. Put $q = \lambda(r)$. We have either $r_k > r_{k+1}$ or $r_{j+1} > r_{j+2}$. If $r_{j+1} > r_{j+2}$, then $q_j > q_{j+1}$ and $q \in \hat{K}_j$. Suppose $r_{j+1} = r_{j+2}$, i.e., $q_j = q_{j+1}$. Then $r_k > r_{k+1}$. This means $q_k + \sum_{i=1}^{n} q_i - k - j + n > 0$.

Furthermore, $q_{k+1} + \sum_{i=1}^{n} q_i - k - j + n = r_{k+2} - r_{k+1} \leq 0$. Now, for $1 \leq l \leq k$,

$$(\alpha_{l,n+1} | q + \rho_K - \rho_P) = q_l + \sum_{i=1}^{n} q_i - l - j + n$$

$$\geq q_k + \sum_{i=1}^{n} q_i - k - j + n > 0;$$

for $k+1 \leq l \leq j$,

$$(\alpha_{l,n+1} | q + \rho_K - \rho_P) = q_l + \sum_{i=1}^{n} q_i - l - j + n$$

$$\leq q_{k+1} + \sum_{i=1}^{n} q_i - k - j + n - 1 < 0;$$

for $j+1 \leq l \leq n$,

$$(\alpha_{l,n+1} | q + \rho_K - \rho_P) = q_l + \sum_{i=1}^{n} q_i - l - j + n + 1$$

$$< q_{k+1} + \sum_{i=1}^{n} q_i - k - j + n \leq 0.$$

Hence, $(\alpha | q + \rho_K - \rho_P) \neq 0 \forall \alpha \in R_p$, and $q \in K_j$.

Quite similarly, we find that for $j < k \leq n$, $r \in S_{jk}$, $q = \lambda(r)$ is in $\hat{K}_j$.

Finally, if $r \in S_{jj}$ and $q = \lambda(r)$, then either $r_j > r_{j+1}$ or $r_{j+1} > r_{j+2}$, so in any case $r_j > r_{j+2}$, i.e., $q_j > q_{j+1}$.

This proves that $\hat{K}^j \subset \hat{K}_j$, $1 \leq j \leq n - 1$.

Take now $q \in \hat{K}_j$. There is unique $k \in \{0, \ldots, n\}$ such that $q_k + n - j - k > -\sum_{i=1}^{n} q_i > q_{k+1} + n - j - k - 1$. Put $r = (q_1, \ldots, q_k, -\sum_{i=1}^{n} q_i + j + k - n, q_{k+1}, \ldots, q_n)$. Then $r \in ((n+1)^{-1} Z)p$ and $\sum_{i=1}^{n+1} r_i = j + k - n$.

If we can show that $r \in S_{jk}$ (or $S_{kj}$ in case $k < j$) we shall have $q = \lambda(r)$; hence, $q \in \hat{K}_j$.

Suppose first $k < j$. If $q_j > q_{j+1}$, then $r_{j+1} > r_{j+2}$; hence, $r \in S_{kj}$. If $q_j = q_{j+1}$, then it must be $(\alpha | q + \rho_K - \rho_P) \neq 0 \forall \alpha \in R_p$. But $r_k - r_{k+1} = (\alpha_{k,n+1} | q + \rho_K - \rho_P)$. Hence $r_k > r_{k+1}$, and again $r \in S_{kj}$.

Similarly, in the case $k > j$, $q_j > q_{j+1}$ implies $r_j > r_{j+1}$, and $(\alpha | q + \rho_K - \rho_P) \neq 0 \forall \alpha \in R_p$ gives $r_{k+2} - r_{k+1} = (\alpha_{k+1,n+1} | q + \rho_K - \rho_P) \neq 0$.

