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CLASSIFICATION THEORY OF ABELIAN GROUPS. I: BALANCED PROJECTIVES

BY

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ABSTRACT. We introduce in this paper a class of Abelian groups which includes the torsion totally projective groups and those torsion-free groups which are direct sums of groups of rank one. Characterizations of the groups in this class are given, and a complete classification theorem, in terms of additive numerical invariants, is proved.

By a *height*, we mean a formal product $h = \Pi_p p^{v(p)}$, where $v(p)$ is either an ordinal or the symbol ∞, and the product is taken over all primes p. If G is an Abelian group, we define $hG = \bigcap_p p^{v(p)}G$. A short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is *balanced* if for every height h, the sequence $0 \longrightarrow hA \longrightarrow hB \longrightarrow hC \longrightarrow 0$ is exact. A group P is a balanced projective if it is projective with respect to all such sequences—i.e., for any such sequence, the induced map $\mathrm{Hom}(P, B) \longrightarrow \mathrm{Hom}(P, C)$ is surjective. A torsion-free group is balanced projective if and and only if it is a direct sum of groups of rank one. A p-group is balanced projective if and only if it is the direct sum of a divisible group and a totally projective group (defined below or in [17], [23], [6]). We will prove a classification theorem for the balanced projectives which will include, as special cases, Baer's 1937 classification of the torsion-free balanced projectives [2], and Hill's classification of totally projective p-groups [7], [23]. We also prove that any balanced projective is a direct sum of groups of torsion-free rank at most one. (Most of these rank one groups are not direct sums of a torsion group and a torsion-free group, so the invariants we need cannot be obtained directly from Baer's invariants and the Ulm invariants.)

An important tool in developing the theory is to work first in the local case—with modules over a discrete valuation ring. One then proves many of the results for an Abelian group M by referring to the localization $M_p = M \otimes Z_p$

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(where Z_p is the ring of integers localized at p and M_p is regarded as a Z_p-module) or to $M \otimes Z_p^*$ (regarded as a Z_p^*-module, where Z_p^* is the ring of p-adic integers). The theory of local balanced projectives over a discrete valuation ring is considered in [25] and [28]. (Some of the results in this paper depend on results in [28].) If R is a discrete valuation ring with prime p, then a sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is balanced if and only if for every ordinal α, the sequence

$$0 \longrightarrow p^\alpha A \longrightarrow p^\alpha B \longrightarrow p^\alpha C \longrightarrow 0$$

is exact. If M is an R-module, the *Ulm factors* of M are the groups $p^\lambda M/p^{\lambda+\omega}M$, where λ is an arbitrary limit ordinal. If M is a reduced R-module which is a balanced projective, then the Ulm factors of M are all direct sums of cyclic modules, and the invariants of these direct sums are exactly the invariants which classify the modules.

We give here a brief historical survey of the previous work which led to the work reported here. Prüfer proved that a countable p-group with no elements of infinite height is a direct sum of cyclic groups, and that these groups can therefore be classified. If G is any countable, reduced p-group, then the invariants of the previously mentioned Ulm factors, (each of which is a direct sum of cyclic groups) gives a set of invariants for G which Ulm [21] proved in 1933 could be used to classify an arbitrary reduced countable p-group. This complete classification of countable, torsion Abelian groups has always been considered the high point of the early development of the theory. A short proof was given by George Mackey, whose argument (published in [10] and [8]) has been essential for subsequent extensions of Ulm's theorem.

In 1937, Baer [2] gave a complete classification of torsion-free groups which could be written as direct sums of groups of rank one. This result was not viewed at the time as being related to Ulm's theorem, presumably partly because of the *ad hoc* description of the groups involved (as direct sums) and because it did not seem possible to prove good closure properties for this class of groups. (It was only in 1958 that Kaplansky [9] proved that a summand of such a group is again a direct sum of groups of rank one, by reducing the problem to the countable case, which had been proved by Kulikov [13] in 1952.)

In 1960, Kolettis [11] extended Ulm's theorem to reduced p-groups which are direct sums of countable groups. In 1967, Nunke [17] gave a homological description of these groups by introducing the totally projective groups, and characterizing the direct sums of countable p-groups as those totally projective p-groups G such that $p^\Omega G = 0$, where Ω is the first uncountable ordinal. E. A. Walker suggested that the entire classification theory should extend to totally projective groups, and showed that if this could be done, the totally projective

groups would be, in some sense, a "natural domain" for Ulm's theorem. He and Parker [18] extended Ulm's theorem to totally projective groups of length less than $\omega\Omega$. The extension of Ulm's theorem to all totally projective groups was proved by Hill [7]. Hill's approach was to give an entirely new characterization of the totally projective groups, and we shall follow Walker [23] in taking this characterization as our definition of these groups.

This paper is the first of a series concerning the classification theory of mixed Abelian groups, (i.e. groups which are not necessarily torsion or torsion free). The possibility of extending Ulm's theorem to mixed groups and including the theory of direct sums of torsion-free groups of rank one was originally suggested to the author by the work of Crawley and Hales [4]. They gave a generators-and-relations description of totally projective p-groups which extends in an obvious way to describe a large class of mixed groups, the "simply presented groups", which are not yet well understood, but which include the groups of this paper.

The author developed the local theory of balanced projectives in 1971 (reported in [25] and [28]). The development of the global theory was partly influenced by conversations with the author's student, Brian Wick. Some contributions contained in his thesis [29] are pointed out in §§2 and 5.

In the first section below, we develop the methods of extending homomorphisms and isomorphisms which we will use. In the second section we introduce the basic building blocks in the theory of balanced projectives, the elementary balanced projectives, and show that there are enough balanced projectives. In the third section we prove a decomposition theorem for objects in an arbitrary additive category. We introduce a category in which Abelian groups of torsion-free rank one are small objects, and use the decomposition theorem in this category to obtain some rough information about summands of a direct sum of groups of rank one. In the fourth section we complete the classification theory in the local case (for modules over a discrete valuation ring). In the fifth section, we extend these results to the global case, introducing the necessary invariants. We close with some comments concerning open questions and related results.

We close this introduction with remarks on notation and basic terminology. We denote by Z_p the ring of integers localized at p—i.e., those rational numbers which can be written as fractions with denominators prime to p. Similarly, if π is any set of primes, we let Z_π be the ring of integers localized at π—those rational numbers which can be written as fractions with denominators prime to the elements of π. It is important to note that if an Abelian group M can be made into a Z_π-module, then this can be done in one and only one way, so that the module structure is inherent in the group structure. (More precisely, a group

has such a module structure if and only if for all primes $p, p \notin \pi, M$ has no p-torsion and $M = pM$.) If M and N are Z_π-modules, then any group homomorphism from M to N is a Z_π-homomorphism, so $\mathrm{Hom}(M, N)$ is unambiguously defined, whether one thinks of M and N as modules or groups, and, similarly, $M \otimes N$ is unambiguously defined. We let Z_p^* be the ring of p-adic integers—the completion of Z_p.

If M is an Abelian group, or a module over a principal ideal ring in which p is a prime, and α is an ordinal, we define the subgroups (or submodules) $p^\alpha M$ inductively by $p^{\alpha+1}M = p(p^\alpha M)$ and, if α is a limit ordinal, $p^\alpha M = \bigcap_{\beta<\alpha} p^\beta M$. We define $p^\infty M = \bigcap_\alpha p^\alpha M$, where the intersection is taken over all ordinals α. (Some may prefer to note that by cardinality considerations, there is some α such that $p^\alpha M = p^{\alpha+1}M$, from which it follows that $p^\alpha M = p^\beta M$ for any ordinal $\beta, \beta > \alpha$, so we can define $p^\infty M$ to be $p^\alpha M$, when α is chosen in this way.)

If M is a group or a module over a discrete valuation ring in which p is a prime, then the *p-height* of an element x of M, written $h_p(x)$, is defined to be α, if $x \in p^\alpha M$ and $x \notin p^{\alpha+1}M$, and to be ∞ if $x \in p^\infty M$. In general, we define a height to be a formal product, $h = \Pi_p p^{v(p)}$, where $v(p)$ is either an ordinal or the symbol ∞, and the product is taken over all primes p. If M is an Abelian group, then the height of an element, $h(x)$ is defined to be $\Pi_p\, p^{v(p)}$ if $x \in p^{v(p)}M$ for all primes p, and in every case in which $v(p)$ is an ordinal, $x \notin p^{v(p)+1}M$. (In other words, $v(p) = h_p(x)$.) In particular, this implies that $x \in hM$, where $hM = \bigcap_p p^{v(p)}M$. We let $d = \Pi_p p^\infty$, so dM is the maximal divisible subgroup of M. dM is a summand of M [6], [8], and M is said to be *reduced* if $dM = 0$.

If M is an Abelian group or a module over a discrete valuation ring R with prime p, we define $M[p] = \{x \in M\colon px = 0\}$. We define

$$U_\alpha(M) = (p^\alpha M)[p]/(p^{\alpha+1}M)[p],$$

and

$$f(\alpha, M) = \dim U_\alpha(M),$$

where the dimension is over the field Z/pZ or R/pR respectively. Similarly

$$U_\infty(M) = (p^\infty M)[p]$$

and

$$f(\infty, M) = \dim U_\infty(M).$$

The numbers $f(\alpha, M)$ and $f(\infty, M)$ are the *Ulm invariants* of M. If M is an Abelian group, and we wish to consider the above notions for various primes p, we will use the notation $U_\alpha^p(M)$ and $f_p(\alpha, M)$ with the obvious meanings.

1. **Extending homomorphisms and isomorphisms.** In the first part of this section we will consider a number of more-or-less well-known results concerning the extending of homomorphisms defined on submodules of a module over a discrete valuation ring. In the following, R will always be a discrete valuation ring and p a prime element. In the second part of this section we will apply these results to obtain similar results for Abelian groups, by reducing them to results involving modules over the rings of p-adic integers, Z_p^*.

1.1. DEFINITION. A submodule S of a module M is *nice* if it satisfies the following two equivalent conditions: (i) for any ordinal α, $p^\alpha(M/S) = \nu(p^\alpha M)$, where ν: $M \longrightarrow M/S$ is the natural homomorphism; (ii) any coset of S contains an element of maximal height.

The next five statements about nice submodules are all trivial except for 1.3, which is proved in [10], and is an easy consequence of the linear compactness of a complete discrete valuation ring. 1.6 is a consequence of 1.3, 1.4 and 1.5.

