ASYMMETRIC MAXIMAL IDEALS IN $M(G)$

BY

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ABSTRACT. Let $G$ be a nondiscrete LCA group, $M(G)$ the measure algebra of $G$, and $M_0(G)$ the closed ideal of those measures in $M(G)$ whose Fourier transforms vanish at infinity. Let $\Delta_G$, $\Sigma_G$ and $\Delta_0$ be the spectrum of $M(G)$, the set of all symmetric elements of $\Delta_G$, and the spectrum of $M_0(G)$, respectively. In this paper this is shown: Let $\Phi$ be a separable subset of $M(G)$. Then there exist a probability measure $\tau$ in $M_0(G)$ and a compact subset $X$ of $\Delta_0 \backslash \Sigma_G$ such that for each $|c| < 1$ and each

$$v \in \Phi \text{ Card } \{f \in X : \hat{\tau}(f) = c \text{ and } |\hat{\tau}(f)| = r(v) \} > 2^c.$$

Here $r(v) = \sup \{ |\hat{\tau}(f)| : f \in \Delta_0 \backslash \hat{\Delta}_G \}$. As immediate consequences of this result, we have (a) every boundary for $M_0(G)$ is a boundary for $M(G)$ (a result due to Brown and Moran), (b) $\Delta_0 \backslash \Sigma_G$ is dense in $\Delta_0 \backslash \hat{\Delta}_G$, (c) the set of all peak points for $M(G)$ is $\hat{G}$ if $G$ is $\sigma$-compact and is empty otherwise, and (d) for each $\mu \in M(G)$ the set $\hat{\mu}(\Delta_0 \backslash \Sigma_G)$ contains the topological boundary of $\hat{\mu}(\Delta_0 \backslash \hat{\Delta}_G)$ in the complex plane.

Throughout the paper, let $G$ be a nondiscrete locally compact abelian group with dual $\hat{G}$, $M(G)$ the convolution measure algebra of $G$, and $M_0(G)$ the ideal in $M(G)$ which consists of all measures with Fourier transforms vanishing at infinity. As is well known, we then have $L^1(G) = M_0(G) \subset M_0(G) \subset M_c(G)$. Let $\Delta_G$ denote the spectrum of $M(G)$, i.e., the space of all nonzero complex homomorphisms of $M(G)$, and let $\hat{\mu}$ denote the Gelfand transform of $\mu \in M(G)$. We define

$$\Delta_0 = \{f \in \Delta_G : \hat{\sigma}(f) \neq 0 \text{ for some } \sigma \in M_0(G)\},$$

$$\Sigma_G = \{f \in \Delta_G : f(\sigma^*) = \overline{f(\sigma)} \text{ for all } \sigma \in M(G)\},$$

where $\sigma^*(E) = \sigma(-E)$ for all Borel sets $E$ in $G$ and $f(\sigma) = \hat{\sigma}(f)$. Then $\Delta_0$ is open (in $\Delta_G$), $\Sigma_G$ is closed, and $\hat{G} \subset \Delta_0 \cap \Sigma_G$. Moreover, $\Delta_0$ may be identified with the spectrum of $M_0(G)$, since $M_0(G)$ is an ideal in $M(G)$.

It is shown in [11] that given $\mu \in M_c(G)$, there exist fairly many elements $f \in \Delta_G$ such that $M_0(G) + L^1(\mu) \subset \text{Ker}(f)$ but $M_0(G) \not\subset \text{Ker}(f)$. In fact, it is not difficult to improve Theorem 2 of [11] as follows.

**Theorem A.** Let $0 \neq \lambda \in M_0(G), \mu \in M_c(G)$, and $H$ a subgroup of $G$ which is a $G_\delta$-set. Then there exists a probability measure $\sigma = \tau * \tau^*$, with $\tau \in \Delta_0$

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$M^+_0(\text{supp } \lambda)$, having the following properties:

(i) Given $0 < r < 1$, the set of all $f \in \Sigma_G$ such that

\[ \hat{\sigma}(f) = r, \quad \text{Ker}(f) \supset L^1(\mu), \quad \text{and} \quad \hat{\nu}(f) = \hat{\nu}(1) \quad \forall \nu \in M_d(G) \]

has cardinality $\geq 2^\mathbb{c}$. Here $\mathbb{c}$ denotes the cardinal number of the continuum.

(ii) Given a complex number $c$ of modulus $< 1$ and $g \in \Delta_G$ with $g(\delta_x) = 1$ for all $x \in H$, the set of all $f \in \Delta_G \setminus \Sigma_G$ such that

\[ \hat{\sigma}(f) = c, \quad \text{Ker}(f) \supset L^1(\mu), \quad \text{and} \quad \hat{\nu}(f) = \hat{\nu}(g) \quad \forall \nu \in M_d(G) \]

has cardinality $\geq 2^\mathbb{c}$.

For some related results, we refer the reader to Izuchi and Shimizu [8], Saka [12], Shimizu [13], and Williamson [15]. Now let $\mu \in M(G)$ be given, and define

\[ r(\mu) = \sup \{|\hat{\mu}(f)| : f \in \Delta_G \setminus \hat{G}\}. \]

Since $\Delta_G \setminus \hat{G}$ is compact, there exists at least one $f$ in this set such that $|\hat{\mu}(f)| = r(\mu)$. It seems to be a natural problem to ask how many $f$ as above there exist. Our answer is as follows.

**Theorem B.** Let $\mu \in M(G)$, and $\Phi$ a separable subset of $L^1(\mu)$. Then there exist a probability measure $\tau$ in $M^0(G)$ and a compact set $X$ in $\Delta_0 \setminus \Sigma_G$ such that

\[ \text{Card } \{ f \in X : \hat{\tau}(f) = c \text{ and } |\hat{\tau}(f)| = r(\nu) \} \geq 2^\mathbb{c} \]

for every complex number $c$ of modulus $< 1$ and every measure $\nu$ in $[L^1(\mu) \cap M^+(G)] \cup \Phi$.