Finally, suppose $k = j$. Then $q_j > q_{j+1}$ means $r_j > r_{j+2}$; hence either
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$r_j > r_{j+1}$ or $r_{j+1} > r_{j+2}$, i.e., $r \in S_{jj}$. On the other hand, $r_j - r_{j+1} = (\alpha_{j,n+1} | q + p_K - p_j)$. Hence, in any case $q \in \hat{K}_j$ implies $r \in S_{jj}$.

Therefore, $\hat{\hat{K}}_j \subset \hat{K}_j$ and Lemma 1 is proven.

Let us return now to the proof of Theorem 3. By the definition of $\hat{\hat{K}}_j$, for any $(q, j) \in \Omega_1$ there is $q \in \hat{G}_0$ such that $q$ is $j$-fundamental corner of $\pi$. Then $\pi = \pi_{j,k,r}$ (or $\pi = \pi_{j,k,r}$) for some $k \in \{0, \ldots, n\}$, where $r \in S_{jk}$ (or $r \in S_{kj}$).

From the proof of Lemma 1 we see that this $k$ is well determined by $q$ and $j$. But then $r$ is also well determined. Hence, this $\pi$ is unique. Therefore, (i) is proven and (ii) follows immediately. (iii) follows from Theorem 2. Q.E.D.

5. Nonelementary unitary representations.

PROOF OF THEOREM 5. Let $T_j (0 < j < n)$ and $T_{jk} (0 < j < k < n)$ be defined as at the beginning of §4. Let $r \in T_{jk}$. From [4, Proposition 11.4(iii)] we know that $\pi_{j,k,r}$ is unitary if and only if $r_{j+1} = \cdots = r_{k+1}$. Let $r \in T_j$. By Proposition 11.4(ii) in [4], $\pi_{j,r}$ is unitary if and only if

$$r_j - l_1(r) \geq - \sum_{i=1}^{n} r_i - n + j \geq r_{j+1} - l_2(r) + 1,$$

where

$$l_1(r) = \max\{l \in \{1, \ldots, j\}; r_{l-1} > r_l\} \quad (r_0 = \infty),$$

$$l_2(r) = \min\{l \in \{j+1, \ldots, n\}; r_i > r_{l+1}\} \quad (r_{n+1} = -\infty).$$

Suppose first that $r_k + \sum_{i=1}^{n} r_i = j + k - n$ for some $k \in \{1, \ldots, j - 1\}$. Then $\pi_{j,r} = \pi_{k,r}$ for $r = (r_1, \ldots, r_k, r_{k+1}, \ldots, r_n)$. Now $0 > r_{j+1} - r_k + k + 1 - l_2(r) = (r_{j+1} - l_2(r) + 1) - (- \sum_{i=1}^{n} r_i - n + j)$; hence the second inequality in (4) is automatically satisfied. Suppose that the first is also satisfied. Then $r_j - l_1(r) \geq r_k - k$, i.e., $0 > r_j - r_k > l_1(r) - k$. Hence, $k \geq l_1(r)$, or $r_k = r_{k+1} = \cdots = r_{j+1}$. Conversely, if $r_k = \cdots = r_j$, then $k \geq l_1(r)$; hence, $r_j - l_1(r) \geq r_j - k = r_k - k = - \sum_{i=1}^{n} r_i - n + j$.

Therefore, $\pi_{j,r}$ is unitary if and only if $r_k = \cdots = r_j$, or equivalently $r_{k+1} = \cdots = r_{j+1}$.

Suppose now that $r_k + \sum_{i=1}^{n} r_i = j + k - n$ for some $k \in \{j+1, \ldots, n - 1\}$. Then $\pi_{j,r} = \pi_{k,r}$ with $r = (r_1, \ldots, r_{k+1}, r_{k+1}, r_{k+2}, \ldots, r_n)$. Similarly as above we find that the unitarity of $\pi_{j,r}$ is equivalent to $r_{k+1} = \cdots = r'_{j+1}$.

Finally, if $r_j \geq 2j - n - \sum_{i=1}^{n} r_i \geq r_{j+1}$, then (4) is satisfied and $\pi_{j,r}$ is unitary.