1.2. LEMMA. *If M is a module, $p^\alpha M$ is a nice submodule. If for all $i \in I$, S_i is nice in M_i, then $\bigoplus_{i\in I} S_i$ is a nice submodule of $\bigoplus_{i\in I} M_i$. If S is nice in M, and ν: $M \longrightarrow M/S$ the natural map, then $p^\infty(M/S) = \nu(p^\infty M)$.*

1.3. LEMMA. *If R is a complete discrete valuation ring, and M is an R-module, then any cyclic submodule is nice.*

1.4. LEMMA. *If R is any discrete valuation ring and M an R-module, then a cyclic submodule of order p is nice.*

1.5. LEMMA. *If M is an R-module and S a nice submodule, and K is a submodule of M containing S and such that K/S is nice in M/S, then K is nice in M.*

1.6. LEMMA. *If S is a nice submodule of a module M and K is a submodule containing S and K/S is finitely generated, then if either K/S is torsion or R is complete, K is also a nice submodule.*

1.7. DEFINITION. A module M is said to satisfy *Hill's condition* if it has a family $\mathcal{C}$ of submodules such that:

(i) If $H_i \in \mathcal{C}$, $i \in I$, then $\Sigma_{i\in I} H_i \in \mathcal{C}$.

(ii) If $H \in \mathcal{C}$ and X is a countable subset of M, then there is a submodule $H' \in \mathcal{C}$ such that $H \subseteq H'$, $X \subseteq H'$, and H'/H is countably generated.

(iii) $\{0\} \in \mathcal{C}$.

(iv) The elements of $\mathcal{C}$ are nice submodules of M.

1.8. DEFINITION. A torsion module is a totally projective module if it is reduced and satisfies Hill's condition.

This condition was first introduced by Hill [7] as a way of analyzing the totally projective p-groups introduced earlier by Nunke, [17]. We follow E. A. Walker [23] in taking this condition as the definition of these groups.

1.9. LEMMA. *If M satisfies Hill's condition and N is a summand of M, then N satisfies Hill's condition.*

PROOF. If $M = N \oplus N'$, let $\mathcal{C}'$ be the elements $C \in \mathcal{C}$ such that $C = C \cap N + C \cap N'$. An easy induction shows that $\mathcal{C}'$ satisfies the conditions in Definition 1.7 if $\mathcal{C}$ does. The submodules $C \cap N$, $C \in \mathcal{C}'$ form a family of submodules of N which enable one to verify easily that N satisfies Hill's condition.

1.10. LEMMA. *A direct sum of totally projective modules is totally projective. If M is a torsion module and α an ordinal such that $p^\alpha M$ is cyclic and $M/p^\alpha M$ is totally projective, then M is totally projective. For any ordinal α, there is a totally projective module H_α such that $p^\alpha H_\alpha = 0$, and α is the smallest ordinal for which this is true. If M is totally projective, then so are $p^\alpha M$ and $M/p^\alpha M$ for any ordinal α.*

PROOF. All of these are easy consequences of the definition, the proof of the existence part being an iteration based on repeated application of the previous parts of the lemma. Totally projective modules of this sort were first constructed by Nunke in [16, Theorem 6.3], from a different point of view.

To state the main theorems concerning extensions of isomorphisms, we need some additional terminology. We recall the Ulm functors

$$U_\alpha(M) = (p^\alpha M)[p]/(p^{\alpha+1}M)[p] \qquad (\alpha \geqslant 0),$$

$$U_\infty(M) = (p^\infty M)[p].$$

If M and N are modules, and S a submodule of M, and we are given a homomorphism $f\colon S \longrightarrow N$ which we would like to extend to a homomorphism $g\colon M \longrightarrow N$, it is reasonable to ask to what extent the maps $U_\alpha(g)\colon U_\alpha(M) \longrightarrow U_\alpha(N)$ are determined by f.

If M is a module and S a submodule, we let $S_\alpha = S \cap p^\alpha M$, and $S_\infty = S \cap p^\infty M$. We let S_α^* be the submodule of S_α consisting of those elements x such that $px \in S_{\alpha+2}$. There is a natural monomorphism [8, p. 28] $\eta_\alpha\colon S_\alpha^*/S_{\alpha+1} \longrightarrow U_\alpha(M)$, for any ordinal α, and we denote the image of this map by $I_\alpha(S)$. (If $x \in S_\alpha^*$ then $px = py$ for some $y \in p^{\alpha+1}M$, and we let $\eta_\alpha(x) = x - y + (p^{\alpha+1}M)[p]$.) Similarly, we let $I_\infty(S) = S[p] \cap p^\infty M$. (Note that this definition would be analogous to that of $I_\alpha(S)$ if we used the convention that $h(0) = \infty + 2$.)

It is clear that if $f\colon S \longrightarrow N$ is a homomorphism and S is a submodule of M, then for f to extend to a homomorphism $g\colon M \longrightarrow N$ it is necessary that

$f(S_\alpha) \subseteq p^\alpha N$. In this case, there is an induced map $S_\alpha^*/S_{\alpha+1} \longrightarrow f(S)_\alpha^*/f(S)_{\alpha+1}$ and we have the obvious formula $U_\alpha(g)\eta_\alpha = \eta_\alpha f$. Hence, the induced map $U_\alpha(g)$ is uniquely determined on the subspace $I_\alpha(S)$ of $U_\alpha(M)$. In particular, if f is an isomorphism of S onto a submodule T of N, and we wish f to extend to an isomorphism of M onto N, then the induced isomorphism $U_\alpha(M) \longrightarrow U_\alpha(N)$ must take $I_\alpha(S)$ onto $I_\alpha(T)$. Hence these two subspaces (necessarily isomorphic if $f(S_\beta) = T_\beta$ for all β) must have the same codimension in $U_\alpha(M)$ and $U_\alpha(N)$ respectively.

We now state the three main extension theorems.

1.11. THEOREM. *Let M and N be modules, and S and T nice submodules such that M/S and N/T are torsion and satisfy Hill's condition. Let ϕ: $S \longrightarrow T$ be an isomorphism such that for each ordinal α, $\phi(S \cap p^\alpha M) = T \cap p^\alpha N$. Then ϕ extends to an isomorphism of M onto N if and only if for every ordinal α, $U_\alpha(M)/I_\alpha(S) \cong U_\alpha(N)/I_\alpha(T)$, and $U_\infty(M)/I_\infty(S) \cong U_\infty(N)/I_\infty(T)$.*

Proofs of essentially this theorem are given in [7], [23], and [6]. These proofs assume that M/S and N/T are reduced. The arguments carry over, however, without significant change. Alternatively one can simply reduce the stated result to the case where the factors are reduced by the following argument. The condition on $I_\infty(S)$ and $I_\infty(T)$ guarantees that the restriction of ϕ to S_∞ extends to an isomorphism $p^\infty M \longrightarrow p^\infty N$. We therefore have a height preserving isomorphism $S + p^\infty M \longrightarrow T + p^\infty N$ and these submodules are clearly nice. We choose a decomposition $M = p^\infty M \oplus M'$, and let $S' = (S + p^\infty M) \cap M'$. S' is a nice submodule of M' and M'/S' is a summand of M/S, and hence satisfies Hill's condition, and if we choose N' and T' similarly, then we have reduced the problem to the case where all of the modules are reduced.

1.12. COROLLARY. *If M is a module and S a nice submodule such that M/S is torsion and satisfies Hill's condition, then any automorphism of S which preserves heights in M extends to an automorphism of M.*

1.13. THEOREM. *Let M and N be modules, S a nice submodule of M such that M/S is torsion and satisfies Hill's condition, and f: $S \longrightarrow N$ a homomorphism such that for all ordinals α, $f(S \cap p^\alpha M) \subseteq p^\alpha N$. Then f extends to a homomorphism of M into N.*

This theorem is actually considerably easier to prove than 1.11, since it only involves a "one-sided" argument. As it stands, however, the result is not in the literature, and we will indicate how it can be derived as a corollary of 1.11. Extend f to a homomorphism ϕ: $S \oplus N \longrightarrow S \oplus N$ by setting $\phi(s, x) = (s, f(s) + x)$. Corollary 1.12 shows that ϕ extends to an automorphism of $M \oplus N$. If this automorphism is σ, θ: $M \longrightarrow M \oplus N$ is the natural inclusion and π: $M \oplus N$

$\longrightarrow N$ the natural projection, then $\pi\sigma\theta$ is the desired extension of f.

The following result is essentially a comment on 1.11 and 1.13, and can be proved by a careful examination of the proofs of these theorems, especially that in [23]. It seems to have been noticed independently by the author and R. Nunke, in each case as a result of examining the various proofs of Ulm's theorem for totally projective groups from a functorial point of view. A proof can also be extracted from the proof of 4.4 below. This result is not used in this paper.

1.14. THEOREM. *If, under the hypotheses of either* 1.11 *or* 1.13, *we have a map* $f\colon S \longrightarrow N$ *where* S *is a submodule of* M, *and for each ordinal* α, $g_\alpha\colon U_\alpha(M) \longrightarrow U_\alpha(N)$ *is any homomorphism such that* $g_\alpha\eta_\alpha = \eta_\alpha f$ *on* $S^*_\alpha/S_{\alpha+1}$, *then the extension* $g\colon M \longrightarrow N$ *guaranteed by those theorems may be chosen so that* $U_\alpha(g) = g_\alpha$, *for all ordinals* α (*and also for* $\alpha = \infty$ *if* $g_\infty\colon U_\infty(M) \longrightarrow U_\infty(N)$ *is similarly chosen*).

In the case of 1.13, this means we have an isomorphism $I_\alpha(S) \longrightarrow I_\alpha(T)$, and any extension that we wish of this map to an isomorphism $U_\alpha(M) \longrightarrow U_\alpha(N)$ may be specified in advance as the map to be induced by the isomorphism of M onto N.

1.15. THEOREM. *If* M *and* N *are torsion totally projective modules, then* $M \cong N$ *if and only if for all ordinals* α, $f(\alpha, M) = f(\alpha, N)$. *If* M *is totally projective*, $p^\alpha M = 0$, *and* α *is the smallest ordinal for which this is true, then* (i) *if* α *is a limit ordinal then* M *is a direct sum of groups of smaller length* (*i.e.*, $M = \bigoplus_{i\in I} M_i$ *where, for each* i, *there is an ordinal* $\beta(i)$, $\beta(i) < \alpha$, *such that* $p^{\beta(i)}M_i = 0$), *and* (ii) *if* $\alpha = \gamma + 1$, *then* $M = \bigoplus_{i\in I} M_i$ *where for each index* i, $p^\gamma M_i \cong R/pR$.

PROOF. The first statement is Hill's theorem, and is an immediate consequence of 1.11. The other two are proved by constructing for such a group M another group N having the desired property and with the same Ulm invariants. This depends on a careful study of just what the invariants of a totally projective group must be, for which we refer to [4], [6], and [28].

We now turn to the global theorems for Abelian groups which follow from these local facts. We recall that the kernel of the natural map $M \longrightarrow M_p$ is exactly the set of elements of M which are torsion of order prime to p. This is also the kernel of the natural map $M \longrightarrow M \otimes Z^*_p$. We also recall that if M is an Abelian group and N a subgroup, then the induced map $N \otimes Z^*_p \longrightarrow M \otimes Z^*_p$ is injective, so $N \otimes Z^*_p$ may be regarded as a submodule of $M \otimes Z^*_p$. If M is a torsion group, then the natural map $M_p \longrightarrow M \otimes Z^*_p$ is an isomorphism, and the map $M \longrightarrow M_p$ takes the p-primary part of M isomorphically onto M_p.