Notice that we can set $\Phi = L^1(\mu)$ if $G$ is metrizable, since then $L^1(\mu)$ is separable. As easy consequences of the last theorem, we have the following results.

**Corollary 1.**

(a) Every boundary of $M^0(G)$ is a boundary of $M(G)$.

(b) The set $\Sigma_G \setminus \hat{G}$ is the topological boundary of $\Delta_G \setminus \Sigma_G$ in $\Delta_G$. In other words, $\Delta_G \setminus \Sigma_G$ is dense in $\Delta_G \setminus \hat{G}$.

(c) If $G$ is o-compact, then the set $P_G$ of all peak points for $M(G)$ is precisely $\hat{G}$. If not, then $P_G = \emptyset$.

**Corollary 2.** For each $\mu \in M(G)$, the set $\hat{\mu}(\Delta_0 \setminus \Sigma_G)$ contains the topological boundary of $\hat{\mu}(\Delta_G \setminus \hat{G})$ in the complex plane $\mathbb{C}$. In particular, we have

(a) If Card $[\hat{\mu}(\Delta_0 \setminus \Sigma_G)] < \mathbb{c}$, then $\hat{\mu}(\Delta_G \setminus \hat{G})$ is (at most) countable and coincides with $\hat{\mu}(\Delta_0 \setminus \Sigma_G)$.

(b) If $\hat{\mu}$ is real on $\Delta_0 \setminus \Sigma_G$, then $\hat{\mu}(\Delta_G \setminus \hat{G}) = \hat{\mu}(\Delta_0 \setminus \Sigma_G)$. 

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Notice that Theorem B implies the result of Brown and Moran [2] and Graham [5]: If $\mu \in M(G)$ and $\hat{\mu} = 0$ off $\Sigma_G$, then $r(\mu) = 0$. Part (a) of Corollary 1 is due to Brown and Moran [3]. We also refer to Brown’s result in [1]: $\Delta_0 \cap \Sigma_G$ is not entirely contained in the Shilov boundary of $M_0(G)$. It may be an interesting problem to ask whether or not we have $\hat{\mu}(\Delta_0 \setminus \Sigma_G) = \hat{\mu}(\Delta_G \setminus \overset{*}{G})$ for all $\mu \in M(G)$.

To prove Theorem B, we shall first construct a measure of a certain type (assuming that $G$ is metrizable). The construction of such a measure is almost the same as the corresponding one in [11], and Körner’s method [9] plays an important role in our construction.

We now introduce some notation. Let $m_G$ denote the Haar measure of $G$, and $\mathbb{Z}$ the group of all integers. For a set $K$ in $G$ and $p \in \mathbb{Z}^+$, we define
\[ pK = \{ x_1 + \cdots + x_p : x_j \in K \text{ for all } 1 \leq j \leq p \} \]
if $p \geq 1$, $pK = \{0\}$ if $p = 0$, and $(-p)K = -(pK)$. The subgroup of $G$ which the set $K$ generates is denoted by $Gp(K)$. We say that a Borel set $K$ is of type $M_0$ if $M_0(K) = M_0(G) \cap M(K)$ is nonzero. Let $q(G)$ denote the supremum of all natural numbers $p$ such that every neighborhood of the identity 0 of $G$ contains an element of order $\geq p$. Then it is easy to see that if $q(G) = \infty$, then $G$ is an $I$-group, and that if $q(G)$ is finite, then $G$ contains an open-and-compact subgroup $H$ such that $\text{ord}(x) \leq q(G)$ for all $x$ in $H$. A set $K$ in $G$ is called strongly independent if it is independent in the usual sense [10, p. 97] and if all of its elements have order $q(G)$. Finally, we denote by $Gp'(K)$ the set of all points $x$ of the form $x = k_1x_1 + \cdots + k_u x_u$, where $u = u_x$ is a natural number, $x_1, \ldots, x_u$ are distinct elements of $K$, $k_j \in \mathbb{Z}$ for all $1 \leq j \leq u$, and $|k_j| = 1$ for at least one index $j$.

**Lemma 1.** Let $\mu_0 \in M_+(G)$, $D$ a compact subset of $G$ with Haar measure zero, and $N$ a natural number. Let also $V_1, V_2, \ldots, V_u$ be nonempty open sets in $G$. Then we can find nonempty open sets $U_j \subset V_j$ (1 $\leq j \leq u$) subject to the following conditions:

(i) If $p_j \in \mathbb{Z}$, $|p_j| < q(G)$, and $1 \leq \sum_{j=1}^u |p_j| \leq N$, then the set $\sum_{j=1}^u p_j U_j$ does not contain $0 \in G$, and

\[ m_G \left[ D + \sum_{j=1}^u p_j U_j \right] < 1/N. \]

(ii) If $q_j \in \mathbb{Z}$, $\sum_{j=1}^u |q_j| \leq N$, and $|q_j| = 1$ for at least one index $j$, then

\[ \mu_0 \left[ D + \sum_{j=1}^u q_j U_j \right] < 1/N. \]
Proof. Let \( P \) be the set of all \( p = (p_1, \ldots, p_u) \in \mathbb{Z}^u \) as in (i). Similarly, let \( Q \) be the set of all \( q = (q_1, \ldots, q_u) \in \mathbb{Z}^u \) as in (ii).

The standard Baire category argument [10, 5.2.3] shows that there are points \( x_j \in V_j \) (1 \( \leq j \leq u \)) of order \( \geq q(G) \) such that \( \{x_j: 1 \leq j \leq u\} \) is independent. Since \( P \) is finite and \( D \) is a compact set with Haar measure zero, we can find a neighborhood \( W \) of \( 0 \in G \) so that

\[
0 \notin \sum_{j=1}^{u} p_j(x_j + W) \quad \text{and} \quad m_G \left[ D + \sum_{j=1}^{u} p_j(x_j + W) \right] < 1/N
\]

for all \( p \in P \). We may assume that \( x_j + W \subset V_j \) (1 \( \leq j \leq u \)).

