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STABILITY THEOREMS IN SHAPE AND PRO-HOMOTOPY

BY

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ABSTRACT. Conditions are given under which a topological space has the pointed shape of a *CW* complex. These are derived from analogous conditions in pro-homotopy.

1. **Introduction.** A pointed connected topological space is *stable* if it is pointed shape equivalent to a pointed *CW* complex. In [5] and [6] we gave necessary and sufficient conditions for a compact metric space (compactum) to be stable. In this paper, we generalize these criteria to arbitrary (pointed, connected) topological spaces. We also prove analogous theorems in pro-homotopy theory, but in this introduction we will only state the shape theorems.

Our first theorem (Theorem 3.2) says that *a pointed connected space is stable if and only if it is pointed shape dominated by a pointed CW complex.* It is an easy matter to deduce this from the compact case in [6]. The details are in §3.

Our second theorem (see Theorem 5.4 for a fuller version) says that *a pointed connected space whose strong shape dimension is finite is stable if and only if its homotopy pro-groups are dominated by groups.* (Among the spaces with finite strong shape dimension are all finite-dimensional separable metric spaces: see §6.) Although the second theorem appears to be a generalization of the compact metric case treated in [5], the proof involves ideas which were not needed there. In the first place, we need the Bousfield-Kan spectral sequence [3]. Secondly, we need to know that if a pro-group $\{G_\alpha\}$ is pro-isomorphic to a group, then the derived limits $\varprojlim^s \{G_\alpha\}$ vanish for all $s \geqslant 1$ (if some of the groups G_α are nonabelian, only $\varprojlim^1 \{G_\alpha\}$ is defined); in the abelian case, this latter result was announced by Verdier in [21]; we give a proof based on the Bousfield-Kan approach in §4. Thirdly, we need a Whitehead Theorem which is slightly different from that given in [5].

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2. **Notation and terminology.** If I is a category, pro-I is a category whose objects are inverse systems in I indexed by directed sets. See [1] or [14] for a description of the morphisms of pro-I. We denote an object of pro-I by $\{X_\alpha\}_{\alpha \in A}$, or simply $\{X_\alpha\}$, where α ranges over some (variable) directed set A, X_α is an object of I, and, whenever $\alpha \leqslant \beta$, an unmarked morphism of I from X_β to X_α is understood to have been chosen in such a way as to make $\{X_\alpha\}$ an inverse system. These morphisms are called *bonds.* If α ranges over the set of natural numbers, $\{X_\alpha\}$ is called a *tower.*

We shall also need the category I^A where A is a directed set. Its objects are inverse systems in I indexed by A. Its morphisms from $\{X_\alpha\}_{\alpha \in A}$ to $\{Y_\alpha\}_{\alpha \in A}$ are collections $\{f_\alpha\colon X_\alpha \longrightarrow Y_\alpha\}_{\alpha \in A}$ of morphisms of I which commute with the bonds.

An object X of I is *dominated by* an object Y if there are morphisms

$$X \underset{u}{\overset{d}{\rightleftarrows}} Y$$

such that $d \circ u = 1$, where 1 stands for the identity morphism.

I_Δ denotes the category whose objects are the commutative triangles in I, and whose morphisms are the commutative prisms in I.

The following categories appear: T_0 (pointed connected spaces and pointed maps); CW_0 (pointed connected CW complexes and pointed maps); HT_0, H_0 (the pointed homotopy categories corresponding to T_0 and CW_0); SS_* (pointed simplicial sets and pointed maps [17]); K_* (pointed Kan complexes and pointed maps [17]); HK_* (pointed Kan complexes and pointed homotopy classes of pointed maps); SS_0, K_0, HK_0 (the full subcategories of SS_*, K_* and HK_* generated by connected objects); Groups (groups and homomorphisms); Abelian Groups (abelian groups and homomorphisms); Pointed Sets (pointed sets and pointed functions).

We always suppress base points when describing objects of these categories of pointed spaces. Similarly in the corresponding pro-categories. If $X = \{X_\alpha\}$ is an object of pro-CW_0 or pro-H_0, $\pi_k(X)$ will denote the corresponding object $\{\pi_k(X_\alpha)\}$ of pro-Groups, where $\pi_k(X_\alpha)$ is the kth homotopy group of X_α. A morphism of pro-CW_0 or pro-H_0 is a *weak equivalence* if it induces an isomorphism on each π_k, $k \geqslant 1$.

If $\{X_\alpha\}$ is in pro-CW_0 we will usually also denote the induced object of pro-H_0 by $\{X_\alpha\}$.

$S\colon CW_0 \longrightarrow K_0$ and $|\cdot|\colon K_0 \longrightarrow CW_0$ denote the singular-complex and geometric-realization functors [17].

The *CW dimension* of a *CW* complex X_α is the integer $CW\text{-dim}\, X_\alpha$ such that the complex contains cells of that dimension, but of no higher dimension. If no such integer exists, $CW\text{-dim}\, X_\alpha = \infty$. If $X = \{X_\alpha\}$ is an object of pro-CW_0, $CW\text{-dim}\, X = \sup_\alpha \{CW\text{-dim}\, X_\alpha\}$. The *homotopy dimension* of X is $h\text{-dim}\, X = \inf\{CW\text{-dim}\, Y | Y$ is isomorphic to X in pro-$H_0\}$. The *strong homotopy* dimension of X is $s\text{-}h\text{-dim}\, X = \inf\{CW\text{-dim}\, Y | Y$ is an object of pro-CW_0 which is isomorphic to X in pro-$H_0\}$.