1.16. LEMMA. *Let M and N be Abelian groups, S and T subgroups such that M/S and N/T are torsion, and $f: S \longrightarrow T$ a homomorphism. If for every prime p, the induced map $f_p: S_p \longrightarrow T_p$ extends to a homomorphism $g(p): M_p \longrightarrow N_p$, then f extends to a homomorphism $g: M \longrightarrow N$ such that $g_p = g(p)$. If each $g(p)$ can be chosen to be an isomorphism, then g will be an isomorphism. Similarly, if M and N are Abelian groups, S and T subgroups such that M/S and N/T are torsion, and $f: S \longrightarrow T$ is a homomorphism, and if for every prime p, the induced map $f_p^*: S \otimes Z_p^* \longrightarrow T \otimes Z_p^*$ extends to a homomorphism $g(p)$: $M \otimes Z_p^* \longrightarrow N \otimes Z_p^*$, then f extends to a homomorphism $g: M \longrightarrow N$ such that $g_p^* = g(p)$ for all primes p. In this case also, if $g(p)$ is an isomorphism for each p, then g will be also. A similar statement holds for modules M and N over a discrete valuation ring R, and its completion R^*.*

PROOF. [27, Lemma 5.1] contains the proof in one case, in the other cases are exactly the same. Roughly, the point is (in the first case) that we can imbed M into the product of the modules M_p, with torsion-free divisible cokernel. The condition extends the map $S \longrightarrow T$ to a map $\Pi_p M_p \longrightarrow \Pi_p N_p$. We can identify M and N inside these products as those elements x such that for some nonzero integer n, $nx \in S$ (or T, respectively). This means that the map on the product groups restricts to a map from M to N. The other cases are proved in the same way.

1.17. LEMMA. *Let M be an Abelian group and $\phi_p: M \longrightarrow M \otimes Z_p^*$ the natural map. Then for each ordinal α, $p^\alpha M = \phi_p^{-1}(p^\alpha(M \otimes Z_p^*))$. If h is a height, $h = \Pi_q q^{v(q)}$, and if M/hM is torsion, then the natural map $(hM) \otimes Z_p^* \longrightarrow M \otimes Z_p^*$ is injective, and its image is $p^{v(p)}(M \otimes Z_p^*)$.*

PROOF. The first statement is an elementary consequence of the fact that a homomorphism preserves p-height if its kernel is p-divisible and its cokernel has no p-torsion.

For the second statement, we let N be those elements $x \in M$ such that for some integer m prime to p, $mx \in hM$. By hypothesis, M/N is a p-group. If $x \in p^{v(p)}M$, then $np^k x \in hM$ for some nonnegative integer k and some integer n prime to p. Since nx and $np^k x$ have the same q-height for all primes q, $q \neq p$, it follows that $nx \in hM$ and $x \in N$. Since, by construction, $N \subseteq p^{v(p)}M$, it follows that $N = p^{v(p)}M$, and $p^{v(p)}(M/N) = 0$. The exactness of the sequence $0 \longrightarrow N \otimes Z_p^* \longrightarrow M \otimes Z_p^* \longrightarrow M/N \longrightarrow 0$ shows that $p^{v(p)}M \otimes Z_p^* \subseteq N \otimes Z_p^*$ and the reverse inclusion follows from the fact that $N = p^{v(p)}M$, so $p^{v(p)}(M \otimes Z_p^*) = N \otimes Z_p^* = (hM) \otimes Z_p^*$, thus proving the last statement of 1.17.

If G is a torsion Abelian group, we will simply say that G satisfies Hill's condition if for every prime p, the p-primary component of G satisfies this con-

dition (1.7). Put differently, G is the direct sum of a divisible group and totally projective p-groups for various primes p.

1.18. COROLLARY. *Let M and N be Abelian groups, h a height such that M/hM is torsion and satisfies Hill's condition, and f: $hM \longrightarrow hN$ a homomorphism. Then f extends to a homomorphism of M into N.*

PROOF. If $h = \Pi_p p^{v(p)}$, then 1.17 allows us to identify $(hM) \otimes Z_p^* = p^{v(p)}(M \otimes Z_p^*)$. We have therefore an induced homomorphism f_p: $p^{v(p)}(M \otimes Z_p^*) \longrightarrow p^{v(p)}(N \otimes Z_p^*)$ (since $(hN) \otimes Z_p^* \subseteq p^{v(p)}(N \otimes Z_p^*)$). This map necessarily does not decrease heights, and $p^{v(p)}(M \otimes Z_p^*)$ is a nice submodule of $M \otimes Z_p^*$ by 1.2. By the argument of 1.17, $(M \otimes Z_p^*)/p^{v(p)}(M \otimes Z_p^*)$ is isomorphic to the p-primary component of M/hM, and hence satisfies Hill's condition. It follows from 1.13 that f_p extends to a homomorphism $M \otimes Z_p^* \longrightarrow N \otimes Z_p^*$, which, by 1.16, proves the desired result.

We remark that the restrictive hypotheses in 1.17 and 1.18 are essential. In general, it is not necessarily true that $p^\alpha(M \otimes Z_p^*) = (p^\alpha M) \otimes Z_p^*$. Furthermore, in [20] are examples of countably generated groups M and N with finitely generated subgroups S and T and a height preserving isomorphism f: $S \longrightarrow T$ which does not extend to a homomorphism from M to N. Here, $S \otimes Z_p^*$ is clearly a nice submodule of $M \otimes Z_p^*$, and $(M \otimes Z_p^*)/(S \otimes Z_p^*)$ satisfies Hill's condition, but the induced map f_p: $S \otimes Z_p^* \longrightarrow T \otimes Z_p^*$ is not height preserving.

2. **Initial results on balanced projectives.** We recall that a short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of Abelian groups is balanced if for all heights h, the sequence $0 \longrightarrow hA \longrightarrow hB \longrightarrow hC \longrightarrow 0$ is exact. Similarly, if $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is a short exact sequence of modules over a discrete valuation ring R with prime p, then the sequence is balanced (or h-pure, as in [25]) if for every ordinal α, the sequence $0 \longrightarrow p^\alpha A \longrightarrow p^\alpha B \longrightarrow p^\alpha C \longrightarrow 0$ is exact. We note that this last condition implies that the sequence $0 \longrightarrow p^\infty A \longrightarrow p^\infty B \longrightarrow p^\infty C \longrightarrow 0$ is exact (and therefore, split), by taking a large enough ordinal α.

We recall that a subgroup A of a group B is *isotype* [14] if for every height h, $hA = A \cap hB$. Clearly, this is equivalent to the condition that for every prime p and ordinal α, $p^\alpha A = A \cap p^\alpha B$.

2.1. LEMMA. *If $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is an exact sequence of groups (or modules over a discrete valuation ring) and ϕ the corresponding map from B to C, then the sequence is balanced if and only if for every height h, $\phi(hB) = hC$, and for every prime p and ordinal α, $\phi((p^\alpha B)[p]) = (p^\alpha C)[p]$.*

PROOF. The first statement is clearly equivalent to the statement that the map $hB \longrightarrow hC$ is surjective. To show that the conditions imply that the sequence is balanced, therefore, we must show that they imply that A is isotype

in B. We show that for each prime p and ordinal α, $p^\alpha A = A \cap p^\alpha B$. We do this by induction on α, the case when α is a limit ordinal being trivial. We assume, then, that $\alpha = \beta + 1$, and that $x \in A \cap p^\alpha B$. There is a $y \in p^\beta B$ such that $x = py$. Clearly $\phi(y) \in (p^\beta C)[p]$, so by condition (ii), there is a $z \in (p^\beta B)[p]$ such that $\phi(z) = \phi(y)$. In this case, $\phi(y - z) = 0$, so $y - z \in A \cap p^\beta B = p^\beta A$. Since $p(y - z) = x, x \in p^\alpha A$. Hence $A \cap p^\alpha B = p^\alpha A$, as desired.

Conversely, if the sequence is balanced, then the first statement is obvious and we verify the second. If $x \in (p^\alpha C)[p]$ then there is a $y \in p^\alpha B$ such that $\phi(y) = x$. Since $px = 0, py \in A \cap p^{\alpha+1}B = p^{\alpha+1}A$. Hence there is a $z \in p^\alpha A$ such that $py = pz$. Clearly, $y - z \in (p^\alpha B)[p]$, and $\phi(y - z) = \phi(y) = x$, thus proving the desired condition.

2.2. DEFINITION. Let h be a height and π the set of primes for which $v(p) < \infty$. An h-elementary balanced projective is a group G such that $hG \cong Z_\pi$ and G/hG is a reduced π-torsion module satisfying Hill's condition.

EXAMPLE. Let G be a torsion-free group of rank one and $x \in G$. Let π be the set of primes such that $pG \neq G$. If $p \in \pi$ there is a unique positive integer $n(p)$ such that $x \in p^{n(p)}G, x \notin p^{n(p)+1}G$. Let h be the height defined by $v(p) = n(p)$ if $p \in \pi$, $v(p) = \infty$ if $p \notin \pi$. Clearly G may be regarded as a Z_π-module (since if $p \notin \pi$, every element of G is uniquely divisible by p), and $hG = (Z_\pi)x$. $G/hG = \bigoplus_{p\in\pi} Z/p^{n(p)}Z$. Hence G is an h-elementary balanced projective.

2.3. THEOREM. *An Abelian group is balanced projective if and only if it is a summand of a direct sum of h-elementary balanced projectives (for various heights h) and torsion groups satisfying Hill's condition. If G is any Abelian group, there is a balanced short exact sequence*

$$0 \longrightarrow K \longrightarrow P \longrightarrow G \longrightarrow 0$$

in which P is a balanced projective.

REMARK. The reader should note that divisible groups are balanced projective and are included in this theorem. The theorem is also valid for modules over a discrete valuation ring (see 4.2 below for the details). The local theorem (for modules over a discrete valuation ring) was among those announced in [25] and proved in notes circulated by the author in 1971. The global form (for abelian groups) was proved by B. Wick in his thesis [29] in 1972. Wick's thesis also contained Definition 2.2, which replaces a much more cumbersome definition, based on generators and relations, given by the author.

PROOF. We first show that the groups in question are, in fact, balanced projectives. It is clear that any divisible group is a balanced projective, since the balanced condition on a short exact sequence implies that the sequence

$$0 \longrightarrow dA \longrightarrow dB \longrightarrow dC \longrightarrow 0$$

is exact. Since divisible groups are injective, this sequence splits. We next suppose that G is a totally projective p-group. We use 1.15 and prove the result by induction on the length of the group. If the length is a limit ordinal, the group is a direct sum of totally projective groups of smaller length, and the result is immediate. If the group has length $\alpha + 1$, then G is a direct sum of groups G_i such that $p^\alpha G_i$ is cyclic, so we may suppose that $p^\alpha G$ is cyclic.