Put \( E = \{x_j: 1 \leq j \leq u\} \), and take a compact neighborhood \( X \) of \( 0 \in G \) such that \( X + X \subset W \). Since \( M_a(G) \) is an ideal of \( M(G) \), it follows from the Fubini theorem and the definition of \( Q \) that

\[
\int_X \sum_{q \in Q} \mu_0 \left[ D + Gp(E) + \sum_{j=1}^{u} q_j t_j \right] dt_1 \cdots dt_u = 0.
\]

Therefore there are \( u \) points \( t_1, t_2, \ldots, t_u \) in \( X \) for which the integrand in (2) is zero. Hence, in particular, we have

\[
\mu_0 \left[ D + \sum_{j=1}^{u} q_j y_j \right] = 0 \quad (q \in Q),
\]

where \( y_j = x_j + t_j \). Upon comparing (1) and (3), we see that if \( U \subset X \) is a sufficiently small neighborhood of \( 0 \in G \), then the sets \( U_j = y_j + U \) have the required properties.

Lemma 2. Suppose that \( G \) is metrizable. Let \( \mu_0 \in M^+(G) \), and let \( C_0 \) be a \( \sigma \)-compact subset of \( G \) with Haar measure zero. Then there exists a strongly independent compact set \( K \) in \( G \) of type \( M_0 \) such that

\[
m_G[C_0 + Gp(K)] = \mu_0[C_0 + Gp'(K)] = 0.
\]

Proof. If \( q(G) \) is finite, we fix an open-and-compact subgroup \( H \) of \( G \) such that \( \text{ord}(x) \leq q(G) \) for all \( x \in H \). In the other case, we set \( H = G \).

Let \( \{D_n\}_{n=1}^{\infty} \) be an increasing sequence of compact subsets of \( G \) with \( C_0 = \bigcup_n D_n \), and \( \{\hat{E}_n\}_{n=1}^{\infty} \) a sequence of compact subsets of \( \hat{G} \) with \( \hat{G} = \bigcup_n \hat{E}_n \). We shall construct a sequence \( \{\sigma_n\}_{n=1}^{\infty} \) of probability measures in \( M_0(H) \), a sequence \( \{I_n\}_{n=1}^{\infty} \) of finite collections of disjoint compact sets in \( H \), and also a sequence \( \{n_p\}_{n=1}^{\infty} \) of natural numbers. They will satisfy the following conditions (and some other conditions):

\[
\supp \sigma_n \subset \bigcup \{\text{int}(I): I \in I_n\}.
\]
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(2) $\sup \{ |a_J(x)| : x \in \hat{G} \setminus \hat{F}_n \} < 2^{-n} a_n(I) \ \forall I \in I_n$.

It is also assumed that each set in $I_{n+1}$ is a subset of some set in $I_n$.

We first take any probability measure $\sigma_1 \in M_0(H)$ with compact support of diameter $<1/2$. Let $I$ be any compact neighborhood of $\text{supp} \ \sigma_1$ such that $\text{diam} I < 1$, $I_1 = \{I\}$, and $n_1 = 1$. Since $\sigma_1 \in M_0(G)$, we can take a compact set $\hat{F}_1$ in $\hat{G}$ subject to (2) with $n = 1$.

Suppose that $p$ is a natural number, and that $n_j (1 \leq j \leq p)$, $I_n$, $\hat{F}_n$ ($1 \leq n \leq m = n_p$) have been defined. Let $M_p$ be the largest natural number such that

(3) $\max \{ \sigma_m(I) : I \in I_m \} \leq M_p^{-2}$,

and write

(4) $\{ A \subset I_m : 1 \leq \text{Card} A \leq M_p \} = \{ A_r : 1 \leq r \leq s_p \}$.

Setting $n_{p+1} = n_p + s_p$, we shall construct $\sigma_n$, $I_n$, and $\hat{F}_n$ for all $m < n < n_{p+1}$ as follows.

Suppose that these objects have been defined for some $n = m + r - 1$ with $1 \leq r \leq s_p$, and put

(5) $K_n = \{ I \in I_n : I \subset J \text{ for some } J \in A_r \}$.

Then, for each set $K$ in $K_n$, there are a finite collection $\{L^K_j\}$ of disjoint compact sets in $K$ and a collection $\{\nu^K_j\}$ of measures in $M_0^+(K)$, with $\text{supp} \ \nu^K_j \subset \text{int}(L^K_j)$, such that

(6) $0 < \| \nu^K_j \| < n^{-1} a_n(K)$;

(7) $\sum_j \| \nu^K_j \| = a_n(K)$;

(8) $\left| \sum_j (\nu^K_j(x) - (\sigma_n|K)^*(x)) \right| < 2^{-n} a_n(K) \ \forall x \in \hat{F}_n$.

To see this, it suffices to apply Lemma 3 of [11] and its obvious modification. By virtue of Lemma 1, we can demand that the sets $L^K_j$ satisfy the following additional conditions:

(9) $\text{diam} \ L^K_j < 1/n$;

(10) $0 \leq \sum_{K \in K_n} \sum_j p^K_j L^K_j \ \forall (p^K_j) \in P_n$.
Here $P_n$ is the set of all tuples $(p_j^K)$ of integers such that $|p_j^K| < q(G)$ for all $j$ and $K$ and $1 \leq \Sigma_{K,j}|p_j^K| < n$. Similarly $Q_n$ is the set of all tuples $(q_j^K)$ of integers such that $|q_j^K| = 1$ for some $(K,j)$ and $\Sigma_{K,j}|q_j^K| < n$. Define

$$
\sigma_{n+1} = \sum_{I \in K_n} \sigma_n |I| + \sum_{K \in K_n} \sum_{I} \nu_j^K,
$$

$$
I_{n+1} = (I_n \setminus K_n) \cup \bigcup_{K \in K_n} \{L_j^K\}_j.
$$

Then (1), with $n$ replaced by $n + 1$, is satisfied. Finally we choose a compact set $\hat{F}_{n+1}$ in $\hat{G}$, with $\hat{F}_{n+1} \supset \hat{F}_n \cup \hat{F}_n$, so that (2) holds for $n + 1$.