The conclusion is that for $0 \leq j \leq k \leq n$ and $r \in S_{jk}$, $\pi_{j,k,r}$ is unitary if and only if $r_{j+1} = \cdots = r_{k+1}$. But this is precisely equivalent to $\lambda(r) = \lambda^k(r),$
that is to the fact that \( \pi_{j,k,r} \) has only one fundamental corner. This proves statement (i) of Theorem 5.

Put

\[
R_{jk} = \{ r \in S_{jk}; r_{j+1} = \cdots = r_{k+1} \}, \quad \hat{G}_{jk}^{0} = \{ \pi_{j,k,r}; r \in R_{jk} \}.
\]

Then \( \hat{G}_{jk}^{0} \) is the disjoint union of \( \hat{G}_{jk}^{0}, 0 \leq j \leq k \leq n \). For \( 0 \leq j \leq k \leq n \), let \( \hat{R}_{jk} \) denote the set of all \( q \in \hat{R} \) with the property: there is \( \pi \in \hat{G}_{jk}^{0} \) such that \( q \) is \( j \)-fundamental corner and \( k \)-fundamental corner of \( \pi \) and \( q \) is not \( i \)-fundamental corner of \( \pi \) if \( i \neq j \) and \( i \neq k \). Obviously, \( \hat{R}_{jk} = \{ \lambda(r); r \in R_{jk} \} \). Furthermore, if \( r, s \in R_{jk} \) and \( \lambda(r) = \lambda(s) \), then \( r_{i} = s_{i} \) for \( 1 \leq i \leq n + 1, i \neq k + 1 \), and \( r_{k+1} = -\sum_{i=k+1}^{j} r_{i} + j + k - n = -\sum_{i=k+1}^{j} s_{i} + j + k - n = s_{k+1} \). Hence, \( r = s \). Therefore, \( r \mapsto \lambda(r) \) is a bijection of \( R_{jk} \) onto \( \hat{R}_{jk} \). So, to prove statement (ii) of Theorem 5 we have to show that \( \hat{R} \) is the disjoint union of \( \hat{R}_{jk}, 0 \leq j \leq k \leq n \).

First of all, we easily see that

\[
\hat{R}_{jk} = \left\{ q \in \hat{R}; q_{j+1} = \cdots = q_{k} = -\sum_{i=1}^{n} q_{i} + k + j - n, q_{j} > q_{k+1} \right\},
\]

(5)

\[
0 \leq j < k \leq n,
\]

\[
\hat{R}_{ij} = \left\{ q \in \hat{R}; q_{j} > -\sum_{i=1}^{n} q_{i} + 2j - n \geq q_{j+1}, q_{j} > q_{j+1} \right\}, \quad 0 \leq j \leq n.
\]

Suppose \( q \in \hat{R}_{jk} \cap \hat{R}_{lm} \), \( 0 \leq j < k \leq n \), \( 0 \leq l < m \leq n \). By (5) we have either \( j \notin \{ j + 1, \ldots, m - 1 \} \) or \( k \notin \{ j + 1, \ldots, m + 1 \} \) and either \( l \notin \{ j + 1, \ldots, k - 1 \} \) or \( m \notin \{ j + 1, \ldots, k - 1 \} \). We can suppose \( j < l \). Then we have the following possibilities.

(a) \( k \leq m \). Then \( \Sigma_{i=1}^{m} q_{i} = -q_{k} + k + j - n = -q_{m} + l + m - n \); hence, \( 0 < q_{k} - q_{m} = k + j - l - m \leq 0 \). Therefore \( q_{k} = q_{m} \), \( k = m \) and \( j = l \).

(b) \( k > m \). Then \( m \in \{ j + 1, \ldots, k - 1 \} \); hence, \( l \notin \{ j + 1, \ldots, k - 1 \} \), i.e., \( l = j \). This implies \( -q_{j+1} + k + j - n = \Sigma_{i=1}^{m} q_{i} = -q_{j+1} + j + m - n \), i.e., \( m = k \).