Suppose that x generates $p^\alpha G$, $px = 0$, $f\colon G \longrightarrow C$ is a homomorphism, and $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is a balanced short exact sequence. By Lemma 2.1, $(p^\alpha C)[p] = \phi((p^\alpha B)[p])$. Since $f(x) \in (p^\alpha C)[p]$, there is a $y \in (p^\alpha B)[p]$, such that $\phi(y) = f(x)$. Let $g\colon G \longrightarrow B$ be a homomorphism such that $g(x) = y$. (Such a homomorphism exists by 1.13.) $f - \phi g$ is now a homomorphism of G to C with $p^\alpha G$ in the kernel, so it may be regarded as a homomorphism of $G/p^\alpha G \longrightarrow C$. Since $G/p^\alpha G$ is totally projective and of length α, by induction $G/p^\alpha G$ is a balanced projective, so there is a homomorphism $g'\colon G \longrightarrow B$ such that $g'(p^\alpha G) = 0$ and $\phi g' = f - \phi g$. We now have $f = \phi(g + g')$, which gives the desired projectivity property for G.

We next suppose that G is an h-elementary balanced projective, $f\colon G \longrightarrow C$ a homomorphism, and $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ a balanced short exact sequence, where $\phi\colon B \longrightarrow C$ is the homomorphism in the sequence. By construction, G is actually a module over the ring Z_π, and hG is a cyclic module over this ring, (where if $h = \Pi_p\, p^{v(p)}$, then π is the set of primes for which $v(p) < \infty$). Let x generate the Z_π-module hG. Since $f(hG) \subseteq hC = \phi(hB)$, there is a $y \in hB$ such that $\phi(y) = f(x)$. The map taking x to y extends to a homomorphism $hG \longrightarrow hB$ since hB is p-divisible for every prime p such that $p \notin \pi$, and this homomorphism extends to a homomorphism $g\colon G \longrightarrow B$ by 1.18. $f - \phi g$ can be regarded as a homomorphism of G/hG to C, and since G/hG is known to be a balanced projective (by the previous argument), there is a $g'\colon G \longrightarrow B$ such that $g'(hG) = 0$ and $\phi g' = f - \phi g$. Hence, as before, $f = \phi(g + g')$, and the projectivity property is proved.

Since a summand of a balanced projective is clearly a balanced projective, the projectivity property of all of the groups mentioned is clear. To show that we obtain all of the balanced projectives in this way, we construct projective resolutions.

We first must note that for any height h, there is an h-elementary balanced projective. For any prime p and ordinal α, there is a group such that $p^\alpha G$ is infinite cyclic and $G/p^\alpha G$ is a totally projective p-group. This can be established either from [16, Theorem 6.3] or from 1.10 and an obvious extension argument. Let $h = \Pi_p\, p^{v(p)}$, and for every prime p such that $v(p) < \infty$, let $G(p)$ be a

group of the type just described, with $p^{v(p)}G(p)$ an infinite cyclic group generated by an element y_p and $G(p)/p^{v(p)}G(p)$ a totally projective p-group. Let $M = Z_\pi \oplus (\bigoplus_{p\in\pi} G(p))$, where π is the set of primes p such that $v(p) < \infty$. Let N be the subgroup of M generated by the elements $1 - y_p$ $(p \in \pi)$ (where we identify Z_π and $G(p)$ as subgroups of M, and 1 is the corresponding element of Z_π). It is clear that M/N is an h-elementary balanced projective.

We now let G be any Abelian group and we construct a balanced short exact sequence $0 \longrightarrow K \longrightarrow P \longrightarrow G \longrightarrow 0$ in which P is a direct sum of a divisible group, a torsion group satisfying Hill's condition, and h-elementary balanced projectives for various heights h. We note that when we have done this, we will not only have shown that enough balanced projectives exist but also that any such projective is a summand of a direct sum of the desired form, since if G is a balanced projective then the sequence we construct will necessarily split.

For every $x \in G$, we let E_x be an $h(x)$-elementary balanced projective, we let y_x be a generator of $h(x)E_x$ as a Z_π-module (where π is the set of primes associated to $h(x)$ as usual), and f_x: $E_x \longrightarrow G$ a homomorphism such that $f_x(y_x) = x$. (Such a homomorphism exists by first noting that $x \in p^\infty G$ for every prime p such that $p \notin \pi$, so the map taking y_x to x extends to a homomorphism of $h(x)E_x$ into $h(x)G$, and the extension to all of E_x follows from 1.18.) If $x \in G[p]$, for some prime p, and $x \in p^\alpha G$, $x \notin p^{\alpha+1}G$, we let H_x be a totally projective p-group (given by 1.10) such that $p^\alpha H$ is cyclic of order p, generated by an element z_x. The correspondence $z_x \longrightarrow x$ extends to a homomorphism g_x: $H_x \longrightarrow G$. We now let $P = d(G) \oplus (\bigoplus_{x\in G} E_x) \oplus (\bigoplus_p \bigoplus_{x\in G[p]} H_x)$ and define a homomorphism F: $P \longrightarrow G$ by letting F be the natural imbedding on $d(G)$, the map f_x on E_x, and the map g_x on H_x. An application of 2.1 shows that the resulting short exact sequence is balanced, and completes the proof of the theorem.

3. **The category $\mathcal{H}$.** In this section we introduce a category associated with the category of Abelian groups which enables us to ignore the torsion of the groups without losing track of such things as the heights of elements of infinite order. In this category, a group whose torsion-free rank is one is actually a small object, so that theorems originally proved for finitely generated modules (for example) can be generalized suitably. We will prove two theorems concerning summands of direct sums of rank one groups in the category $\mathcal{H}$, and conclude that any balanced projective group is isomorphic in the category $\mathcal{H}$ to a direct sum of h-elementary balanced projectives for various heights h. (It will take entirely different methods to translate this into an actual group theoretic isomorphism in the following sections of the paper.)

3.1. Definition. If M and N are Abelian groups, and A is a subgroup of

M, we let $H_A(M, N)$ be the subset of $\mathrm{Hom}(A, N)$ consisting of homomorphisms $f: A \longrightarrow N$ such that $f(A \cap hM) \subseteq hN$ for all heights h. If $B \subseteq A$ there is a natural restriction map $H_A(M, N) \longrightarrow H_B(M, N)$. We define

$$\mathcal{H}(M, N) = \varinjlim H_A(M, N)$$

where the limit is taken over all subgroups A of M such that M/A is torsion. We define $\mathcal{H}$ to be the category whose objects are Abelian groups and whose morphisms are the groups $\mathcal{H}(M, N)$.

3.2. LEMMA. *$\mathcal{H}$ is an additive category with kernels and infinite direct sums. A group is small as an object in $\mathcal{H}$ if and only if its torsion-free rank is finite.*

PROOF. It is clear that $\mathcal{H}$ is an additive category with infinite direct sums. If $f \in \mathcal{H}(M, N)$ then there is a subgroup A of M such that M/A is torsion and a homomorphism $f': A \longrightarrow N$ which represents f. Let K be the kernel of f' and let $K' = \{x \in M$: for some nonzero integer n, $nx \in K\}$. The group K' with its natural imbedding into M is easily seen to be a kernel for f.

It is clear that groups of finite torsion-free rank are small in $\mathcal{H}$, since such a group has a finitely generated subgroup A such that M/A is torsion. Conversely, if the torsion-free rank of M is infinite, then the map $M \longrightarrow M \otimes Q$, together with the fact that $M \otimes Q$ is an infinite direct sum, show that M is not small in $\mathcal{H}$.

3.3. LEMMA. *If M and N are Abelian groups, then M and N are isomorphic in $\mathcal{H}$ if and only if there are torsion-free subgroups A and B of M and N such that M/A and N/B are torsion, and an isomorphism $f: A \longrightarrow B$ such that $f(A \cap hM) = B \cap hN$, for all heights h.*

The proof of this lemma is an easy exercise.

3.4. DEFINITION. If $h = \Pi_p\, p^{v(p)}$ and $k = \Pi_p\, p^{w(p)}$ are heights, then h and k are equivalent if $v(p) = w(p)$ for all but a finite number of primes p, and for the remaining primes, $v(p) \neq \infty$ and either $v(p) = w(p) + n$ for some nonnegative integer n or $w(p) = v(p) + n$ for some nonnegative integer n. The height g.c.d.(h, k) is $\Pi_p\, p^{x(p)}$ where $x(p) = \min(v(p), w(p))$.

We note that if we use multiplicative notation, and think of heights as operators acting on the left of a group, then h and k are equivalent if for some positive integers n and m, $nh = mk$. (Here $nhG = n(hG)$, so if h is as above and q is a prime, then $qh = \Pi_p\, p^{u(p)}$ where $u(p) = v(p)$ for all primes except q, and $u(q) = v(q) + 1$.) We denote the equivalence class of a height h by $[h]$. An equivalence class of heights is called a *type*, and if M is an h-elementary balanced projective, then $[h]$ is the *type* of M. The point of this is that in this

situation, h is not an invariant of M but $[h]$ is. In the case where M is torsion free of rank one, this notion coincides with the traditional notion of the type of M as in [2], and [6, Chapter 13].

3.5. LEMMA. *If M is an Abelian group and h a height, then M is isomorphic in $\mathcal{H}$ to an h-elementary balanced projective if and only if M has torsion free rank one and there is an element $x \in M$ such that $h(x) = h$ and, for all positive integers n, $h(nx) = nh(x)$. If A is an h-elementary balanced projective and B is a k-elementary balanced projective, then A and B are isomorphic in $\mathcal{H}$ if and only if $[h] = [k]$. If M is an Abelian group, M is isomorphic in $\mathcal{H}$ to a direct sum of h-elementary balanced projectives, for various heights h, if and only if M has a subset X such that X is a basis (i.e., X is a set of free generators for the free Abelian group $[X]$, and $M/[X]$ is torsion) and if x_i $(1 \leqslant i \leqslant k)$ are distinct elements of X, then $h(n_1x_1 + \cdots + n_kx_k) = \text{g.c.d.}_{i=1}^{k} n_ih(x_i)$, where the heights are computed in M.*

REMARK. When dealing the p-height, for a single prime, we recall that $h_p(x)$ is actually the exponent (i.e., an ordinal or ∞), so the group theoretic condition above could be expressed as follows: if $x \in X$ then $h_p(p^kx) = h_p(x) + k$ and, for every prime p, $h_p(\Sigma_{i=1}^{k} n_ix_i) = \min_{i=1}^{k} h_p(n_ix_i)$.

PROOF. The first and third statements follow from 3.3. For the second statement, if M and N were h- and k-elementary balanced projectives with $[h] \neq [k]$, and M and N were isomorphic in $\mathcal{H}$, there would be nonzero elements x and y in hM and kN such that $h(x) = h(y)$. It is clear that no such elements exist if $[h] \neq [k]$.

3.6. LEMMA. *Let h be a height, and M a group which is a direct sum of h-elementary balanced projectives, and N a group which is $\mathcal{H}$-isomorphic to a summand of a group which is $\mathcal{H}$-isomorphic to M. Then N is isomorphic in $\mathcal{H}$ to a direct sum of h-elementary balanced projectives.*

REMARK. This theorem is proved by a category equivalence argument similar to that used in [24] to give a new proof of the special case in which all groups involved are torsion free (a result proved in a different way by Baer in 1937 [2]). The category equivalence argument has been used in similar ways in other recent papers in Abelian group theory, as in [1].