An object $\{X_\alpha\}$ of pro-CW_0 is *compact* if each X_α is a finite complex.

Our shape theory (pointed) is that of [13]; with very little change it could be that of [19]: the two agree on paracompact Hausdorff spaces [12], [18]. Thus it agrees with that of [8] on metric spaces [18], with that of [15] on compact Hausdorff spaces [13], and with that of [2] on compact metric spaces [16].

Following Morita [18], we say that an object $\{X_\alpha\}$ of pro-H_0 is *associated with* a pointed connected space Z if (i) there are morphisms of H_0, $p_\alpha: Z \longrightarrow X_\alpha$ such that bond $\circ\, p_\beta = p_\alpha$ whenever $\alpha \leqslant \beta$; (ii) each morphism of H_0, $m: Z \longrightarrow P$ (where P is an object of H_0) factorizes as $m = m_\alpha \circ p_\alpha$; and (iii) if $m_\alpha \circ p_\alpha = m'_\alpha \circ p_\alpha$ are two factorizations, then there exists $\beta \geqslant \alpha$ such that $m_\alpha \circ$ bond $= m'_\alpha \circ$ bond as morphisms of H_0 from X_β to P.

Every pointed topological space Z has a *canonical object* of pro-H_0 associated with it, namely the inverse system built from the nerves of all open locally-finite normal (= numerable) covers of Z exactly one of whose elements contains the base point [18, §6].

Two objects of pro-H_0 are both associated with Z if and only if they are isomorphic [18]. Two pointed spaces Z and Z' are *pointed shape equivalent* if some (and hence any) object of pro-H_0 associated with Z is isomorphic to an object associated with Z'. Z is *pointed shape dominated* by Z' if an object associated with Z is dominated in pro-H_0 by an object associated with Z'. Note, in this connection, that a pointed *CW* complex is associated with itself.

We define pro-$\pi_k(Z)$ to be the pro-group $\{\pi_k(X_\alpha)\}$ where $\{X_\alpha\}$ is the canonical object of pro-H_0 associated with Z. Up to isomorphism in pro-Groups, any object associated with Z would do as well.

Other terminology will be introduced as required.

3. Domination criteria for stability in pro-homotopy and shape.

The following observation is elementary but important:

LEMMA 3.1. *Let Y be an object of a category I, let X be an object of pro-I and let*

$$X \underset{u}{\overset{d}{\rightleftarrows}} Y$$

be morphisms of pro-I with $d \circ u = 1_X$. *Then* X *is isomorphic in pro-I to the tower*

$$\{Y \xleftarrow{f} Y \xleftarrow{f} Y \xleftarrow{f} \cdots\}$$

where f *is the morphism of* I *defined by* $u \circ d$.

PROOF. Routine. Compare with Proposition 3.1 of [6].

THEOREM 3.2. *If an object* X *of pro-*H_0 *is dominated in pro-*H_0 *by a pointed CW complex* Y, *then* X *is isomorphic in pro-*H_0 *to a pointed CW complex.*

PROOF. By Lemma 3.1 we may assume without loss of generality that X is a tower $\{X_n\}$ in pro-H_0. By choosing representatives of the bonding homotopy classes, we get a tower in pro-CW_0 which we also denote by $\{X_n\}$. The tower $\{S(X_n)\}$ in pro-K_0 is isomorphic in pro-HK_0 to a tower $\{Q_n\}$ of Kan fibrations (an object of pro-K_0: compare Corollary 2.3 of [6]). Let Q be the inverse limit of $\{Q_n\}$. Since the bonds are fibrations, Q is clearly a Kan complex.

Let $p\colon Q \longrightarrow \{Q_n\}$ be the canonical projection. For each $i \geqslant 0$ there is a short exact sequence (see [3, p. 254])

$$* \longrightarrow \underset{\underset{n}{\longleftarrow}}{\lim}{}^1\, \pi_{i+1}(Q_n) \longrightarrow \pi_i(Q) \xrightarrow{p_\#} \underset{\longleftarrow}{\lim}\, \pi_i(Q_n) \longrightarrow *.$$

Since $\{Q_n\}$ is dominated in pro-HK_0 by a complex, all the $\underset{\longleftarrow}{\lim}{}^1$ terms vanish. Hence Q is connected and $p_\#$ is an isomorphism for $i \geqslant 1$.

There are morphisms

$$\{Q_n\} \underset{u}{\overset{d}{\rightleftarrows}} S(Y)$$

of pro-HK_0 such that $d \circ u$ is the identity. By the Covering Homotopy Property [17, p. 30] d is induced by a morphism of pro-K_0, which necessarily maps $S(Y)$ into the inverse limit, Q. Hence $d = p \circ d'$ in pro-HK_0, where $d'\colon S(Y) \longrightarrow Q$ is a map. Thus, in pro-HK_0, $p \circ (d' \circ u) = 1$, so p has a right inverse. To see that $d' \circ u$ is also a left inverse, note that

$$(d' \circ u)_\#\colon \underset{\underset{n}{\longleftarrow}}{\lim}\, \pi_i(Q_n) \longrightarrow \pi_i(Q)$$

is a right inverse for $p_\#$, hence a two-sided inverse ($p_\#$ being an isomorphism). So $(d' \circ u) \circ p\colon Q \longrightarrow Q$ is a weak homotopy equivalence, hence a homotopy equivalence, hence the identity. Hence X is isomorphic to $|Q|$. □

We will need the next proposition in §5.