Hence, in any case \( j = l \) and \( k = m \).

Suppose now \( q \in \hat{R}_{ij} \cap \hat{R}_{lm} \), \( 0 \leq j \leq n \), \( 0 \leq l < m \leq n \). By (5) we have \( j \notin \{ j + 1, \ldots, m - 1 \} \), i.e., the following possibilities.

(a) \( j < l \). Then \( -\Sigma_{i=1}^{l} q_{i} + 2j - n \geq q_{j+1} \geq q_{l+1} = -\Sigma_{i=1}^{l} q_{i} + l + m - n \); hence, \( l \geq j \geq \frac{1}{2}(l + m) \), i.e., \( l \geq m \), a contradiction.

(b) \( j \geq m \). Then \( -\Sigma_{i=1}^{m} q_{i} + 2j - n \leq q_{j} \leq q_{m} = -\Sigma_{i=1}^{m} q_{i} + l + m - n \) which gives again \( l \geq m \).

Suppose finally \( q \in \hat{R}_{ij} \cap \hat{R}_{kk} \), \( 0 \leq j < k \leq n \). Suppose \( j < k \). Then by (5), \( 0 > q_{k} - q_{j+1} > 2(k - j) > 0 \), which is impossible. Hence, \( j = k \).

Therefore, \((j, k) \neq (l, m) \) implies \( \hat{R}_{jk} \cap \hat{R}_{lm} = \emptyset \).
Now, we shall show $\tilde{K} = \bigcup_{0 < j < k < n} \tilde{K}_{j/k}$. Take $q \in \tilde{K}$ and suppose $q \notin \tilde{K}_{j/l}$ for any $j \in \{0, \ldots, n\}$. $q \notin \tilde{K}_{0/0}$ implies $q_1 > -\sum_{i=1}^{n} q_i - n$; $q \notin \tilde{K}_{n/n}$ implies $-\sum_{i=1}^{n} q_i - n > q_n - 2n$; $q \notin \tilde{K}_{j/l}$ for any $j \in \{1, \ldots, n - 1\}$ implies that if $q_j - 2j > -\sum_{i=1}^{n} q_i - n \geq q_{j+1} - 2j$, $j \in \{1, \ldots, n - 1\}$, then $q_j = q_{j+1}$. Hence, there are $0 < j < k < n$ such that $q_j > q_{j+1} = \cdots = q_k > q_{k+1}$ ($q_0 = \infty$, $q_{n+1} = -\infty$) and $q_k - 2j > -\sum_{i=1}^{n} q_i - n > q_k - 2k$.

Suppose that $q_k - 2j > -\sum_{i=1}^{n} q_i - n \geq q_k - k - j$. Then there is $s \in \{j + 1, \ldots, k \}$ such that $-\sum_{i=1}^{n} q_i + s + j - n, q_j > q_s \geq q_{s+1}$; hence $q \in \tilde{K}_{js}$.

Suppose now that $q_k - k - j > -\sum_{i=1}^{n} q_i - n > q_k - 2k$. Then there is $s \in \{j + 1, \ldots, k - 1\}$ such that $-\sum_{i=1}^{n} q_i - n = q_k - k - s$. Then $q_{s+1} = \cdots = q_k = -\sum_{i=1}^{n} q_i + k + s - n$; hence $q \in \tilde{K}_{sk}$.

This proves assertion (ii) of Theorem 5.