PROOF. If A is an h-elementary balanced projective, then $\mathcal{H}(A, A) = Z_\pi$, as a ring. (Any endomorphism in $\mathcal{H}$ of A is determined by a representative homomorphism defined on hA, since A/hA is torsion. If B is a subgroup of hA and $f\colon B \longrightarrow A$ a homomorphism, then $f \in H_B(A, A)$ if and only if f is given by multiplication by some element of Z_π.) We now look at the functors $X \longrightarrow \mathcal{H}(A, X)$ (from the category $\mathcal{H}$ to the category of Z_π-modules) and $Y \longrightarrow Y \otimes A$ (from

the category of Z_π-modules to the category H). These functors induce a category equivalence between the full subcategory of H consisting of objects H-isomorphic to summands of direct sums of copies of A (regarded as an object in H), and the category of projective Z_π-modules. To show this we must show that the natural transformations $X \longrightarrow H(A, X) \otimes A$ and $Y \longrightarrow H(A, Y \otimes A)$ are isomorphisms (in H and the category of Z_π-modules respectively) whenever X is H-isomorphic to a summand of a direct sum of copies of A and Y is a projective Z_π-module. These maps are isomorphisms when $X = A$ and $Y = Z_\pi$. It follows that they are also isomorphisms when X is a direct sum of copies of A and Y is a direct sum of copies of Z_π (using the smallness of A in H), and the isomorphism property is also clearly inherited by direct summands. The stated result now follows from the fact that any projective Z_π-module is free, since Z_π is a principal ideal domain.

If h and k are heights, $h = \Pi_p\, p^{v(p)}$ and $k = \Pi_p\, p^{w(p)}$, then we say $h \leqslant k$ if $v(p) \leqslant w(p)$ for all primes p. We note that this notation gives us a new description for the set π of primes associated to a height h: they are exactly those primes for which it is not true that $h \geqslant p^\infty$. We extend this to a partial ordering on the types by defining $[h] \leqslant [k]$ to mean that for some $h' \in [h]$ and $k' \in [k]$, $h' \leqslant k'$. It is easy (but necessary) to verify that this does, in fact, define a partial ordering on the types.

This partial ordering is the key fact which is used to get information about the structure in the category H of summands of direct sums of h-elementary balanced projectives for various heights h. We will prove the relevant theorems in the context of an arbitrary additive category satisfying certain minimal additional conditions which seem to be essential.

Throughout the following we work in an additive category with kernels and infinite direct sums, and satisfying the following "weak Grothendieck condition" [22]: For every index set I and every nonzero subobject $A \subseteq \bigoplus_{i \in I} B_i$, there is a finite subset $J \subseteq I$ such that

$$A \cap \bigoplus_{i \in J} B_i \neq 0.$$

(We note that if A is a subobject of B and K is a subobject of B which is the kernel of a homomorphism f, and ϕ is the homomorphism imbedding A in B, then $A \cap K$ is the kernel of the map $f\phi$. Hence $A \cap K$ exists.)

An object M of the category has a *countable small approximation* if there is a countable family of homomorphisms f_i: $X_i \longrightarrow M$ such that each X_i is small, with the property that a monic f: $L \longrightarrow M$ is an isomorphism if and only if each f_i can be factored through f, [22].

3.7. LEMMA. *If S_i $(i \geqslant 1)$ are subobjects of an object M with a countable*

small approximation, and (i) *the induced map* $S_1 \oplus \cdots \oplus S_n \longrightarrow M$ *is monic for all* n, *and* (ii) *for each of the small objects* X_i *and maps* f_i *in the small approximation, there is an integer* n *such that* f_i *factors through* $S_1 \oplus \cdots \oplus S_n$, *then the induced map* $\bigoplus_{i=1}^{\infty} S_i \longrightarrow M$ *is an isomorphism.*

PROOF. The map is monic, since otherwise it would have a nonzero kernel, which is impossible by the weak Grothendieck condition and the fact that the map $S_1 \oplus \cdots \oplus S_n \longrightarrow M$ is monic. The countable approximation property implies that the map is an isomorphism.

3.8. LEMMA. [22]. *A small object has a countable small approximation. A countable direct sum of objects with countable small approximations has a countable small approximation. A summand of an object with a countable small approximation has a countable small approximation.*

3.9. LEMMA. *If* $M = A \oplus B = A' \oplus B$, *then* $A \cong A'$. *If* $M = A \oplus B$ *and* F *is a fully invariant subobject of* M, *then* $F = (F \cap A) \oplus (F \cap B)$. *If* $M = A \oplus B$ *and* X *is a subobject of* M, $X \supseteq A$, *then* $X = A \oplus (X \cap B)$. *If* $M = A \oplus B$ *and* X *is a summand of* M *such that* $X = (X \cap A) \oplus (X \cap B)$, *then* X *has a complementary summand which splits along* A *and* B–*i.e., if* $M = X \oplus X'$ *then* $M = X \oplus A' \oplus B'$ *where* $A' = A \cap (X' \oplus (X \cap B))$ *and* $B' = B \cap (X' \oplus (X \cap A))$.

All of Lemma 3.9 is true in any additive category with kernels.

3.10. DEFINITION. Let $\mathcal{C}$ be a set of objects and $\tau\colon \mathcal{C} \longrightarrow P$ a function from $\mathcal{C}$ to a partially ordered set P. We say that τ makes $\mathcal{C}$ into a semirigid system if for all $A \in \mathcal{C}, B \in \mathcal{C}$, $\mathrm{Hom}(A, B) = 0$ unless $\tau(A) \leqslant \tau(B)$. If M is any object, and $\alpha \in P$, we will say that M is homogeneous of type α if M isomorphic to a summand of a direct sum of objects A in $\mathcal{C}$ such that $\tau(A) = \alpha$.

3.11. THEOREM. *Let* $\mathcal{C}$ *be a set of objects and* $\tau\colon \mathcal{C} \longrightarrow P$ *a function such that* P *is a partially ordered set and* τ *makes* $\mathcal{C}$ *into a semirigid system. Suppose that* M *is an object with a countable small approximation and that* $M = \bigoplus_{\alpha \in P} M_\alpha$, *where* M_α *is homogeneous of type* α. *Then any summand of* M *also has such a decomposition into homogeneous summands.*

REMARK. This theorem and proof is a direct descendent of the theorem of Kulikov [13] that a summand of a countable direct sum of torsion-free groups of rank one is again a direct sum of torsion-free groups of rank one, which was the major step in solving the famous problem concerning such summands posed by Baer in [2]. The countability restriction was finally removed by Kaplansky [9]. Kulikov's argument was improved by Fuchs in [5], and then given two very smooth renditions by B. Charles, [3], and G. Kolettis, [12]. We have based

our categorical version primarily on Kolettis's version of the argument, but we also borrow some of Charles's point of view. We refer to [12] for the application to torsion-free modules over a Dedekind domain, in which C is the class of modules of rank one, the category is the category of torsion-free modules, and each type represents (in general) several distinct isomorphism types. In [15] is a quite different application to Abelian group theory.

PROOF. We first note that if the original object has a finite decomposition into n homogeneous summands, then the result is immediate. We write $M = G \oplus H$ where H is homogeneous of a single type, which is maximal among the types of the summands in the finite decomposition of M, and G has a finite decomposition into $n-1$ homogeneous summands (of types distinct from, and not greater than, the type of H). H is a fully invariant subobject, so if $M = A \oplus B$, then $H = (A \cap H) + (B \cap H)$, (3.9), so $A = (A \cap H) \oplus [A \cap (G \oplus (B \cap H))]$. If we write this decomposition as $A = (A \cap H) \oplus A'$, then A' is isomorphic to a summand of G (3.9), and so, by induction, the result is proved.

We now consider the general case, where M has a countable decomposition into homogeneous summands, and $M = A \oplus B$. We will show in the next paragraph that if X is a small object, $f\colon X \longrightarrow A$ a homomorphism, and $\phi_A\colon A \longrightarrow M$ the natural inclusion, then we can find a decomposition $M = U \oplus T$ such that $\pi_T \phi_A f = 0$, U has a decomposition into a finite number of homogeneous summands, and both U and T split along A and B. Once we have done this, the theorem is proved by the following argument. We choose an approximating family $f_i\colon X_i \longrightarrow A$ (as we may, by 3.8, since M has a countable small approximation). By induction, we suppose that we have a decomposition $A = S_1 \oplus \cdots \oplus S_n \oplus A_n$, where each S_i has a finite decomposition into homogeneous summands and for each i, $1 \leqslant i \leqslant n$, f_i factors through the summand $S_1 \oplus \cdots \oplus S_i$. Let $\phi_n\colon A_n \longrightarrow M$ be the natural imbedding, η the projection of A onto A_n, and $g = \phi_n \eta f_{n+1}$. By the previous (as yet unproved) statement, applied to the decomposition $M = A_n \oplus [S_1 \oplus \cdots \oplus S_n \oplus B]$, we can write $M = U_{n+1} \oplus M_{n+1}$ where g factors through U_{n+1}, U_{n+1} has a finite decomposition into homogeneous summands, and both U_{n+1} and M_{n+1} split along A_n and $[S_1 \oplus \cdots \oplus S_n \oplus B]$. Let $S_{n+1} = U_{n+1} \cap A_n$. Since U_{n+1} splits along A_n, S_{n+1} is a summand of U_{n+1}, and hence, by the first part of the proof of this theorem, S_{n+1} has a finite decomposition into homogeneous summands. This gives the inductive construction of S_{n+1}. We now look at the induced homomorphism $\bigoplus_{i=1}^{\infty} S_i \longrightarrow A$. This is an isomorphism by Lemma 3.7, and each S_i has a decomposition into homogeneous summands by construction, so the theorem is proved.

We still must show that if M has a countable homogeneous decomposition and $M = A \oplus B$, ϕ_A is the imbedding of A into M, X is a small object and f:

$X \longrightarrow A$ a homomorphism, then $M = U \oplus T$, where U has a finite decomposition into homogeneous summands, both U and T split along A and B, and $\pi_T \phi_A f = 0$. We first notice that it will be enough to show that $M = U \oplus T$ where $\pi_T \phi_A f = 0$, T splits along A and B, and U has a finite decomposition into homogeneous summands, since, in that case, we can replace U by

$$U' = [A \cap (U \oplus (T \cap B))] \oplus [B \cap (U \oplus (T \cap A))]$$

as in 3.9.