This completes our induction. It is a routine matter to prove that the sequence $\{\sigma_n\}_n$ converges to some probability measure $\sigma \in M_0(H)$ in the weak-$*$ topology of $M(G)$, that

$$
K = \text{supp} \sigma \subset \bigcap_{n=1}^{\infty} \bigcup \{I: I \in I_n\},
$$

and that $K$ is strongly independent. (See the proof of Lemma 4 of [11], and notice that every element of $H$ has order $\leq q(G)$.)

Now we want to confirm

$$
m_G[C_0 + GP(K)] = \mu_0[C_0 + GP'(K)] = 0.
$$

Let $0 \neq x \in GP(K)$ be given. We have $x = \Sigma k_i x_i$ for some $(k_1, \ldots, k_u) \in \mathbb{Z}^u$ and some distinct elements $x_1, \ldots, x_u$ of $K$. By (9), (14), and (15), there exists a natural number $N_x$ such that the points $x_i$ belong to distinct sets in $I_n$ whenever $n > N_x$. Choose any natural number $p$ so that

$$
n_p > N_x + \sum_{1}^{u} |k_i| \quad \text{and} \quad M_p > u,
$$

and let $A$ be the collection of all $I$ in $I_{n_p}$ which contain some $x_i$ $(1 \leq i \leq u)$. Then $1 \leq \text{Card} A = u < M_p$, and so $A = A_r$ for some $1 \leq r \leq s_p$ by (4). Setting $n = n_p + r - 1$, we therefore infer from (5), (14) and (15) that $x$ belongs to the set

$$
\bigcup_{p_n} \left( \sum_{I \in K_n} \sum_{j} p_j^K L_j^K \right).
$$
Since \( p \) can be chosen as large as one pleases, we conclude that

\[
G_p(K) \{0\} \subset \bigcup_{n=1}^{\infty} \bigcup_{P_n} \sum_n \sum_{K_n} p^K_j L_j^K \quad (N = 1, 2, \ldots).
\]

Similarly we have

\[
G_p'(K) \subset \bigcup_{n=1}^{\infty} \bigcup_{Q_n} \sum_n \sum_{K_n} q^K_j L_j^K \quad (N = 1, 2, \ldots).
\]

It follows from (11) and (16) that

\[
m_G[D_N + G_p(K)] \leq \sum_{n=N}^{\infty} \sum_{P_n} m_G \left[ D_N + \sum_{K_n} \sum_{j} p^K_j L_j^K \right]
\]

\[
\leq \sum_{n=N}^{\infty} \sum_{P_n} m_G \left[ D_N + \sum_{K_n} \sum_{j} p^K_j L_j^K \right] < 2^{-N+1}
\]

for all \( N \geq 1 \). (Notice that \( m_G(D_N) = 0 \).) Letting \( N \to \infty \) in (18), we have \( m_G[C_0 + G_p(K)] = 0 \). Similarly we have \( \mu_0[C_0 + G_p'(K)] = 0 \) by (12) and (17). This completes the proof.

**Lemma 3.** Let \( \mu_0 \in M^+(G) \), \( C_0 \) a \( \sigma \)-compact subset of \( G \) which carries \( \mu_0 \), and \( K \) a compact subset of \( G \) such that

\[
(\ast) \quad \mu_0[C_0 + G_p(K)] = 0.
\]

Suppose that \( K_1, K_2, \ldots, K_p \) are disjoint compact subsets of \( K \) and that \( \sigma_j \in M_c(K_j \cup (-K_j)) \) for all \( 1 \leq j \leq p \).

(a) If \( m = (m_j)_1^p \) and \( n = (n_j)_1^p \) are different tuples of nonnegative integers, then

\[
\mu_0 \ast \sigma_1^m \ast \cdots \ast \sigma_p^m \perp \mu_0 \ast \sigma_1^n \ast \cdots \ast \sigma_p^n.
\]

(b) If \( \sigma_j \in M_c(K_j) \) for all \( 1 \leq j \leq p \) and \( \nu \in L^1(\mu_0) \), then

\[
\| \nu \ast \sigma_1^n \ast \cdots \ast \sigma_p^n \|^p = \| \nu \|^p \ast \| \sigma_1 \|^n_1 \cdots \| \sigma_p \|^n_p.
\]

**Proof.** To prove (a), we use the well-known method of Hewitt and Kakutani [6] (see also [10, 5.4.2]). Without loss of generality, assume that \( \sigma_j \geq 0 \) for all \( 1 \leq j \leq p \) and that \( m_1 < n_1 \). Write \( \tau_m = \sigma_1^m \ast \cdots \ast \sigma_p^m \), and similarly for \( \tau_n \). Putting \( E_j = K_j \cup (-K_j) \) for \( 1 \leq j \leq p \), we then see that \( \mu_0 \ast \tau_m \) is carried by the set \( A_m = C_0 + m_1 E_1 + \cdots + m_p E_p \). Therefore it suffices to show \( (\mu_0 \ast \tau_n) (A_m) = 0 \). Let \( \lambda_j \in M(E_j^n) \) be the \( n_j \)-fold product of \( \sigma_j \), and let \( B_j \) be the set of all points \( x_j = (x_{j1}, \ldots, x_{jn_j}) \) of \( E_j^n \) such that \( x_{jl} \neq \pm x_{jk} \) whenever \( 1 \leq i < k \leq n_j \). Since \( \sigma_j \) is a continuous measure, we then have
\[ \lambda_j(G^n \cap B_j) = 0 \] by the Fubini theorem. On the other hand, \((x_{ij}) \in B_1 \times \cdots \times B_p\) implies
\[
(1) \quad \mu_0 \left[ A_m - \sum_{j=1}^p \sum_{i=1}^{n_i} x_{ij} \right] \leq \mu_0 \left[ C_0 + m_1 E_1 - \sum_{i=1}^{n_1} x_{1i} + \sum_{j=2}^p Gp(K_j) \right]
\]
\[
\leq \mu_0 [C_0 + Gp'(K)] = 0
\]
by (\(^*\)) and the definition of \(Gp'(K)\). Evidently these two facts imply
\[ (\mu_0 * \tau_n)(A_m) = 0, \]
as was required.