PROPOSITION 3.3. *If $\{G_\alpha\}$ is dominated in pro-Groups by a group H, then the projection p: $\varprojlim\{G_\alpha\} \longrightarrow \{G_\alpha\}$ is an isomorphism in pro-Groups.*

PROOF. Let

$$\{G_\alpha\} \underset{u}{\overset{d}{\leftrightarrows}} H$$

be morphisms of pro-Groups such that $d \circ u = 1$. d necessarily factorizes as $d = p \circ d'$ where d': $H \longrightarrow \varprojlim\{G_\alpha\}$. Thus p has a right inverse $(d' \circ u)$. It is easy to check that $(d' \circ u) \circ p$ is an automorphism of G, and hence the identity.

We now use Theorem 3.2 to obtain a stability theorem in shape:

THEOREM 3.4. *A pointed connected space Z is pointed shape equivalent to a CW complex if and only if Z is pointed shape dominated by a CW complex.*

PROOF. "Only if" is obvious. To prove "if" observe that an object of pro-H_0 associated with Z will be dominated in pro-H_0 by a complex. Use Theorem 3.2. □

For compact pro-complexes and spaces we can say a little more:

THEOREM 3.5. *Let X be a compact object of pro-H_0. The following are equivalent*:

(i) *X is dominated in pro-H_0 by a finite complex*;

(ii) *X is isomorphic in pro-H_0 to a complex*;

(iii) *X is dominated in pro-H_0 by a complex.*

PROOF. (ii) is equivalent to (iii) by Theorem 3.2. The proof that (i) is equivalent to (iii) is the same as the corresponding part of the proof of Theorem 1.1 of [6]. □

From this, we deduce

THEOREM 3.6. *Let Z be a pointed connected compact space. The following are equivalent* (*in pointed shape theory*):

(i) *Z is shape dominated by a finite complex*;

(ii) *Z is shape equivalent to a complex*;

(iii) *Z is shape dominated by a complex.*

REMARK 3.7. There remain the questions: *when is a pro-complex isomorphic to a finite complex*? and when is a space shape equivalent to a finite complex? By Theorems 3.5 and 3.6 we see that domination by a finite complex is necessary. But by Lemma 3.1, domination by a finite complex implies isomorphism to a tower (or shape equivalence to a compact metric space). We have explained in 1.1 and 3.3 of [6], and in 4.2 of [5] that for finitely dominated towers (and compact metric spaces) the vanishing of a "Wall obstruction" is necessary and sufficient for isomorphism (or shape equivalence) to a finite complex; and all possible obstructions are realized. Thus, our questions are answered.

REMARK 3.8. There is also the question: *when is a pro-complex isomorphic to a tower?* A modification of Lemma 3.1 implies that an object of pro-H_0 is isomorphic to a tower if and only if it is dominated in pro-H_0 by a tower.

4. **Homotopy limits and derived limits.** If A is a directed set (or more generally a small category) Bousfield and Kan define a homotopy inverse limit functor $\underleftarrow{\mathrm{holim}}_A\colon (SS_*)^A \longrightarrow SS_*$ which associates a "best approximating" simplicial set with each inverse system indexed by A; see [3, pp. 295 and 301].(2) It follows easily from Lemma 5.5, p. 303, of [3] that the homotopy inverse limit of Kan complexes is a Kan complex, so we may write the restricted functor as

$$\underleftarrow{\mathrm{holim}}_A\colon (K_*)^A \longrightarrow K_*.$$

Let $i\colon (K_*)^A \longrightarrow \text{pro-}K_*$ and $p\colon K_* \longrightarrow HK_*$ be the natural "inclusion" and "projection" functors. The principal theorem of this section is

THEOREM 4.1. *There exists a functor* $\underleftarrow{\mathrm{holim}}\colon \text{pro-}K_* \longrightarrow HK_*$ *such that for any directed set A,* $\underleftarrow{\mathrm{holim}} \circ i = p \circ \underleftarrow{\mathrm{holim}}_A$.

PROOF. If $\{X_\alpha\}_{\alpha\in A}$ is an object of pro-K_*, define $\underleftarrow{\mathrm{holim}}\{X_\alpha\}$ to be $\underleftarrow{\mathrm{holim}}_A\{X_\alpha\}$. Next, let $f\colon \{X_\alpha\}_{\alpha\in A} \longrightarrow \{Y_\beta\}_{\beta\in B}$ be a morphism of pro-K_*. We will assume familiarity with the *proof* of the Artin-Mazur Reindexing Lemma [1] as it appears in §2.2 of [14]. From it we get a commutative diagram in pro-K_*

$$\begin{tikzcd} \{X_\alpha\}_{\alpha\in A} \arrow[r, "f"] \arrow[d, "d'"'] & \{Y_\beta\}_{\beta\in B} \arrow[d, "r'"] \\ \{X'_\gamma\}_{\gamma\in C} \arrow[r, "f'"] & \{Y'_\gamma\}_{\gamma\in C} \end{tikzcd}$$

where C is a directed set, f' is induced by a morphism $\{X'_\gamma \xrightarrow{f'_\gamma} Y'_\gamma\}$ of $(K_*)^C$,