Let $M = \bigoplus_{\alpha \in P} M_\alpha$ be our original decomposition into homogeneous summands. For any type α, we let $M(\alpha) = \bigoplus_{\beta \geqslant \alpha} M_\beta$ and $M(\alpha)^* = \bigoplus_{\beta > \alpha} M_\beta$. Let $M = U \oplus T_1$, where U is a direct sum of a finite number of the original summands M_α, and $\phi_A f$ factors through U. We will show that any such summand U has a complementary summand which splits along A and B, by induction on the number of homogeneous summands of U, thus completing the proof of the theorem. U is a finite direct sum of summands from the original decomposition of M, and we let μ be a type which is maximal among those appearing in the decomposition of U. $U = M_\mu \oplus G$, where G is the direct sum of a smaller number of the original summands M_α. By induction, G has a complement, F, which splits along A and B. By the maximality property of μ, $M(\mu) \subseteq F$, and we also have $M(\mu) = M_\mu \oplus M(\mu)^*$. Since $F = (F \cap A) \oplus (F \cap B)$ and $M(\mu)$ is a fully invariant subobject of F, $M(\mu)$ splits along $F \cap A$ and $F \cap B$, and, by 3.9, since $M(\mu)$ is a summand of F, it has a complement E (in F) which also splits along $F \cap A$ and $F \cap B$. We thus have $M = G \oplus M_\mu \oplus E \oplus M(\mu)^*$. Since $U = G \oplus M_\mu$, if we let $T = E \oplus M(\mu)^*$, then T is the desired complement for U which splits along A and B. We have now completed the essential step which was used (in the second paragraph of the proof) to prove the theorem.

3.12. COROLLARY. *Let M be an Abelian group which is a direct sum of h-elementary balanced projectives, for various heights h, and let N be a summand of M. Then in the category $\mathcal{H}$, N is isomorphic to a direct sum of h-elementary balanced projectives, for various heights h. In group-theoretic terms, N has a subset X such that X is a basis (i.e., the set X is a set of free generators for the subgroup $[X]$, which is free Abelian, and $N/[X]$ is torsion), and for all sets $\{x_i\}$ $(1 \leqslant i \leqslant k)$ of distinct elements of X and n_i $(1 \leqslant i \leqslant k)$ of integers, $h(n_1 x_1 + \cdots + n_k x_k) = \text{g.c.d.}_{i=1}^{k} n_i h(x_i)$, where the heights are computed in N.*

PROOF. By [22, Theorem 3] N is $\mathcal{H}$-isomorphic to a direct sum of modules, each of which is $\mathcal{H}$-isomorphic to a summand of a countable direct sum of h-elementary balanced projectives, for various heights h. (This is the additive category version of the Kaplansky lemma [9], and it works in any additive category with kernels, infinite direct sums, and a weak Grothendieck condition, all

of which are true for H.) It follows that we may assume that M is a countable direct sum of h-elementary balanced projectives for various heights h, from which it follows that M and N have countable small approximations (by 3.8).

We have previously defined the necessary partial ordering on types, and it is clear that the conditions of definition 3.10 are satisfied. We conclude that N is H-isomorphic to a direct sum of groups, each of which is H-isomorphic to a summand of a direct sum of h-elementary balanced projectives for some *particular* height h. An application of 3.6 completes the proof.

3.13. DEFINITION. A basis X of a group N satisfying the conditions of the previous theorem is called a K-basis for the group. Similarly, if M is a module over a discrete valuation ring R with prime p, then a subset X of M is a K-basis if it is a basis ($M/[X]$ is torsion and the elements of X are independent), if $x \in X$ then $h_p(p^n x) = h_p(x) + n$, and if x_i $(1 \leqslant i \leqslant n)$ are distinct elements of X and r_i $(1 \leqslant i \leqslant n)$ elements of R, then $h_p(\Sigma_{i=1}^n r_i x_i) = \min_{i=1}^n h_p(r_i x_i)$.

4. **The local classification theorem.** In this section we develop the local theory of balanced projectives. We will therefore consider modules over an arbitrary discrete valuation ring R, with a single prime p. We denote the completion of R by R^*. We recall that a short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of R-modules is *balanced* if for every ordinal α, the sequence

$$0 \longrightarrow p^\alpha A \longrightarrow p^\alpha B \longrightarrow p^\alpha C \longrightarrow 0$$

is exact.

4.1. DEFINITION. If λ is a limit ordinal, we say a module is a λ-elementary KT-module if $p^\lambda M \cong R$ and $M/p^\lambda M$ is torsion and totally projective.

Here and throughout this paper, it is convenient to consider 0 as a limit ordinal.

If M is a module, we let

$$h(0, M) = \dim(M/pM + T)$$

where T is the torsion submodule of M. Note that if M is a direct sum of cyclic modules, then $h(0, M)$ is the number of infinite cyclic summands in a decomposition of M. More generally, $h(0, M)$ is the number of infinite cyclic summands in a basic submodule of M [8, rev. ed., Lemma 21]. For any limit ordinal λ, we define $h(\lambda, M) = h(0, p^\lambda M)$. If M is a module which is a direct sum of a torsion group and α-elementary KT-modules for various ordinals α, then it is clear that $h(\lambda, M)$ is the number of λ-elementary summands in a decomposition of M. We define $h(\infty, M) = \dim p^\infty M \otimes Q$ where Q is the quotient field of R. (It is well known that the invariants $f(\infty, M)$ and $h(\infty, M)$ classify the divisible R-modules.)

We note that a λ-elementary KT-module could also be described as a p^λ-elementary balanced projective. The KT-module terminology is retained to agree with that in [25] and [28].

4.2. LEMMA. *If M is a direct sum of a divisible module, a torsion totally projective module and λ-elementary KT-modules, for various limit ordinals λ, then M is a balanced projective. If λ is any limit ordinal, there is a λ-elementary KT-module. A module is a balanced projective if and only if it is a summand of a module which is a direct sum of a divisible module, a torsion totally projective module, and λ-elementary KT-modules for various limit ordinals λ.*

PROOF. The proof of the first assertion is the same as that of 2.3. The existence statement follows from the arguments of 2.3 above, or [28, 4.4 and 4.1]. As in the proof of 2.3, one constructs for any module M a balanced sequence $0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$ where P is a balanced projective of the specified form. The proof is exactly the same as that of 2.3 except that if $x \in M$ and $h(x) = \alpha$, then we write $\alpha = \lambda + n$, where n is an integer and λ is a limit ordinal. We then let E_x be a λ-elementary KT-module and y_x a generator of $p^\alpha E_x$ and proceed as before.

4.3. THEOREM. *If M is a balanced projective module, then there is a module N which is the direct sum of a divisible module, a torsion totally projective module, and λ-elementary KT-modules for various ordinals λ, such that for all ordinals α, $f(\alpha, M) = f(\alpha, N)$; $f(\infty, M) = f(\infty, N)$; for all limit ordinals λ, $h(\lambda, M) = h(\lambda, N)$; and $h(\infty, M) = h(\infty, N)$.*

PROOF. We write $M = D \oplus M'$, where D is divisible and M' is reduced. In the terminology of [28], M' is a summand of a KT-module. By the results of §4 of [28], this means that M' has the same invariants as some KT-module, so that there is a reduced module N' which is a direct sum of a torsion totally projective module and λ-elementary KT-modules for various limit ordinals λ, such that for all ordinals α and limit ordinals λ, $f(\alpha, M') = f(\alpha, N')$ and $h(\lambda, M') = h(\lambda, N')$. If we let $N = D \oplus N'$, the result is established.

4.4. THEOREM. *Let R be a complete discrete valuation ring, and M and N reduced R-modules with K-bases, X and Y, and suppose that M and N have the same Ulm invariant and both satisfy Hill's condition. Then if ϕ: $X \longrightarrow Y$ is a height preserving bijection, ϕ extends to an isomorphism of M onto N.*

REMARK. We refer to 1.7 and 3.13 for the definitions of Hill's condition and K-basis. The entire result would be a consequence of 1.11 if we knew that X and Y generated nice submodules and that $M/[X]$ and $N/[Y]$ were totally projective. It is not hard (see [29] for example) to verify that $[X]$ and $[Y]$

are actually nice, but the total projectivity of the quotients is less obvious, so we approach the problem differently.

PROOF. We remind the reader that if S is a nice submodule of M and S' a submodule containing S such that S'/S is finitely generated, then S' is also nice by 1.6. This is used constantly in this proof, and is the only way in which the completeness of R is used.

We let Q be the quotient field of R. Given any element in M, the *associated basis elements* in any given basis are those basis elements needed in an expansion of the image of the element in $M \otimes Q$.

We now let $\mathcal{C}$ and $\mathcal{C}'$ be the families of nice submodules of M and N respecitively, satisfying the conditions of 1.7. We choose in advance isomorphisms $g_\alpha: U_\alpha(M) \longrightarrow U_\alpha(N)$. We now look at all triples (S, T, f) satisfying the following conditions:

(i) $S \in \mathcal{C}$, $T \in \mathcal{C}'$, and $f: S \longrightarrow T$ is a height preserving isomorphism (i.e., $f(S \cap p^\alpha M) = T \cap p^\alpha N$ for all ordinals α).

(ii) For every ordinal α, the induced homomorphism

$$f_\alpha: I_\alpha(S) \longrightarrow I_\alpha(T)$$

is the restriction of g_α to $I_\alpha(S)$.

(iii) If $x \in S$, then any basis element in X associated to x is in S; and if $y \in T$, then any basis element in Y associated to y is in T.

(iv) If $x \in S \cap X$ then $f(x) = \phi(x)$.

We now choose f maximal in this class and will show that the corresponding S and T are, in fact, M and N. It suffices to show that if we have such a triple (S, T, f) and $S \neq M$, then f can be extended to another such map defined on a submodule larger than S. (We use here the symmetry between M and N.)

We assume then, that we have a triple (S, T, f) satisfying our four conditions, and that $x \in M, x \notin S$. We find a new triple (S', T', f') such that $S \subseteq S', x \in S'$, and the restriction of f' to S is f. We construct S' and T' inductively, by what is sometimes called a back-and-forth argument.

Suppose first that we have submodules A and B of M and N respectively, containing S and T, and such that A/S and B/T are finitely generated. Suppose that we have extended f to a height preserving isomorphism from A to B, and that A and B (in place of S and T) satisfy (ii), (iii), and (iv) of our previous conditions. We suppose that $z \in M, z \notin A$, and that either (a) $z \in X$ or (b) $pz \in A$. In case (b), by (iii), all basis elements in X associated to z are already in A. We show that in each of these cases, f can be further extended to the submodule A' generated by A and z, and that the extended map f_0 will be a height preserving isomorphism of A' onto a submodule B' of N, and that the triple (A', B', f_0) will be a triple satisfying conditions (ii), (iii) and (iv).