To prove (b), we need the following fact: Given \(\mu \in M(G)\) and \(\epsilon > 0\), there is a neighborhood \(V\) of \(0 \in G\) such that
\[
(2) \quad \sigma \in M^+(G), \quad \text{supp } \sigma - \text{supp } \sigma \subset V = \|\mu * \sigma\| > (\|\mu\| - \epsilon)\|\sigma\|.
\]
Suppose by way of contradiction that this is false for some \(\mu\) and \(\epsilon\). Then, to each neighborhood \(V\) of \(0\) there corresponds a probability measure \(\sigma_V \in M(G)\) such that
\[ \|\mu * \sigma_V\| < \|\mu\| - \epsilon \quad \text{and} \quad \text{supp } \sigma_V \subset V - x_V \quad \text{for some } x_V \in G. \]
Upon replacing \(\sigma_V\) by \(\sigma_V * \delta_{x_V}\), we may assume that \(x_V = 0\). But then the net \(\{\mu * \sigma_V\}\) converges to \(\mu\) in the weak-* topology of \(M(G)\). Hence
\[
\|\mu\| \leq \lim \inf \|\mu * \sigma_V\| \leq \|\mu\| - \epsilon,
\]
a contradiction.

We now prove (b) as follows. By the continuity of convolution, we can retain generality in assuming that each \(\sigma_j\) has the form \(\sigma_j = \sum_{k=1}^q c_{jk} \tau_{jk}\), where the \(c_{jk}\) are complex numbers of absolute modulus one and the \(\tau_{jk}\) are mutually singular measures in \(M_c^+(K_j)\). Expanding \(\sigma_j^n = (\sum_{k=1}^q c_{jk} \tau_{jk})^{n_j}\) for all \(1 \leq j \leq p\) and applying part (a), we reduce (b) to the case where \(\sigma_j \geq 0\) (\(1 \leq j \leq p\)), and hence to the case where \(c_{jk} = 1\) for all \(j\) and \(k\). Since we can demand that every \(\tau_{jk}\) has support of sufficiently small diameter, part (b) follows from (2). This completes the proof.

**Proof of Theorem B.** Let \(\mu \in M(G)\), and \(\Phi\) a separable subset of \(L^1(\mu)\). Given \(\sigma \in M(G)\), we let \(\sigma_s\) denote the singular part of \(\sigma\) with respect to \(m_G\). Notice that
\[
(\ast) \quad r(\sigma) = \lim_{n \to \infty} \|\sigma^n + M_\sigma(G)\|^{1/n} = \lim_{n \to \infty} \|(\sigma^n)_s\|^{1/n},
\]
since \(M_\sigma(G)\) is an ideal in \(M(G)\) with spectrum \(\hat{G}\). Now define \(\mu_0\) to be the singular part of \(\exp(|\mu|)\), and choose a \(\sigma\)-compact subset \(C_0\) of \(G\) so that \(m_G(C_0) = \mu_0(G \setminus C_0) = 0\). Then \(\nu \in L^1(\mu)\) implies \((\nu^n)_s \in L^1(\mu_0)\) for all \(n \in \mathbb{Z}^+\).

We first assume that \(G\) is metrizable, and take a compact subset \(K\) of \(G\) as
where $H = H_\Gamma$ is the annihilator of $\Gamma$ in $G$ and $m_H$ denotes the Haar measure of $H$ of norm one. This can be proved by considering the Fourier transform of $\nu$ and by applying Theorem 1.9.1 of [10]. Since $\Phi \subset L^1(\mu)$ is separable, there is a $\sigma$-compact open subgroup $\Gamma$ of $G$ such that

$$\| (\nu^n)_* \| = \| (\nu^n)_* \cdot m_H \| \quad \forall \nu \in \Phi \text{ and } \forall n \in \mathbb{Z}^+.$$  

By Lemma 6 of [11], we may assume that $G_0 = G/H$ is metrizable and $m_G(C_0 + H) = 0$. Let $\pi : G \rightarrow G_0$ be the natural quotient map, and let

$$\nu \mapsto \pi^*(\nu) = \nu \circ \pi^{-1} : M(G) \rightarrow M(G_0)$$

be the measure algebra homomorphism induced by $\pi$. Then it is easy to check that $\pi^*$ maps $M_a^+(G)$ onto $M_a^+(G_0)$, $M_r^+(G)$ onto $M_r^+(G_0)$, and $L^1(\mu_0)$ onto $L^1(\mu^*(\mu_0))$ (cf. [14, 2.2.4]). Moreover, we have $\| \pi^*(\nu) \| = \| \nu \cdot m_H \|$ for all $\nu \in M(G)$, as is easily seen. It follows from (9) that

$$\| \pi^* [(\nu^n)_*] \| = \| (\nu^n)_* \| \quad \forall n \in \mathbb{Z}^+$$

for all $\nu \in \Phi$. Obviously (10) is satisfied for every $\nu \in M^+(G)$ as well.

Since $m_G \{ \pi(C_0) \} = m_G(C_0 + H) = 0$ and $\pi^*(\mu_0)$ is carried by the set $\pi(C_0)$, we have $L^1(\pi^*(\mu_0)) \subset M_a(G_0)$. In particular $\pi^*[(\nu^n)_*]$ is the singular part of $(\pi^*(\nu))^n = \pi^*(\nu^n)$ for every $\nu \in L^1(\mu)$ and every $n \in \mathbb{Z}^+$. Hence $r(\pi^*(\nu)) = r(\nu)$ for all $\nu \in [L^1(\mu) \cap M^+(G)] \cup \Phi$, by (10). To complete the proof, it therefore suffices to note that $\pi^*[M_a^+(G)] = M_a^+(G_0)$, that $\pi^*[M_0(G)] = M(G_0)$, and that the adjoint map of $\pi^*$ sends $\Delta_G \setminus \Sigma_G$ into $\Delta_G \setminus \Sigma_G$ in a one-to-one way. This establishes Theorem B for all nondiscrete groups.