6. Discrete series representations. As $G = SU(n, 1)$ is a real form of a simply connected complex semisimple Lie group, the set of discrete series representations is parametrized by $\bigcup_{p=0}^{n} (\tilde{K} \cap D_j)$ [2]. Let $\theta_\lambda$ denote the discrete series representation corresponding to $\lambda \in \tilde{K} \cap D_j, j \in \{0, \ldots, n\}$. By Blattner’s conjecture (which is proven for the hermitian symmetric case in [8]), if $\lambda \in \tilde{K} \cap D_j$, then $\theta_\lambda$ has $\lambda - \rho_K + \rho_j$ as a $j$-fundamental corner. Hence $\theta_\lambda = \pi_q$ for $q = \lambda - \rho_K + \rho_j$. Then $q + \rho_K - \rho_j = \lambda \in D_j$. Conversely, if $q \in \tilde{K}$ and $j \in \{0, \ldots, n\}$ are such that $\lambda = q + \rho_K - \rho_j \in D_j$, then $\theta_\lambda = \pi_q$. This proves assertion (i) in Theorem 6.

7. Subquotients of reducible elementary representations. In the following we write $\pi_{i,k}(r)$ instead of $\pi_{i,k,r}$. Using Theorem 7.5 in [4] and the notation preceding this theorem, we easily find

**Proposition 3.** (i) Let $p \in \tilde{M}$ be such that $\pi^{p,0}$ is reducible (i.e., $0 \in K(p)$). Put $j = j(p, 0)$. Then $\pi^{p,0}$ is direct sum of the following two irreducible representations:

\[
\pi_{j/k}(p_1, \ldots, p_{j-1}, j - 1/2s(p), j - 1/2s(p), p_j, \ldots, p_{n-1}),
\]
\[
\pi_{j-1,j-1}(p_1, \ldots, p_{j-1}, j - 1/2s(p) - 1, j - 1/2s(p) - 1, p_j, \ldots, p_{n-1}).
\]

(ii) Let $p \in \tilde{M}$ and $\lambda > 0$ be such that $\lambda \in K(p), -\lambda \in S(p), j(p, \lambda) = j \leq k \leq n - 1, -\lambda = s_k(p)$. Then $\pi^{p,\lambda}$ has the following two irreducible subquotients:

\[
\pi_{j,k}(p_1, \ldots, p_{j-1}, j + k - s(p) - p_{k}, p_j, \ldots, p_{k}, p_{k}, \ldots, p_{n-1}),
\]
\[
\pi_{j-1,k}(p_1, \ldots, p_{j-1}, j + k - s(p) - p_{k-1}, p_j, \ldots, p_{k-1}, p_{k-1}, \ldots, p_{n-1}).
\]

(iii) Let $p \in \tilde{M}$ and $\lambda > 0$ be such that $-\lambda \in S(p), \lambda \in K(p), j(p, -\lambda) = j,$
$\lambda = s_k(p)$, $1 \leq k \leq j - 1 \leq n - 1$. Then $\pi^{p,\lambda}$ has the following two irreducible subquotients:

\[
\pi_{k,j}(p_1, \ldots, p_{k-1}, p_j - s(p) - \frac{1}{2}\lambda, p_{j+1}, \ldots, p_{n-1}),
\]

\[
\pi_{k,j-1}(p_1, \ldots, p_{k-1}, p_j - s(p) - \frac{1}{2}\lambda, p_{j+1}, \ldots, p_{n-1}).
\]

(iv) Let $p \in \hat{M}$ and $\lambda > 0$ be such that $\lambda \in K(p)$, $-\lambda \in K(p)$. Put $j(p, \lambda) = j$, $j(p, -\lambda) = k$ ($1 \leq j \leq k \leq n$). Then $\pi^{p,\lambda}$ has the following irreducible subquotients:

\[
\pi_{j,k-1}(p_1, \ldots, p_{j-1}, j - \frac{1}{2} s(p) + \frac{1}{2}\lambda, p_j, \ldots, p_{k-1}, k - \frac{1}{2}s(p) - \frac{1}{2}\lambda,
\]

\[
p_k, \ldots, p_{n-1}),
\]

\[
\pi_{j-1,k}(p_1, \ldots, p_{j-1}, j - \frac{1}{2}s(p) + \frac{1}{2}\lambda - 1, p_j, \ldots, p_{k-1}, k - \frac{1}{2}s(p) - \frac{1}{2}\lambda,
\]

\[
p_k, \ldots, p_{n-1}),
\]

\[
\pi_{j-1,k-1}(p_1, \ldots, p_{j-1}, j - \frac{1}{2}s(p) + \frac{1}{2}\lambda - 1, p_j, \ldots, p_{k-1},
\]

\[
k - \frac{1}{2}s(p) - \frac{1}{2}\lambda - 1, p_k, \ldots, p_{n-1}).
\]