(2) The homotopy inverse limit functor discussed explicitly in [3] is the unpointed version. However, as explained on p. 301 of [3], all the results we shall use from [3] have pointed analogues. See [7] for an alternative treatment of the material in Chapter XI of [3].

and d' and r' are induced by cofinal functors $d\colon C \longrightarrow A$ and $r\colon C \longrightarrow B$ (see [3, pp. 316–317]: we may regard a directed set as a small category). By the Cofinality Theorem [3, p. 317], d' and r' induce pointed homotopy equivalences $d_*\colon \underleftarrow{\mathrm{holim}}_A \{X_\alpha\} \longrightarrow \underleftarrow{\mathrm{holim}}_C \{X'_\gamma\}$ and $r_*\colon \underleftarrow{\mathrm{holim}}_B \{Y_\beta\} \longrightarrow \underleftarrow{\mathrm{holim}}_C \{Y'_\gamma\}$. There is of course a pointed map $\underleftarrow{\mathrm{holim}}_C \{f'_\gamma\}$. We define $\underleftarrow{\mathrm{holim}}\, f$ to be the morphism of HK_* induced by $(r_*)^{-1} \circ \underleftarrow{\mathrm{holim}}_C \{f'_\gamma\} \circ d_*$, where $(r_*)^{-1}$ is homotopy inverse to r_*.

We must now show that $\underleftarrow{\mathrm{holim}}$ preserves identities and compositions.

Starting with $1_{\{X_\alpha\}}$, we have, as above, the commutative diagram

$$\begin{tikzcd}
\{X_\alpha\}_{\alpha\in A} \arrow[r, "1"] \arrow[d, "d'"'] & \{X_\alpha\}_{\alpha\in A} \arrow[d, "r'"] \\
\{X'_\gamma\}_{\gamma\in C} \arrow[r, "\{f'_\gamma\}"] & \{Y'_\gamma\}_{\gamma\in C}
\end{tikzcd}$$

with d' and r' induced by cofinal functors $d, r\colon C \longrightarrow A$. By referring to the definition of C in §2.2 of [14], one sees at once that there is a cofinal "inclusion" functor $e\colon A \longrightarrow C$ such that $d \circ e = r \circ e = 1_A$. Furthermore, the definition implies that the following diagram commutes in pro-K_*

$$\begin{tikzcd}
\{X'_\gamma\}_{\gamma\in C} \arrow[r, "\{f'_\gamma\}"] \arrow[d] & \{Y'_\gamma\}_{\gamma\in C} \arrow[d] \\
\{X_\alpha\}_{\alpha\in A} \arrow[r, "1"] & \{X_\alpha\}_{\alpha\in A}
\end{tikzcd}$$

where the vertical morphisms are induced by e. The "naturality properties" of $\underleftarrow{\mathrm{holim}}_A$ and $\underleftarrow{\mathrm{holim}}_C$ (see p. 296 of [3]) then give a commutative diagram in K_*

$$\begin{tikzcd}
\underleftarrow{\mathrm{holim}}_A \{X_\alpha\} \arrow[r, "\underleftarrow{\mathrm{holim}}\ 1"] \arrow[d, "d_*"] \arrow[dd, bend right=60, "1"'] & \underleftarrow{\mathrm{holim}}_A \{X_\alpha\} \arrow[d, "r_*"] \arrow[dd, bend left=60, "1"] \\
\underleftarrow{\mathrm{holim}}_C \{X'_\gamma\} \arrow[r, "\underleftarrow{\mathrm{holim}}_C \{f'_\gamma\}"] \arrow[d, "e_*"] & \underleftarrow{\mathrm{holim}}_C \{Y'_\gamma\} \arrow[d, "e_*"] \\
\underleftarrow{\mathrm{holim}}_A \{X_\alpha\} \arrow[r, "\underleftarrow{\mathrm{holim}}_A \equiv 1"] & \underleftarrow{\mathrm{holim}}_A \{X_\alpha\}
\end{tikzcd}$$

from which it follows that $\underleftarrow{\mathrm{holim}}\, 1 = 1$ in HK_*.

The proof that $\underleftarrow{\mathrm{holim}}$ preserves compositions uses the naturality properties in a similar way. We will give an outline which will enable the reader to construct the necessary diagrams and check that they commute.

Let $h = g \circ f$ where $f\colon \{X_\alpha\}_{\alpha\in A} \longrightarrow \{Y_\beta\}_{\beta\in B}$ and $g\colon \{Y_\beta\}_{\beta\in B} \longrightarrow \{Z_\gamma\}_{\gamma\in C}$ are morphisms of pro-K_*. Reindex f, g and h as above to get $\{X'_\delta \xrightarrow{f'_\delta} Y'_\delta\}_{\delta\in D}$, $\{Y''_\epsilon \xrightarrow{g'_\epsilon} Z'_\epsilon\}_{\epsilon\in E}$ and $\{X''_\zeta \xrightarrow{h'_\zeta} Z''_\zeta\}_{\zeta\in F}$ where we have cofinal functors $d_1\colon D \longrightarrow A$, $r_1\colon D \longrightarrow B$, $d_2\colon E \longrightarrow B$, $r_2\colon E \longrightarrow C$, $d_3\colon F \longrightarrow A$ and $r_3\colon F \longrightarrow C$. As explained in §2.2 of [14], D is a directed set consisting of those morphisms $X_\alpha \longrightarrow Y_\beta$ which can be "refined by" f, and the functors d_1 and r_1 pick out the domains and ranges. E, d_2 and r_2 are similarly related to g, as are F, d_3 and r_3 to h. Let G be the directed set consisting of those compositions $X_\alpha \longrightarrow Y_\beta \longrightarrow Z_\gamma$ which can be "refined by" h, with the obvious partial ordering. There are obvious cofinal functors $m_1\colon G \longrightarrow D$, $m_2\colon G \longrightarrow E$ and $m_3\colon G \longrightarrow F$, and we have $r_1m_1 = d_2m_2$, $d_3m_3 = d_1m_1$ and $r_3m_3 = r_2m_2$. These equalities allow us to write