**Proof of Corollary 1.** Let $Y \subset \Delta_0$ be a boundary of $M_0(G)$, and $\mu \in M(G)$. Choose any $f \in \Delta_G$ such that $|\hat{\mu}(f)| = \| \hat{\mu} \|_{\Delta_G}$. If $f \notin \hat{G}$, we take $\lambda \in M_a(G)$ so that $0 \leq \lambda \leq 1$ on $\hat{G}$ and $\hat{\lambda}(f) = 1$. Then we have $\lambda \ast \mu \in M_a(G)$ and $\| \lambda \ast \mu \|_{\Delta_G} = |\hat{\mu}(f)|$; hence $|\hat{\mu}(g)| = |\lambda \ast \mu(g)| = |\hat{\mu}(f)|$ for some $g \in Y$. If $f \in \hat{G}$, then $\rho(\mu) = |\hat{\mu}(f)|$. By Theorem B, we can find a probability measure $\tau \in M_0(G)$ such that $r(\tau \ast \mu) = r(\mu)$. Then $|\hat{\mu}(g)| = |\hat{\tau} \ast \mu(g)| = r(\mu) = |\hat{\mu}(f)|$ for some $g \in Y$, which establishes part (a).

To prove (b), first notice that $\Delta_G \setminus \Sigma_G \subset \Delta_G \setminus \hat{G}$ since $\hat{G}$ is open and is contained in $\Sigma_G$. If the above two sets were different, there would exist a nonempty open set $U$ in $\Delta_G$ such that $U \cap \Delta_G \setminus \Sigma_G = \emptyset \neq U \setminus \hat{G}$. Since the space of all Gelfand transforms of measures is closed under the complex conjugation on $\Sigma_G$, it would follow from the Stone-Weierstrass theorem that there would exist a $\hat{\mu} \in M(G)$ such that $0 \leq \hat{\mu} \leq 1$ on $\Sigma_G$, $\hat{\mu}(f) = 1$ for some $f \in U \setminus \hat{G}$, and $\hat{\mu} < 1/2$ on $\Sigma_G \setminus U$. Then the set $U \cap \hat{\mu}^{-1}(1)$ would be a local peak set for $M(G)$, and therefore would be a peak set for $M(G)$ by Rossi's theorem [4]. Consequently
in Lemma 2. Let \( \sigma_1, \sigma_2, \ldots, \sigma_p \) be mutually singular measures in \( M_\circ(K) \), and let \( z_1, z_2, \ldots, z_p \) be complex numbers satisfying \( |z_j| \leq \|\sigma_j\| \) \((1 \leq j \leq p)\). We then claim that given \( \nu \in L^1(\mu) \) there exists an element \( f \) in \( \Delta_G \setminus \hat{G} \) such that

\[
|f(\nu)| = r(\nu) \quad \text{and} \quad f(\sigma_j) = z_j \quad (1 \leq j \leq p).
\]

There is no loss of generality in assuming \( \|\sigma_j\| = 1 \) for all \( j \). Let \( \tau_{2j-1} \) and \( \tau_{2j} \) be mutually singular measures in \( L^1(\sigma_j) \) such that \( \sigma_j = (\tau_{2j-1} + \tau_{2j})/2 \) and \( \|\tau_{2j-1}\| = \|\tau_{2j}\| = 1 \), and write \( z_j = (w_{2j-1} + w_{2j})/2 \) with \( |w_{2j-1}| = |w_{2j}| = 1 \). Since \( m_G[C_0 + Gp(K)] = 0 \), it follows from Lemma 3 that

\[
\|\left[ v \ast \left( \delta_0 + \sum_{k=1}^{2p} \overline{w_k} \tau_k \right) \right]^n + M_a(G) \|
\]

\[
= \left\| (\nu^n)_x \ast \left( \delta_0 + \sum_{k=1}^{2p} \overline{w_k} \tau_k \right) \right\|^n
\]

\[
= \| (\nu^n)_x \| \left( 1 + \sum_{k=1}^{2p} \| \tau_k \| \right)^n = \| (\nu^n)_x \| (1 + 2p)^n,
\]

which yields

\[
r \left[ v \ast \left( \delta_0 + \sum_{k=1}^{2p} \overline{w_k} \tau_k \right) \right] = r(\nu) \cdot (1 + 2p).
\]

We can therefore find an element \( f \in \Delta_G \setminus \hat{G} \) such that

\[
|f(\nu)| = r(\nu) \quad \text{and} \quad f(\tau_k) = w_k \quad (1 \leq k \leq 2p).
\]

By the choices of \( \tau_k \) and \( w_k \), \((1)'\) implies \((1)\), which establishes our claim.

We next assert that, given \( \nu \in L^1(\mu) \), every linear functional on \( M_\circ(K) \), of norm \( \leq 1 \), extends to an element \( f \in \Delta_G \setminus \hat{G} \) such that \( |f(\nu)| = r(\nu) \). In fact, this is an easy consequence of \((1)\) and the arguments of Hewitt and Kakutani in [6]. We leave the details to the reader.

Now choose three disjoint compact sets \( K_i \) in \( K \) \((i = 1, 2, 3)\), each of type \( M_0 \), and fix two probability measures \( \lambda \in M_0(K_1) \) and \( \tau \in M_0(K_2) \). We now prove that \( \tau \) and the set

\[
X = \{ f \in \Delta_G : f(\lambda) = 1, |f(\lambda^*)| \leq 1/2 \} \cup \{ f \in \Delta_G : |1 - f(\lambda \ast \lambda^*)| \geq 3/2 \}
\]