($\pi_{j,k-1}(\cdots)$ does not appear if $k = j$.)

From this we can easily identify all elementary representations in which a given $\pi \in \hat{G}^0$ appears as subquotient. Especially, for unitary representations we get the following statement (notation is from §5):

**Theorem 7.** (i) If $q \in \hat{K}_{jk}$, $0 \leq j < k < n$, then $\pi_q$ appears as a subquotient in at most four different elementary representations $\pi^{p,\lambda}$, $p \in \hat{M}$, $\lambda > 0$. These are $\pi^{p,\lambda}$ for the following values of $p$ and $\lambda$:

1. $p = (q_1, \ldots, q_{j-1}, q_{j+1}, \ldots, q_n)$, $\lambda = q_j - q_k + k - j$ (only if $j \geq 1$).

2. $p = (q_1, \ldots, q_{j-1}, q_{j+1}, \ldots, q_k, q_k, q_{k+2}, \ldots, q_n)$, $\lambda = q_j - q_{k+1} + k - j$ (only if $1 \leq j \leq k < n - 1$).

3. $p = (q_1, \ldots, q_j, q_{j+2}, \ldots, q_n)$, $\lambda = k - j$.

4. $p = (q_1, \ldots, q_j, q_{j+1}, \ldots, q_k, q_{k+2}, \ldots, q_n)$, $\lambda = q_{j+1} - q_{k+1} + k - j$ (only if $k \leq n - 1$).

(Notice, that in cases $q_j = q_{j+1}$ or $q_k = q_{k+1}$ some of these elementary representations coincide.)

(ii) If $q \in \hat{K}_{jj}$, $0 \leq j < n$, then $\pi_q$ appears as a subquotient in at most three different elementary representations $\pi^{p,\lambda}$, $p \in \hat{M}$, $\lambda > 0$. These are $\pi^{p,\lambda}$ for the following values of $p$ and $\lambda$:
(1) $p = (q_1, \ldots, q_{j-1}, 2j - n - \sum_{i=1}^{n} q_i, q_{j+2}, \ldots, q_n)$, $\lambda = q_j - q_{j+1}$ (only if $1 \leq j \leq n - 1$).

(2) $p = (q_1, \ldots, q_{j-1}, q_{j+1}, \ldots, q_n)$, $\lambda = q_j - 2j + n + \sum_{i=1}^{n} q_i$ (only if $j \geq 1$).

(3) $p = (q_1, \ldots, q_j, q_{j+2}, \ldots, q_n)$, $\lambda = 2j - n - \sum_{i=1}^{n} q_i - q_{j+1}$ (only if $j \leq n - 1$).

(Notice that in cases $q_j = 2j - n - \sum_{i=1}^{n} q_i$ or $q_{j+1} = 2j - n - \sum_{i=1}^{n} q_i$ some of these elementary representations coincide.)

We can prove now statement (ii) of Theorem 6. If $p \in \hat{K}$ is such that $\pi_p$ is not a discrete series representation, then we have the following possibilities:

(a) $q \in \hat{K}_j$, $q_j = 2j - n - \sum_{i=1}^{n} q_i$, $j > 1$. By Theorem 7(ii) $\pi_q$ appears as a subrepresentation of $\pi_{p,0}$, $p = (q_1, \ldots, q_{j-1}, q_{j+1}, \ldots, q_n)$; obviously $\lambda_p = 0$.