$$(\underleftarrow{\operatorname{holim}}\, g) \circ (\underleftarrow{\operatorname{holim}}\, f) = (m_{2*}r_{2*})^{-1}\ \underleftarrow{\operatorname{holim}}_G \{g''_\eta\}\ \underleftarrow{\operatorname{holim}}_G\ \{f''_\eta\} m_{1*}d_{1*}$$

and

$$\underleftarrow{\operatorname{holim}}(g \circ f) = (m_{3*}r_{3*})^{-1}\ \underleftarrow{\operatorname{holim}} \{g''_\eta f''_\eta\} m_{3*}d_{3*}.$$

A diagram similar to the one used in proving that $\underleftarrow{\operatorname{holim}}$ preserves identities is used to show that $\underleftarrow{\operatorname{holim}} \circ i = p \circ \underleftarrow{\operatorname{holim}}_A$. The argument contains no new ideas. □

REMARK. Our pro-K_* only contains inverse systems indexed by directed sets. But Theorem 4.1 also holds for the more general pro-K_* defined in the Appendix to [1]; one must, of course, refer to pp. 160–162 of [1], rather than to [14] in the proof. (In fact for any category I, pro-I using directed sets is equivalent to pro-I using filtered categories: see [7].)

If $\{G_\alpha\}$ is an object of pro-(Abelian Groups) there exists, for each integer $s \geqslant 0$, the *derived limit* abelian group $\underleftarrow{\lim}^s \{G_\alpha\}$: see [3, p. 305], for the definition and references; $\underleftarrow{\lim}^0 \{G_\alpha\}$ is the ordinary inverse limit abelian group. If $\{G_\alpha\}$ is an object of pro-Groups, the *derived limits* $\underleftarrow{\lim}^0 \{G_\alpha\}$ and $\underleftarrow{\lim}^1 \{G_\alpha\}$ are introduced in [3, p. 307]; in this latter case, $\underleftarrow{\lim}^0 \{G_\alpha\}$ is the ordinary inverse limit group, and $\underleftarrow{\lim}^1 \{G_\alpha\}$ is a pointed set.

COROLLARY 4.2. (i) *If $\{G_\alpha\}$ is isomorphic in pro-(Abelian Groups) to an abelian group G, then $\underleftarrow{\lim}^s \{G_\alpha\}$ is trivial for all $s \geqslant 1$.*

(ii) *If $\{G_\alpha\}$ is isomorphic in pro-Groups to a group G, then $\underleftarrow{\lim}^1 \{G_\alpha\}$ is trivial.*

PROOF. We prove (i): (ii) is proved similarly. For $n \geqslant 1$ we have the Eilenberg-Mac Lane functor $K(\,\cdot\,, n)$: (Abelian Groups) $\longrightarrow K_*$: see [17, pp. 88

and 98–100]; this functor automatically extends to pro-categories. Using Theorem 4.1, we have $\pi_1\ \underleftarrow{\mathrm{holim}}\{K(G_\alpha, s+1)\}$ isomorphic to $\pi_1\ \underleftarrow{\mathrm{holim}}\,K(G, s+1)$. The first of these groups is isomorphic to $\underleftarrow{\lim}^s\{G_\alpha\}$ while the second is isomorphic to $\underleftarrow{\lim}^s G$: see [3, p. 309]. Here G stands for an inverse system indexed by a one-element directed set; for such a system $\underleftarrow{\lim}^s$ vanishes when $s \geqslant 1$: see [3, p. 306]. So $\underleftarrow{\lim}^s\{G_\alpha\}$ vanishes. □

5. **Algebraic criteria for stability in pro-homotopy.** If $\{Y_\alpha\}_{\alpha\in A}$ is an object of $(K_*)^A$, there are canonical maps $p_{\alpha_0}\colon \underleftarrow{\mathrm{holim}}_A\{Y_\alpha\} \longrightarrow Y_{\alpha_0}$ for each $\alpha_0 \in A$, such that p_{α_0} is pointedly homotopic to bond $\circ\, p_{\alpha_1}$; see Proposition 3.4, p. 296 of [3]. These define a morphism $p\colon \underleftarrow{\mathrm{holim}}\{Y_\alpha\} \longrightarrow \{Y_\alpha\}$ in pro-HK_*. In this section we will discuss conditions which make p an isomorphism.