have the required property. It is obvious that \( X \) is a compact subset of \( \Delta_0 \setminus \Sigma_G \). Let \( c \) be a complex number of modulus \( \leq 1 \), and \( \nu \in L^1(\mu) \). Let also \( \varphi \) be an arbitrary (linear) functional on \( M_\circ(K_3) \), of norm \( \leq 1 \). By the Hahn-Banach theorem, \( \varphi \) extends to a functional \( \psi \) on \( M_\circ(K) \), of norm one, such that \( \psi(\lambda) = 1 \) and \( \psi(\tau) = c \). It follows from the result asserted in the last paragraph that there
is an \( f \) in \( \Delta_G \setminus \hat{G} \) such that \( |f(\nu)| = r(\nu), f(\lambda) = 1, f(\varphi) = e \) and \( f = \varphi \) on \( M_c(K_3) \). We want to show that such an \( f \) can be chosen from the set \( X \). If \( |f(\lambda^\ast)| \) is less than 1/2, then there is nothing to prove; so assume \( |f(\lambda^\ast)| \geq 1/2 \). Setting \( \tau_1 = \lambda \ast \lambda^\ast \), we then have

\[
\|(\nu^m)_g \ast \tau_1^n\| \geq |f(\nu^m \ast \tau_1^n)| \geq r(\nu)^m(1/2)^n
\]

for all \( m \) and \( n \in \mathbb{Z}^+ \), so that

\[
(4) \quad r [\nu^m \ast (\delta_0 - \tau_1)] \geq r(\nu)^m(3/2) \quad (m \in \mathbb{Z}^+) \tag{*}
\]

by (\( \ast \)) and Lemma 3. Putting \( \mu_1 = \mu_0 \ast \text{exp}(\tau_1) \), we also see that \( \mu_1 \) is carried by the \( \sigma \)-compact set \( C_1 = C_0 + Gp(K_1) \) and that

\[
\mu_1 [C_1 + Gp'(K_2 \cup K_3)] = \int \mu_0 [C_1 + Gp'(K_2 \cup K_3) - y] \, d\theta(y)
\]

\[
\leq \mu_0 [C_0 + Gp'(K)] \cdot e = 0,
\]

where \( \theta = \text{exp}(\tau_1) \). Therefore, if \( \tau_2, \ldots, \tau_p \) are mutually singular probability measures in \( M_c(K_2 \cup K_3) \) and if \( m, n, n_2, \ldots, n_p \in \mathbb{Z}^+ \), then

\[
(5) \quad \|(\nu^m \ast (\delta_0 - \tau_1))^n \ast \tau_2^{n_2} \ast \cdots \ast \tau_p^{n_p} + M_a(G)\| \geq r [\nu^m \ast (\delta_0 - \tau_1)]^n \tag{5}
\]

by Lemma 3 (applied to \( \mu_1 \) and \( C_1 \)). Consequently, one more application of Lemma 3, combined with (5), yields

\[
(6) \quad r \left[ \nu^m \ast (\delta_0 - \tau_1) \ast \left( \delta_0 + \sum_{j=2}^p z_j \tau_j \right) \right] = r [\nu^m \ast (\delta_0 - \tau_1)] \cdot p
\]

for all complex numbers \( z_2, \ldots, z_p \) of absolute modulus one. (Notice that the left-hand side of (6) cannot be larger than the right-hand one.) Therefore there is a \( g_m \in \Delta_G \setminus \hat{G} \) such that

\[
|g_m [\nu^m \ast (\delta_0 - \tau_1)]| = r [\nu^m \ast (\delta_0 - \tau_1)], \quad g_m(\tau_j) = z_j \quad (2 \leq j \leq p).
\]

It follows from (4) and the first equality of (7) that \( |1 - g_m(\tau_1)| \geq 3/2 \), and so \( g_m \in X \); moreover \( |g_m(\nu)| = |g_m(\nu^m)|^{1/m} \geq r(\nu) (3/4)^{1/m} \) by (7) and (4). Recalling that \( X \) is compact and letting \( m \to \infty \), we find an element \( h \in X \) such that

\[
(8) \quad |h(\nu)| = r(\nu) \quad \text{and} \quad h(\tau_j) = z_j \quad (2 \leq j \leq p).
\]

We repeat almost the same argument as before to obtain an \( f \in X \) with the required property. Since it is easy to prove that the conjugate space of \( M_c(K_3) \) has cardinality equal to \( 2^c \), this establishes Theorem B for metrizable groups.

The proof for the nonmetrizable case is now easy. We first note that given \( \nu \in M(G) \) there is a \( \sigma \)-compact open subgroup \( \Gamma \) of \( \hat{G} \) such that \( \|\nu \ast m_\nu\| = \|\nu\| \),
there would exist a $\nu \in \mathcal{M}(G)$ such that $\hat{\nu}(f) = 1,$ $|\hat{\nu}| < 1$ on $\Delta_G,$ and $|\hat{\nu}| < 1/2$ on $\Delta_G \setminus \Sigma_G.$ But then $r(\nu) = 1,$ which contradicts Theorem B. This establishes part (b).

By Theorem B, no element of $\Delta_G \setminus \hat{G}$ can be a peak point for $\mathcal{M}(G);$ hence $P_G \subset \hat{G}.$ Therefore part (c) is an easy consequence of the fact that $G$ is $\sigma$-compact if and only if $\hat{G}$ is metrizable [7]. This completes the proof.

**Proof of Corollary 2.** Let $\mu \in \mathcal{M}(G)$ be given. Notice that $\Delta_G \setminus \hat{G}$ is the spectrum of the quotient algebra $\mathcal{M}(G)/\mathcal{M}_d(G).$ Choose a countable dense subset $D$ of $C \setminus \hat{\mu}(\Delta_G \setminus \hat{G}).$ For each $c \in D,$ there is a $\nu_c \in \mathcal{M}(G)$ such that $\hat{\nu}_c = (c - \hat{\mu})^{-1}$ on $\Delta_G \setminus \hat{G}.$ Setting $\Phi = \{ \nu_c : c \in D\},$ we apply Theorem B to find a compact set $X$ in $\Delta_0 \setminus \Delta_G$ such that

$$\sup \{|c - \hat{\mu}(f)|^{-1} : f \in X\} = \sup \{|c - \hat{\mu}(g)|^{-1} : g \in \Delta_G \setminus \hat{G}\}$$

for all $c \in D.$ Since $\hat{\mu}(X)$ is compact, this implies that $\hat{\mu}(X)$ contains all the boundary points of $\hat{\mu}(\Delta_G \setminus \hat{G})$ in $\mathbb{C}.$