(b) $q \in \hat{K}_j$, $q_{j+1} = 2j - n - \sum_{i=1}^{n} q_i, j \leq n - 1$. By Theorem 7(ii) $\pi_q$ is a subrepresentation of $\pi_{p,0}$, $p = (q_1, \ldots, q_j, q_{j+2}, \ldots, q_n)$ and then $\lambda_p = 0$.

(c) $q \in \hat{K}_k$, $0 \leq j < k \leq n$. Then, by Theorem 7(i), $\pi_q$ appears as a subquotient of $\pi_{p,0}$, $p = (q_1, \ldots, q_j, q_{j+2}, \ldots, q_n)$. We have to prove that $\lambda_p = k - j$. To do so, we use the notation from [4, §7]. We have $q_{j+1} = \cdots = q_k = -\sum_{i=1}^{k} q_i + j + k - n$. Hence, $s(p) = j + k - 2q_k$ and $s(p) = j + k - 2i$ for $j + 1 \leq i < j + k - 1$. Therefore, $K(p)$ is contained in $k - j + 2\mathbb{Z}$ and $\{k - j - 2, k - j - 4, \ldots, j - k + 2\} \cap K(p) = \emptyset$. Hence

$$\lambda_p = \min\{\lambda \geq 0; \lambda \in K(p) \cup (- K(p))\} \geq k - j.$$ On the other hand, $\pi_{p, k-j}$ is reducible; hence, $\lambda_p = k - j$. This proves Theorem 6.

Finally, we state a theorem giving the exact value of $\lambda_p$ for any $p \in \hat{M}$. This can be easily deduced from Theorems 7.5 and 8.5 in [4]. We use the notation from [4, §7], and for $p \in \hat{M} = ((n + 1)^{-1} \mathbb{Z})_{>0}^{-1}$ we put $p_j = \infty$ for $j \in \mathbb{Z}$, $j \leq 0$, and $p_j = -\infty$ for $j \in \mathbb{Z}$, $j \geq n$.

**Theorem 8.** (i) Let $p \in \hat{M}$ be such that $S(p) = \{s_1(p), \ldots, s_{n-1}(p)\} \subset 2\mathbb{Z}$. If $s_j(p) \neq 0$ for $1 \leq j \leq n - 1$, then $\lambda_p = 0$. If $s_j(p) = 0$ for some $j \in \{1, \ldots, n - 1\}$, then

$$\lambda_p = \max\{2k + 2; k \in \mathbb{Z}_+, p_{j-k} = p_{j+k}\}.$$ (ii) Let $p \in \hat{M}$ be such that $S(p) = \{s_1(p), \ldots, s_{n-1}(p)\} \subset 2\mathbb{Z} + 1$. If $\{1, -1\} \not\subset S(p)$, then $\lambda_p = 1$. If $s_j(p) = 1, s_{j+1}(p) = -1$ for some $j \in \{1, \ldots, n - 2\}$, then $\lambda_p = \max\{2k + 3; k \in \mathbb{Z}_+, p_{j-k} = p_{j+k+1}\}$.

**Remark.** When this paper was finished, I heard about the preprint *On an infinitesimal characterization of the discrete series* by T. J. Enright and V. S.
Varadarajan. In this paper, they have constructed for any \( q \in \hat{K} \) and \( D \in C \) an irreducible admissible representation \( \pi(q, D) \) of \( G \) with the property that \( q \) is a \( D \)-fundamental corner of \( \pi(q, D) \). The only assumption about the group is rank \( G = \text{rank} \, K \). Our results show that these representations exhaust all of \( \hat{G}^0 \) in the case \( G = SU(n, 1) \). Furthermore, it is not difficult to check that if \( q \in \hat{K} \setminus \hat{K}^1 \) then \( \pi(q, D^1) \) is equivalent to some irreducible elementary representation having a fundamental corner and all such elementary representations are also exhausted.

REFERENCES


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