Lemma 5.1. *Assume each Y_α is connected. Then $\underleftarrow{\mathrm{holim}}\{Y_\alpha\}$ is connected and p is a weak equivalence if and only if $\{\pi_i(Y_\alpha)\}$ is dominated in pro-Groups by a group, $i \geqslant 1$.*

Proof. "Only if" is obvious. In fact, by Proposition 3.3, we can conclude that $\{\pi_i(Y_\alpha)\}$ is isomorphic to $\underleftarrow{\lim}\{\pi_i(Y_\alpha)\}$. We prove "if". The following diagram commutes (in pro-Groups if $i \geqslant 1$, in pro-(Pointed Sets) if $i = 0$):

$$\begin{tikzcd}
\pi_i(\underleftarrow{\mathrm{holim}}\{Y_\alpha\}) \arrow[rr, "p_\#"] \arrow[dr, "p_\#"'] & & \underleftarrow{\lim}\{\pi_i(Y_\alpha)\} \arrow[dl, "\text{projection}"] \\
& \{\pi_i(Y_\alpha)\} &
\end{tikzcd}$$

By Proposition 3.3, "projection" is an isomorphism if $i \geqslant 1$; and "projection" is trivially an isomorphism if $i = 0$. Hence it will be enough to show that the horizontal $p_\#$ is an isomorphism. That this is so follows from the convergence of the Bousfield-Kan spectral sequence [3, p. 309], together with Corollary 4.2 above. The details of the argument are given by Porter in [20] (where, of course, the *conclusion* of Corollary 4.2 is assumed). For the reader's convenience we quote them.

We use the notation of [3]. Let $Z_n = \mathrm{Tot}_n\, \Pi^*\{Y_\alpha\}$, with the natural base point. There is a pointed tower of fibrations $\{Z_n\}$ whose simplicial inverse limit is $Z \equiv \underleftarrow{\mathrm{holim}}\{Y_\alpha\}$. The spectral sequence to be used is that associated with $\{Z_n\}$, [3, p. 259]. $E_2^{s,t} = \underleftarrow{\lim}^s\{\pi_t(Y_\alpha)\}$ if $0 \leqslant s \leqslant t$. By Proposition 3.3 and Corollary 4.2, $E_2^{s,t} = 0$ unless $s = 0$. The differential has bi-degree $(r, r-1)$, so $E_2^{s,t} \cong E_r^{s,t} \cong E_\infty^{s,t}$, for all r.

The case $i = 0$ is treated by the Connectivity Lemma, p. 261 of [3], from which it follows that $Z \equiv \underleftarrow{\mathrm{holim}}\{Y_\alpha\}$ is connected.

From now on, we assume $i \geqslant 1$. By Adams' Lemma, p. 263 of [3], the spectral sequence is completely convergent. Hence the natural homomorphisms

$$\pi_i(\operatorname{\underleftarrow{holim}}\{Y_\alpha\}) \equiv \pi_i(Z) \longrightarrow \underleftarrow{\lim}\{\pi_i(Z_n)\}$$

and

$$e_\infty^{s,s+i} \longrightarrow E_\infty^{s,s+i} \longrightarrow E_2^{s,s+i}$$

are isomorphisms. Thus the natural homomorphism

$$Q_s\pi_i(Z) \longrightarrow Q_{s-1}\pi_i(Z), \qquad s \neq 0$$

is a monomorphism and hence is an isomorphism (since it is clearly onto); when $s = 0$, $Q_{s-1}\pi_i(Z) = 0$, so $Q_0\pi_i(Z) \equiv e_\infty^{0,i}$ is naturally isomorphic to $\underleftarrow{\lim}\{\pi_i(Y_\alpha)\}$. But there are natural isomorphisms

$$\underleftarrow{\lim}\{\pi_i(Z_n)\} \longleftarrow \underleftarrow{\lim}\{Q_n\pi_i(Z)\} \longrightarrow Q_0\pi_i(Z).$$

Combining, we find that p induces an isomorphism $\pi_i \operatorname{\underleftarrow{holim}}\{Y_\alpha\} \longrightarrow \underleftarrow{\lim}\{\pi_i(Y_\alpha)\}$. □

Next we recall some well-known facts about mapping cylinders. If f_α: $X_\alpha \longrightarrow Y_\alpha$ is a morphism of T_0, the following diagram commutes in T_0:

$(*_\alpha)$
$$\begin{tikzcd} X_\alpha \arrow[rr, "f_\alpha"] \arrow[dr, "i(f_\alpha)"'] & & Y_\alpha \arrow[dl, "p(f_\alpha)"] \\ & M(f_\alpha) & \end{tikzcd}$$

where $M(f_\alpha)$ is the reduced mapping cylinder, $i(f_\alpha)$ is the natural inclusion and $p(f_\alpha)$ is the natural projection map. $p(f_\alpha)$ is a pointed homotopy equivalence. If the following diagram in T_0 commutes on passing to HT_0

$$\begin{tikzcd} X_\alpha \arrow[d, "f_\alpha"'] & X_\beta \arrow[l, "x_{\alpha\beta}"'] \arrow[d, "f_\beta"] \\ Y_\alpha & Y_\beta \arrow[l, "y_{\alpha\beta}"'] \end{tikzcd}$$

then, in order to get an induced morphism $(*_\beta) \longrightarrow (*_\alpha)$ in $HT_{0,\Delta}$, one must choose a pointed homotopy $F_{\alpha\beta}$: $X_\beta \times I \longrightarrow Y_\alpha$ between $f_\alpha \circ x_{\alpha\beta}$ and $y_{\alpha\beta} \circ f_\beta$. Define $m_{\alpha\beta}$: $M(f_\beta) \longrightarrow M(f_\alpha)$ by $m_{\alpha\beta}([x, t]) = [x_{\alpha\beta}(x), 2t]$ if $0 \leqslant t \leqslant 1/2$, $m_{\alpha\beta}([x, t]) = F_{\alpha\beta}(x, 2t-1)$ if $1/2 \leqslant t \leqslant 1$. The maps $x_{\alpha\beta}$, $y_{\alpha\beta}$ and $m_{\alpha\beta}$ then induce a morphism of $HT_{0,\Delta}$ as required.