In case (a) we have no choice but to let $A' = \{A, z\}$ and $B' = \{B, \phi(z)\}$ and to define f_0 to be equal to f on A and $f_0(z) = \phi(z)$. The fact that X is an independent set and condition (iii) of the hypothesis on f clearly imply that $A' = A \oplus [z]$ and $B' = B \oplus [\phi(z)]$, so f_0 is well defined. To show that f_0 is height preserving, we show that if $a \in A$ and $r = R$, then $h_p(a + rz) = \min\{h_p(a), h_p(rz)\}$, and, similarly, $h_p(f(a) + r\phi(z)) = \min\{h_p(f(a)), h_p(r\phi(z))\}$. This will clearly prove the result. If this equation should fail, so that $h_p(a + rz) > \min\{h_p(a), h_p(rz)\}$, and if $h_p(z) = \alpha$, and $r = p^k u$, where u is a unit, then, necessarily, $h_p(a) = \alpha + k$. For some integer n, $p^n a \in [X]$. By condition (iii) on A, $h_p(p^n a + p^n rz) = \min\{h_p(p^n a), h_p(p^n rz)\} = \alpha + k + n$. However, $h_p(a + rz) > \alpha + k$, so $h_p(p^n(a + rz)) > \alpha + k + n$, a contradiction. This shows that f_0 is, indeed, height preserving. We note that $I_\beta(A) = I_\beta(A')$ for all β, so no problems arise from (ii).

In case (b) we may assume that z has maximal height in $z + A$ (the coset $z + A$ contains such an element since A is a nice submodule), and, subject to this restriction, we assume that $h_p(pz) > h_p(z) + 1$ if any element of maximal height in the coset has this property. If $h_p(pz) = h_p(z) + 1$ then find an element $w \in N$ such that $h_p(w) = h_p(z)$ and $f(pz) = pw$. If we extend f by setting $f_0(z) = w$, then we obtain the desired extension. (See [23, Lemma 2.6, case (b)] for details.) In this case, $I_\alpha(A) = I_\alpha(A')$, for all ordinals α, so conditions (ii), (iii), and (iv) are no problem. If $h_p(pz) > h_p(z) + 1$ then $pz = py$ for some y of height greater than $h_p(z)$. If $\alpha = h_p(z)$, then $g_\alpha(z - y) = w + (p^{\alpha+1}N)[p]$, where $w \notin p^{\alpha+1}N$. Let $f(pz) = pw'$, where $h_p(w') > \alpha$, and define the extension of f by letting $f_0(z) = w + w'$. (See [23, Lemma 2.6, case (a)] for details to show that this works.) In this case $I_\beta(A) = I_\beta(A')$ for all ordinals β, $\beta \neq \alpha$, and $I_\alpha(A) \neq I_\alpha(A')$. The use of g_α in the construction guarantees, however, that condition (ii) is still satisfied.

The only problem remaining is to set up a procedure whereby we extend f in stages, each stage being one of the type described in (a) or (b) above, and so that at the end of a countable number of steps, condition (i) is satisfied. (The other conditions will follow immediately if we have succeeded in proceeding in a manner which allows us to use cases (a) or (b) at each step.)

We begin with a specific element z not in S. Choose $S_1 \in C$ such that $S_1 \supseteq S$ and S_1/S is countably generated, and such that $z \in S_1$, and S_1 satisfies condition (iii). Choose a countable set of elements of S_1, including all elements of X in S_1 which are not in S, such that the set chosen together with S generates S_1. Assign to each of these elements a power of 2 (so that the generators are elements $x(n)$ where the integers n are all powers of 2, the first one being $x(1) = x(2^0)$), in such a way that no two elements correspond to the same integer, and for any element $x(n)$, any new associated basis elements in X for

$x(n)$ are in the set $\{x(k): k < n\}$. In addition to this, we require that if $x(n)$ is not in X then $p(x(n))$ is in the submodule generated by S and the elements $\{x(k): k < n\}$. All of this guarantees that any element on this list can be adjoined in turn according to the procedures (a) and (b).

We now extend our map to the submodule generated by S and the first element on our list, $x(1)$: We let A_1 be this submodule and map it to a submodule B_1 containing T. We now similarly find a nice submodule $T_1 \in \mathcal{C}'$ such that $T_1 \supseteq B_1$, T_1/T is countably generated, and T_1 satisfies condition (iii). We choose, as before, a set of generators for T_1 calling these generators $y(n)$, where n is restricted to be a power of 2, and the set of generators so chosen satisfies the same restrictions as those imposed upon the elements $x(n)$. We let B_1' be the the submodule generated by B_1 and $y(1)$, and find a height preserving isomorphism of B_1 onto a submodule A_1' of M, such that $A_1' \supseteq A_1$ and the new map is an extension of the previous isomorphism between B_1 and A_1.

We now consider the general inductive step. We assume that m is a positive integer, and we have nice submodules S_m and T_m in $\mathcal{C}$ and $\mathcal{C}'$, containing S and T, such that S_m/S and T_m/T are countably generated and S_m and T_m satisfy condition (iii). We assume that we have for S_m and T_m countable sets of elements $x(n)$ and $y(n)$, indexed by integers which are all powers of the first m primes, and such that if $x(n) \notin X$, then $px(n)$ is in the submodule generated by S and the elements $x(k)$, $k < n$, and that the elements $x(k)$, $k < n$ include all of the elements of X which are associated basis elements of $x(n)$ and not in S. We assume further that S_m is generated by S and the elements $x(n)$. We assume that the corresponding conditions hold for the elements $y(n)$ and T_m. We assume that A_m and B_m are submodules of M and N respectively, containing S and T, such that A_m contains the elements $x(i)$, $i \leqslant m$, and B_m contains the elements $y(i)$, $i < m$. We assume that A_m/S and B_m/T are finitely generated, and that f has been extended to a height preserving isomorphism from A_m to B_m. We let B_m' be the submodule of N generated by B_m and $y(m)$. (If $y(m)$ does not exist, which will happen, for example, if m is not a power of a prime, then $B_m' = B_m$. We can, of course, have $B_m' = B_m$ for other reasons.) We extend our previous map to a height preserving isomorphism $B_m' \longrightarrow A_m'$ (thus defining A_m'). Bearing in mind that A_m' may not be contained in S_m, we find an $S_{m+1} \in \mathcal{C}$, $S_{m+1} \geqslant S_m$, such that S_{m+1}/S_m is countably generated, S_{m+1} satisfies (iii) and $A_m' \leqslant S_{m+1}$. We find a countable set of elements which together with S_m generate S_{m+1}, and such that this set includes all elements of X in S_{m+1} which are not in S_m. We regard the prime numbers as listed $p_1, p_2, \ldots, p_m, p_{m+1}$ where $p_1 = 2$, etc., (none of these to be confused with the prime p of R). We index the new set of generators as elements $x(n)$, where n is restricted to be a positive power of p_{m+1}, and this indexing satisfies the previous

conditions (the same as those imposed on the previously chosen elements $x(n)$). We let A_{m+1} be generated by A'_m and $x(m+1)$, if $x(m+1)$ exists. We extend our isomorphism between A'_m and B'_m to an isomorphism from A_{m+1} to a submodule B_{m+1} (defined by this process). We extend T_m to a (possibly) larger submodule T_{m+1} satisfying the usual conditions and containing B_{m+1}. This completes the inductive construction.

Finally, we let $S' = \bigcup_{m=1}^{\infty} S_m$, $T' = \bigcup_{m=1}^{\infty} T_m$, and we note that by construction, $S' = \bigcup_{m=1}^{\infty} A_m$ also, so we have an induced isomorphism between S' and T' which clearly satisfies all of the requisite conditions. In particular, $S' \in C$ because of condition (i) of 1.7. This completes the proof that if (S, T, f) is one of our specified triples, and $S \neq M$, then (S, T, f) is not maximal, and thus completes our proof of Theorem 4.4.

4.5. THEOREM. *Let R be a discrete valuation ring. An R-module M is a balanced projective if and only if it is the direct sum of a divisible module, a torsion, totally projective module, and λ-elementary KT-modules for various limit ordinals λ. If M and N are balanced projectives, then $M \cong N$ if and only if $f(\infty, M) = f(\infty, N)$, $h(\infty, M) = h(\infty, N)$, and for all ordinals α and limit ordinals λ, $f(\alpha, M) = f(\alpha, N)$ and $h(\lambda, M) = h(\lambda, N)$.*

PROOF. Divisible modules cause no difficulty, so, we may assume that M and N are reduced. M is a summand of a module P which is a direct sum of a totally projective module and λ-elementary KT-modules for various limit ordinals λ, by 4.2. We now refer to the decomposition theory in the category developed in §3. (This theory is equally valid over any principal ideal domain.) 3.11 and 3.12 imply that M is H-decomposable into λ-elementary KT-modules, that is, M has a K-basis.

We let X be this K-basis. For every $x \in X$, we let $h_p(x) = \lambda + n$ where λ is a limit ordinal and n is a nonnegative integer. Let $g(x)$ be an element of $p^\lambda M$ such that $p^n g(x) = x$. The elements $g(x)$ so obtained form a set X', which is easily shown to be another K-basis with the additional property that the height of every element is a limit ordinal. For a given limit ordinal λ, if X'_λ is the set of elements of X' of height λ, then one easily verifies that $p^\lambda M = [X'_\lambda] \oplus M^0_\lambda$, where M^0_λ is the set of elements of $p^\lambda M$ which are torsion modulo $p^{\lambda+\omega} M$. From this one sees easily that $h(\lambda, M)$ is the cardinality of X'_λ. By 4.3 there is a module N which is a direct sum of a totally projective torsion module and λ-elementary KT-modules for various limit ordinals λ, such that $f(\alpha, M) = f(\alpha, N)$ and $h(\lambda, M) = h(\lambda, N)$ for all ordinals α and limit ordinals λ. We let Y be a K-basis for N, with exactly $h(\lambda, N)$ elements of height λ for each limit ordinal λ. (This is obtained by choosing a suitable element in each of the summands of

N.) Since $h(\lambda, M) = h(\lambda, N)$ for all limit ordinals λ, there is a bijective map ϕ: $X' \longrightarrow Y$ which is height preserving.

If we can show that ϕ extends to an isomorphism of M onto N, then the entire theorem will be proved, in a strong form. By 1.16 it suffices to show that ϕ extends to an isomorphism of $M \otimes R^*$ onto $N \otimes R^*$, where R^* is the completion of the valuation ring R. This will be an application of Theorem 4.4, as soon as we show that $M \otimes R^*$ and $N \otimes R^*$ satisfy Hill's condition and that X and Y form K-bases for these R^*-modules.

Hill's condition is easy to verify. It is easy to see that if A is a λ-elementary KT-module over R, then $A \otimes R^*$ is a λ-elementary KT-module over R^*, and also (by 1.3 and 1.5) that a λ-elementary KT-module over R^* satisfies Hill's condition. By 4.2 and the above remark, if M is a reduced balanced projective, then $M \otimes R^*$ is a summand of a direct sum of elementary KT-modules over R^*, whence, by 1.9, we infer that $M \otimes R^*$ satisfies Hill's condition.