If $\text{Card} \left[ \hat{\mu}(\Delta_0 \setminus \Sigma_G) \right] < c,$ then $\hat{\mu}(\Delta_G \setminus \hat{G})$ has a countable boundary since it is compact. Therefore $\hat{\mu}(\Delta_G \setminus \hat{G})$ itself is countable, so that $\hat{\mu}(\Delta_G \setminus \hat{G}) = \hat{\mu}(\Delta_0 \setminus \Sigma_G)$ by the result already established. If $\hat{\mu}$ is real on $\Delta_0 \setminus \Sigma_G,$ then $\hat{\mu}$ must be real on $\Delta_G \setminus \hat{G},$ hence $\hat{\mu}(\Delta_G \setminus \hat{G})$ has no interior point, and hence $\hat{\mu}(\Delta_G \setminus \hat{G}) = \hat{\mu}(\Delta_0 \setminus \Sigma_G).$ This establishes Corollary 2.

**Remarks.** (a) Theorem A implies $\hat{\mu}(\Delta_G) \subset \hat{\mu}(\Delta_0 \setminus \Sigma_G)$ for all $\mu \in \mathcal{M}(G).$ Moreover, we can prove that $\hat{\mu}_d(\Delta_G) \subset \hat{\mu}(\Delta_0 \setminus \Sigma_G)$ by applying the methods in [11].

(b) Notice that $\delta_0(C_1 + C_2) = 0$ if and only if $C_1 \cap (-C_2)$ is empty. If we only require that $C_0 \cap G\nu'(K) = \emptyset$ in Lemma 2 instead of that $m_G [C_0 + G\nu(K)] = \mu_0 [C_0 + G\nu'(K)] = 0,$ then the assumption that $C_0$ is a $\sigma$-compact set with $m_G(C_0) = 0$ can be weakened to be that $C_0$ is a set of the first category in $G$ (cf. [5, 2.1]).

(c) In some special cases, the proof of Theorem B can be somewhat simplified and we have a result slightly stronger than Theorem B.

Let $H_0$ be an open subgroup of $G$ of the form $H_0 = \mathbb{R}^n \times H_1,$ where $n$ is a nonnegative integer and $H_1$ is a compact subgroup of $G$ (cf. [10, 2.4.1]). Let $P$ be the set of all $p \in \mathbb{Z}$ such that $1 \leq p < q(H_1)$ and $\text{Card} \{ \chi \in \hat{H}_1 : \chi^p = 1 \} < \infty.$ Then the last condition in Lemma 2 can be strengthened to be that $m_G [C_0 + G\nu(K)] = \mu_0 [C_0 + K(P)] = 0.$ Here $K(P)$ denotes the set of all points $x$ of the form $x = \Sigma_k k_j x_j,$ where $u = u_x$ is a natural number, $x_1, x_2, \ldots, x_u$ are distinct elements of $K,$ and $k_1, k_2, \ldots, k_u$ are integers such that $|k_j| \in P$ for some $1 \leq j \leq u.$ The case where $2 \in P$ is particularly interesting.

Suppose in Lemma 3 that $\mu_0, C_0$ and $K$ are such that $\mu_0 [C_0 + K([1, 2]) ] = 0.$ Then we can prove that...
for all $v \in L^1(\mu_0)$ and all $\sigma_j \in M_c(K_j \cup (-K_j))$. Therefore a moment's glance at the proof of Theorem B yields this result: If either $q(G) = 2$, or $G$ contains an open subgroup $H_0$ as above with $2 \in P$, then the measure $\tau$ in Theorem B can be taken so that $\tau = \lambda \ast \lambda_*$ for some $\lambda \in M_0^+(G)$. We omit the details.

(d) If $\mu \in M^+(G)$, then the number $r(\mu)$ is in $\hat{\mu}(\Delta_0 \setminus \Sigma_G)$. To see this, choose a complex number $z$ of absolute modulus one so that $z\mu \in (AG \setminus G)$. Then we have

$$r(\delta_0 + \mu) = \lim_{n \to \infty} \|[(\delta_0 + \mu)^n]_s\|^{1/n} \geq \lim_{n \to \infty} \|[(\delta_0 + z\mu)^n]_s\|^{1/n} = 1 + r(\mu),$$

and so $r(\delta_0 + \mu) = 1 + r(\mu)$. Thus our assertion follows from Theorem B with $\Phi = \{\delta_0 + \mu\}$.

(e) Let $M_0^\infty(G)$ denote the $L$-ideal in $M(G)$ generated by all measures $\mu$ of the form $\mu = \mu_1 \ast \mu_2 \ast \cdots$, where the $\mu_j$ are probability measures in $M_0(G)$ and the infinite convolution product is assumed to converge in the weak-* topology of $M(G)$. Let also $\Delta_0^\infty$ denote the spectrum of $M_0(G)$ identified with an open subset of $\Delta_G$. Then it is not difficult to prove that $\text{Card}(\Delta_0 \setminus \Delta_0^\infty) \geq 2^\infty$. Moreover, in Theorem B, we can replace $M_0(G)$ and $\Delta_0$ by $M_0^\infty(G)$ and $\Delta_0^\infty$, respectively. Using this result, we can prove that if $Y$ is a boundary of $M_0^\infty(G)$, then $(Y \setminus \Sigma_G) \cup (Y \cap \hat{G})$ is a boundary of $M(G)$, which of course improves part (a) of Corollary 1. Similarly the set $\Delta_0$ in Corollary 2 can be replaced by $\Delta_0^\infty$.

(f) Finally we list three problems which the author has been unable to solve.

(i) Is it true that $\hat{\mu}(\Delta_G \setminus \hat{G}) = \hat{\mu}(\Delta_0 \setminus \Sigma_G)$ for all $\mu \in M(G)$?

(ii) Does $\hat{\mu}(\Sigma_G \setminus \hat{G}) = \{0\}$ imply $r(\mu) = 0$?

(iii) Does $\Sigma_G \setminus \hat{G}$ contain any strong boundary point for $M(G)$?

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