Now, suppose that in the following diagram in T_0, the triangles commute in T_0 while the squares commute on passing to HT_0.

$$\begin{tikzcd}
 & X_\beta \arrow[dl, "x_{\alpha\beta}"'] \arrow[dd, "f_\beta"] & \\
X_\alpha \arrow[dd, "f_\alpha"'] & & X_\gamma \arrow[ul, "x_{\beta\gamma}"'] \arrow[ll, "x_{\alpha\gamma}"] \arrow[dd, "f_\gamma"] \\
 & Y_\beta \arrow[dl, "y_{\alpha\beta}"'] & \\
Y_\alpha & & Y_\gamma \arrow[ul, "y_{\beta\gamma}"'] \arrow[ll, "y_{\alpha\gamma}"]
\end{tikzcd}$$

Let $F_{\alpha\beta}$, $F_{\beta\gamma}$ and $F_{\alpha\gamma}$ be pointed homotopies making the squares commute, and let $m_{\alpha\beta}$, $m_{\beta\gamma}$ and $m_{\alpha\gamma}$ be the corresponding maps between the mapping cylinders. Then there is an induced diagram in $HT_{0,\Delta}$:

$$\begin{tikzcd}
 & (*_\beta) \arrow[dl] & \\
(*_\alpha) & & (*_\gamma) \arrow[ul] \arrow[ll]
\end{tikzcd}$$

This last diagram will commute in $HT_{0,\Delta}$ provided the homotopies $F_{\alpha\beta}$, $F_{\beta\gamma}$ and $F_{\alpha\gamma}$ are "coherent", i.e. provided there is a "higher" pointed homotopy $F_{\alpha\beta\gamma}$: $X_\gamma \times \Delta \longrightarrow Y_\alpha$ where Δ is a standard 2-simplex, which agrees with $F_{\alpha\gamma}$, $F_{\alpha\beta} \circ (x_{\beta\gamma} \times 1)$ and $y_{\alpha\beta} \circ F_{\beta\gamma}$ on the appropriate faces of Δ.

If $\{X_\alpha\}$ and $\{Y_\alpha\}$ are objects of $(T_0)^A$, a morphism $\{X_\alpha \xrightarrow{f_\alpha} Y_\alpha\}$ of $(HT_0)^A$ will be called *coherent* if for every $\alpha \leqslant \beta$ there is $F_{\alpha\beta}$: $X_\beta \times I \longrightarrow Y_\alpha$, and for every $\alpha \leqslant \beta \leqslant \gamma$ there is $F_{\alpha\beta\gamma}$: $X_\gamma \times \Delta \longrightarrow Y_\alpha$, as above. We have proved:

LEMMA 5.2. *With notation as above, if $f \equiv \{X_\alpha \xrightarrow{f_\alpha} Y_\alpha\}$ is coherent, then the following diagram commutes in $(HT_{0,\Delta})^A$*

$$\begin{tikzcd}
\{X_\alpha\} \arrow[rr, "f"] \arrow[dr, "i"'] & & \{Y_\alpha\} \\
 & \{M(f_\alpha)\} \arrow[ur, "p"'] &
\end{tikzcd}$$

and p is invertible. □

We can now state the appropriate Whitehead Theorem:

THEOREM 5.3. *Let* $\{X_\alpha\}$ *and* $\{Y_\alpha\}$ *be objects of* $(CW_0)^A$ *of finite CW dimension. Let* $f \equiv \{X_\alpha \xrightarrow{f_\alpha} Y_\alpha\}$ *be a coherent morphism of* $(H_0)^A$ *such that for all* $i \geqslant 1$, $\{\pi_i(X_\alpha) \xrightarrow{f_{\alpha\#}} \pi_i(Y_\alpha)\}$ *induces an isomorphism in pro-Groups. Then* f *induces an isomorphism in pro-*H_0.

PROOF. This follows from Lemma 5.2 together with the *proof* of the Whitehead Theorem in §3 of [5] (Lemma 5.2 allows one to "enter" that proof at Lemma 3.7 of [5]). □

The main theorem of this section is

THEOREM 5.4. *Let* $X \equiv \{X_\alpha\}_{\alpha \in A}$ *be an object of pro-*CW_0.