We now let x_i $(1 \leqslant i \leqslant n)$ be elements of X and r_i $(1 \leqslant i \leqslant n)$ be elements of R^*, and show that

$$(*) \qquad h_p\left(\sum_{i=1}^{n} r_i x_i\right) = \min_{i=1}^{n} h_p(r_i x_i).$$

We write $r_i = s_i(1 + pu_i)$, where $s_i \in R$, $u_i \in R^*$. We then have

$$\sum_{i=1}^{n} r_i x_i = \sum_{i=1}^{n} s_i x_i + p\left(\sum_{i=1}^{n} u_i s_i x_i\right).$$

Since the imbedding $M \longrightarrow M \otimes R^*$ is height preserving, the height of the first term on the right is exactly $\min_{i=1}^{n} h_p(s_i x_i)$, since X is a K-basis in M. Since $h(r_i x_i) = h(s_i x_i)$, we will be done if we can show that the height of the second term on the right is greater than the height of the first term. Since $h_p(s_i x_i) \leqslant h_p(u_i s_i x_i)$, it is clear, in fact, that the height of the second term on the right is at least one greater than that of the first term. This establishes equation (*), thus completing the proof that X is a K-basis for $M \otimes R^*$. From our previous remarks, it is clear that this completes the proof of Theorem 4.5.

For convenience later, we put together parts of the previous two results to obtain a result useful in localization arguments.

4.6. COROLLARY. *Let M and N be balanced projective modules over a discrete valuation ring R which have the same Ulm invariants. Then if ϕ: $X \longrightarrow Y$ is a height preserving bijection between K-bases of M and N respectively, ϕ extends to an isomorphism of M onto N.*

We should remark that the main result of this section, 4.5, can be proved in other ways involving somewhat less potent methods than those involved in 4.4. However, 4.4 is essential for 4.6, and 4.6 is exactly what is required to ex-

tend these local results to global results, which we do in the next section.

5. The global classification theorems.

5.1. DEFINITION. If h is a height and $[h]$ the corresponding type, we let $[h]M = \Sigma_{k \in [h]} kM$. In other terms, $[h]M = \{x \in M\colon [h(x)] \geqslant [h]\}$. We define $[h]^*M = \Sigma_{[k]>[h]} kM$. We define $g([h], M) = \dim Q \otimes ([h]M/[h]^*M)$.

If M is a group which is a direct sum of h-elementary balanced projectives for various heights h, then for a given height h, the number of summands whose type is $[h]$ is clearly just $g([h], M)$. Since this number is clearly an isomorphism invariant, this remark shows that the number of summands of a given type is an invariant. (That this number is an invariant for direct sums of h-elementary balanced projectives was shown by the author, but this nice description of the invariant, in a context that makes sense for arbitrary groups, was given by Wick [29].)

We should remark that if M is a module over a discrete valuation ring R, then any type corresponds to some limit ordinal λ, but in general $g([p^\lambda], M) \neq h(\lambda, M)$ (as defined in §4). This equality will, of course, be true for balanced projective modules. We recall that the Ulm factor of M corresponding to λ is $p^\lambda M/p^{\lambda+\omega}M$. It is easy to verify that $g([p^\lambda], M)$ is the torsion-free rank of this Ulm factor, while $h(\lambda, M)$ is the rank of the torsion-free part of a basic submodule of the Ulm factor. These agree for balanced projectives, since the Ulm factor is then a direct sum of cyclic modules.

We remark that it is very easy to show that these invariants plus the Ulm invariants classify groups which are direct sums of torsion groups satisfying Hill's condition and h-elementary balanced projectives for various ordinals h. The difficult problem is to show that they also classify summands of these groups. In the end, it turns out that all such summands are direct sums of this form, but to show this it is necessary to extend the classification theorem first, and then to conclude, by examining the invariants, that all of the summands so obtained are actually direct sums of torsion groups satisfying Hill's condition and elementary balanced projectives.

5.2. THEOREM. *Let M and N be balanced projective Abelian groups with the same Ulm invariants, and such that for all heights h, $g([h], M) = g([h], N)$. Then $M \cong N$.*

PROOF. Let X and Y be K-bases of M and N (3.12, 3.13). Since $M/[X]$ is torsion, one can easily see that

$$g([h], M) = \dim [Q \otimes ([X] \cap [h]M)/([X] \cap [h]^*M)].$$

Since the heights of elements in $[X]$ are computed easily, using Definition 3.13,

we see that $g([h], M)$ is exactly the number of elements $x \in X$ such that $h(x) \in [h]$. Applying this reasoning to M and N, and using the fact that $g([h], M) = g([h], N)$ for all heights h, we obtain a bijective map ϕ: $X \longrightarrow Y$ such that $[h(x)] = [h(\phi(x))]$. Replacing each element of X and Y by a suitable multiple if necessary, we can assume that, in fact, $h(x) = h(\phi(x))$.

The natural map ν: $M \longrightarrow M_p$ preserves p-height, and $M_p/\nu(M)$ is torsion, with no p-torsion. It follows easily from this that $\nu(X) = X(p)$ is a K-basis for M_p and we have an induced, height preserving bijection ϕ_p: $X(p) \longrightarrow Y(p)$ between K-bases of M_p and N_p. We note that a balanced short exact sequence of Z_p-modules is balanced when regarded as a sequence of Abelian groups. This implies (by the standard universal mapping properties) that if M is a balanced projective group then for every prime p, M_p is a balanced projective module. We can therefore apply 4.6, since the Ulm invariants of M_p and N_p are equal, and conclude that ϕ_p extends to an isomorphism $g(p)$: $M_p \longrightarrow N_p$. By 1.16, this shows that ϕ extends to an isomorphism of M onto N.

5.3. THEOREM. *Let M be a balanced projective group. Then M is the direct sum of a divisible group, a torsion group satisfying Hill's condition, and h-elementary balanced projectives for various heights h.*

PROOF. We may clearly assume that M is reduced, and not torsion. Let X be a K-basis for M. For every prime p, M_p is a balanced projective Z_p-module (as we remarked in the proof of 5.2). If ϕ_p: $M \longrightarrow M_p$ is the localization map, ϕ_p takes X bijectively to a K-basis $X(p)$ for M_p, such that for $x \in X$, $h_p(x) = h_p(\phi_p(x))$. From the proof of 4.5 we know we may decompose M_p as $\bigoplus_{x \in X} M(p, x)$ where $\phi_p(x) \in M(p, x)$ for all $x \in X$. Let $M(p, x)' = \{y \in M(p, x)$: for some positive integer k, $p^k y \in Zx\}$. That is, $M(p, x)'$ is the p-closure of the subgroup (not the submodule) of $M(p, x)$ generated by x. Let $G(x) = \bigoplus_p M(p, x)'$. Let K be the subgroup of $G(x)$ generated by the elements $\phi_p(x) - \phi_q(x)$, where p and q are distinct primes. Let $M(x) = G(x)/K$. It is clear that $M(x)$ is an $h(x)$-elementary balanced projective. We let the element of $M(x)$ which is the common image of the elements $\phi_p(x)$ be denoted $\phi(x)$. Let $N = \bigoplus_{x \in X} M(x)$. We claim that it is now obvious that the map ϕ extends to an isomorphism $M \longrightarrow N$, thus proving the theorem.

These two theorems complete the structure theory which is the main purpose of this paper. To summarize, we have found a class of Abelian groups defined by a natural projectivity property. The torsion elements of this class coincide with groups whose reduced part is a direct sum of totally projective p-groups for various primes p. The torsion-free groups in our class are exactly the direct sums of groups of rank one, classified by Baer in 1937. The groups in our new class are classified by numerical, functorial invariants, and the groups are shown

all to be direct sums of groups whose torsion-free rank is at most one.

It is reasonable to ask whether this class is in some sense the largest possible class for which such a generalization is possible. If we restrict ourselves to the invariants used here, it is not hard to see that this is so, in the sense that there is no larger class containing the balanced projectives and closed with respect to direct sums which can be classified by these invariants. (For a proof in the local case, see [28, 4.5].) If one extends the invariants, then the theory can quite possibly be extended. A considerable extension in the local case is outlined in [25], where a classification theorem is stated for modules over a discrete valuation ring which are summands of modules which have presentations by generators and relations in such a way that all of the relations are of the form $px = y$ or $px = 0$. This theory, however, does not yet have a satisfactory generalization to the global case (see [26]).

In [28] it is proved that if M is a module over a discrete valuation ring R, and α is an ordinal, then M is a balanced projective if and only if $p^\alpha M$ and $M/p^\alpha M$ are both balanced projectives. The proof in [28] that M has this property if $p^\alpha M$ and $M/p^\alpha M$ do is quite difficult. We remark that if one used the projective characterization of these modules, which was not available in [28], then a very easy proof can be given using the ideas from the proof of 2.3. The corresponding theorem in the global case is false, as was shown by Wick in [29]. In particular, if G is a group such that $2^\Omega 3^\Omega G \cong Z$ and $G/2^\Omega 3^\Omega G$ is the direct sum of a totally projective 2-group of length Ω and a totally projective 3-group of length Ω, (here, Ω is the first uncountable ordinal), then $3^{\Omega+\omega} G$ is exactly the 2-torsion subgroup of G, which is not totally projective. It is, in fact, an Ω-elementary S-group (using the terminology of [28]). In [29], Wick considers reduced groups which are projective with respect to all short exact sequences of reduced groups such that the induced sequence of cotorsion hulls is balanced. The class of groups so obtained includes the reduced balanced projectives and also the S-groups of [28]. Wick's results suggest that this larger class of groups should have a good classification theory and that it should have stronger closure properties than the class of balanced projectives. None of this has yet been proved.

The proof of 4.4 suggests that balanced projective modules over a discrete valuation ring should have the properties which are the analogues for mixed groups of the transitive and fully-transitive properties for torsion modules, as defined by Kaplansky in [8]. One should therefore be able to characterize the characteristic and fully invariant submodules. Fuchs and Walker showed [6, Vol. II, p. 101] that if G is a totally projective p-group and H is a fully invariant subgroup, then H and G/H are also totally projective. This will not generalize naively, since the torsion submodule of a balanced projective module is not gen-

erally balanced projective [28], but a suitable generalization should still be possible. Presumably there should also be global forms of these results. The fact that the S-groups of [28] keep rearing their ugly heads in such considerations suggests that the balanced projectives may not be the natural domain for questions of this sort, and that one should carry on all of this discussion for a larger class of groups—presumably the class introduced by Wick in [29] and discussed in the previous paragraph.

Theorem 4.4 says that a module over a complete discrete valuation ring is a balanced projective if and only if it is H-isomorphic to a balanced projective and it satisfies Hill's condition. We have not proved the corresponding result for modules over an arbitrary discrete valuation ring or for Abelian groups. It would be interesting to know whether it is true.

We have proved that two balanced projectives are isomorphic if and only if they have the same Ulm invariants and are isomorphic in the category H. We remark that the theorems of Rotman and Yen in [19], show that countably generated modules of finite torsion-free rank over a complete discrete valuation ring also have this property—two of them are isomorphic if and only if they are isomorphic in H and they have the same Ulm invariants. The category H, in fact, gives a natural categorical expression for the "invariant" defined in [19] for these modules. It is shown in [20] that the completeness of the valuation ring is essential for the Rotman-Yen theorem. This suggests the following scheme for classifying mixed Abelian groups: one should be working with a class of groups such that two of them are isomorphic if they are isomorphic in H and certain additional invariants are equal, and the main labor should then be the classification problem in H (i.e., as in §3 of this paper).

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