(i) *There exists a pointed connected CW complex* Q *and a weak equivalence* $q\colon Q \longrightarrow X$ *in pro-*H_0 *if and only if* $\{\pi_i(X_\alpha)\}$ *is dominated in pro-Groups by a group, for each* $i \geqslant 1$. *In case the condition in* (i) *holds* Q *and* q *may be chosen so that*:

(ii) CW-dim $Q = \max\{3, h\text{-dim}\, X\}$, *and if* h-dim $X = 1$, Q *can be a bouquet of circles*;

(iii) *if* s-h-dim $X < \infty$, q *induces an isomorphism in pro-*H_0;

(iv) *if* s-h-dim $X < \infty$, *and* X *is compact then* Q *is dominated* (*in* H_0) *by a finite complex.*

PROOF OF (i). By Lemma 5.1, the required Q is $|\underleftarrow{\text{holim}}\{S(X_\alpha)\}|$, and q is the composition

$$Q \xrightarrow{|p|} \{|S(X_\alpha)|\} \xrightarrow{\psi} \{X_\alpha\}$$

where ψ is the isomorphism of pro-H_0 induced by the canonical maps ψ_α: $|S(X_\alpha)| \longrightarrow X_\alpha$.

PROOF OF (ii). Let $q\colon Q \longrightarrow X$ be as in (i). The argument used in the proof of Theorem 4.2(ii) of [5] (which is based on Theorems D and E of [22]) shows that Q is pointed homotopy equivalent to a complex Q^* with the required properties. If $q^*\colon Q^* \longrightarrow Q$ is a pointed homotopy equivalence then $q \circ q^*\colon Q^* \longrightarrow X$ is a weak equivalence in pro-H_0.

PROOF OF (iii). We may assume CW-dim $X < \infty$. p is defined by means of maps

$$p_{\alpha_0}\colon \underleftarrow{\text{holim}}\{S(X_\alpha)\} \longrightarrow S(X_{\alpha_0}).$$

Let $\{Q_\alpha\}_{\alpha \in A}$ be the constant system defined by Q, i.e. $Q_\alpha = Q$ for all α, and all bonds are identity maps. By applying Proposition 3.4, p. 296, of [3], and then taking geometric realizations, we note that collection of maps

$\{Q_\alpha \xrightarrow{|p|} S(X_\alpha)\}$ induces a coherent morphism of $(H_0)^A$. Now $\{|S(X_\alpha)| \xrightarrow{\psi_\alpha} X_\alpha\}$ is a morphism of $(CW_0)^A$, as is $\{Q_\alpha^* \xrightarrow{q^*} Q_\alpha\}$. Hence the collection

$$\{Q_\alpha^* \xrightarrow{q^*} Q_\alpha \xrightarrow{|p_\alpha|} S(X_\alpha) \xrightarrow{\psi} X_\alpha\}$$

induces a coherent morphism of $(H_0)^A$ between objects $\{Q_\alpha^*\}$ and $\{X_\alpha\}$ of finite CW dimension. By Theorem 5.3, the resulting morphism of pro-H_0 is invertible.

PROOF OF (iv). Similar to the proof of 4.2(iv) in [5]. One needs s-h-dim $X < \infty$ to use (iii); as in [5], one needs the fact that $\varprojlim\{X_\alpha\}$ is a compact space. □

REMARK 5.5. Theorem 5.4(iv) should be read in conjunction with Remark 3.7.

REMARK 5.6. In the spirit of Remark 3.8, we conjecture that, for an object X of pro-CW_0, there exist a tower Q in pro-CW_0 and a weak equivalence q: $Q \longrightarrow X$ if and only if each $\pi_k(X)$ is dominated in pro-Groups by a tower in pro-Groups.

6. **Algebraic criteria for stability in shape.** A pointed connected space Z has *strong shape dimension* [resp. *shape dimension*] $\leqslant n$ if there is an object X of pro-CW_0 [resp. pro-H_0] associated with Z such that CW-dim $X \leqslant n$. Although we shall not use the fact, it is worth noting that there is an object of pro-CW_0 associated with every topological space Z: one applies the Vietoris functor [19] based on locally finite open normal covers of Z exactly one of whose elements contains the base point [18], together with [4].

So that our Theorem 6.3 may be relevant, we prove

PROPOSITION 6.1. *If a (pointed connected) separable metric space Z has covering dimension $\leqslant n$, then Z has strong shape dimension $\leqslant 2n+1$.*

PROOF. Embed Z in euclidean $(2n+1)$-space [9]. The system of all connected open neighborhoods of Z, pointed by the base point of Z and bonded by inclusions, is an object of pro-CW_0. Even if Z is not closed, this object is associated with Z in the sense of Fox [8], see [10], and hence [18, Theorem 2.5] the induced object of pro-H_0 is associated with Z. □

REMARK 6.2. An n-dimensional compact metric space, being the inverse limit of nerves of covers, has strong shape dimension n.

Here is our stability theorem:

THEOREM 6.3. *Let Z be a pointed connected space whose strong shape dimension is finite. Then Z is pointed shape equivalent to a CW complex if and* *only if each pro-*$\pi_k(Z)$ *is dominated in pro-Groups by a group. This complex may be chosen to have CW-dimension* $\max\{3,$ *shape dimension of* $Z\}$, *and to be a bouquet of circles if the shape dimension of* Z *is* 1. *If, in addition,* Z *is compact, then* Z *is pointed shape dominated by a finite complex. In particular, the theorem holds when* Z *is a finite-dimensional separable metric space.*

PROOF. Immediate from Theorem 5.4 and Proposition 6.1. □

REMARK 6.4. When Z is compact, the theorem should be read in conjunction with Remark 3.7.

Note (added November 1975). Since this paper was submitted, a Whitehead Theorem in pro-H_0 more general than Theorem 5.3 has appeared, due to Morita [23]. As a result, Theorem 6.3 now holds for spaces of finite shape dimension.

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