CENTRALISERS OF $C^\infty$ DIFFEOMORPHISMS

BY

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ABSTRACT. It is shown that if $F$ is a hyperbolic contraction of $\mathbb{R}^n$, coordinates may be chosen so that not only is $F$ a polynomial mapping, but so is any diffeomorphism which commutes with $F$. This implies an identity principle for diffeomorphisms $G_1$ and $G_2$ commuting with an arbitrary Morse-Smale diffeomorphism $F$ of a compact manifold $M$: if $G_1, G_2 \in Z(F)$, then $G_1 = G_2$ on an open subset of $M$ implies $G_1 = G_2$ on $M$.

Finally it is shown that under a certain linearisability condition at the saddles of $F$, $Z(F)$ is in fact a Lie group in its induced topology.

Introduction. Let $f$ be a $C^\infty$ diffeomorphism of $\mathbb{R}^n$ onto itself which fixes the origin, and let $Df_0: \mathbb{R}^n \to \mathbb{R}^n$ be its first derivative at 0. We shall describe $f$ as a sink on $\mathbb{R}^n$ if it is hyperbolic and a topological contraction: i.e., (i) every eigenvalue $\lambda$ of $Df_0$ satisfies $|\lambda| < 1$ and (ii) $\cap_{n=0}^\infty U^n(0) = \{0\}$ for any bounded set $U$ containing the origin. The $k$-jet of $f$, denoted $f_k$, is an element of $L^k(\mathbb{R}^n)$, the group of $k$-jets of local diffeomorphisms of $\mathbb{R}^n$ which preserve the origin. We denote by $L^\omega(\mathbb{R}^n)$ the group of formal power series; $L^\omega(\mathbb{R}^n) = \lim\limits_{k \to \infty} L^k(\mathbb{R}^n)$.

The theorem of K.-T. Chen and S. Sternberg [2], [7] applied to sinks, implies that there is $k$ (computed from the eigenvalues of $Df_0$) such that the germ of $f$ is conjugate by a $C^\infty$ diffeomorphism germ $g$, to some $L^k$-conjugate of $f_k$, $g f g^{-1} = f_k$.

We describe the normal form $f_k$ in §2. Two sinks $f$ and $h$ are conjugate if and only if $f_k$ and $h_k$ are conjugate in $L^\omega(\mathbb{R}^n)$.

It is our purpose to show that the space of diffeomorphisms commuting with $f$ admits a finite dimensional parametrisation. The first main result is

**THEOREM 1.** Let $h$ be any local diffeomorphism of $\mathbb{R}^n$ such that $f_k h = h f_k$. Then $h = h_k$.

An obvious corollary is that conjugating functions $g$ in equation (I) are unique up to elements of the centraliser in $L^k(\mathbb{R}^n)$ of $f_k$.

The theorem is a generalisation of a corresponding theorem of N. Kopell [3] in which $\bar{f}_k = \bar{f}_1$, i.e., $f$ is linearisable. Thus by an argument in her paper,
there follows an identity principle for a sink, or for that matter, for a \( C^\infty \)
Morse-Smale diffeomorphism of a compact \( C^\infty \)-manifold \( M \) without boundary.
(For definitions, see [6].)

**Corollary 1.** If \( g_1 \) and \( g_2 \) both commute with \( f \), and if \( U \) is an open
set of \( R^n(M) \), then \( g_1 = g_2 \) on \( U \) implies \( g_1 \equiv g_2 \).

A second corollary may be adduced which describes the space of solutions
to the equation \([X, Y] = 0\), where \( X \) and \( Y \) are \( C^\infty \) vector fields on \( R^n \), and
the flow \( \phi_t \) generated by \( Y \) consists of hyperbolic sinks.

**Corollary 2.** Let \( Y \) be a vector field germ such that \( Y \) is an elementary
contracting critical point: i.e., the eigenvalues \( \mu \) of \( DZ_0 \) satisfy \( \text{Re}(\mu) < 0 \). Let
\( \mathfrak{g}(Y) = \{ X \mid [X, Y] = 0 \} \). Then \( \mathfrak{g}(Y) \) is the Lie algebra of \( Z(\phi_1) \) the centraliser
in \( L^k(n) \) of \( \phi_1 \) (\( k \) chosen as above).

In order to prove the theorem, we give in §2 a normal form \( \tilde{f}_k \) for \( f \); it
is related to the real Jordan form which the matrix \( f_k \) has in a certain faithful
linear representation of \( L^k(n) \). The jet \( \tilde{f}_k \) is an invariant of \( C^\infty \)-conjugacy of
sinks. Indeed, using the representation one can give an alternative proof of the
formal content of Sternberg's theorem.

The second main result is an extension of Theorem 1 to Morse-Smale dif-
feomorphisms. To a Morse-Smale diffeomorphism \( f \), and to an orbit \( \{ x, f(x),
\ldots, f^m(x) \} \) in \( \Omega(f) \), there is associated the spectrum of \( D(f^m)_x \). At least
one such orbit is a sink, for which all the eigenvalues of \( D(f^m)_x \) have absolute
value less than 1. A source for \( f \) is a sink for \( f^{-1} \) and any other point of
\( \Omega(f) \) is called a saddle.

Now let Diff\(^{\infty}\)(\( M \)) denote the topological group of \( C^\infty \) diffeomorphisms
of a compact manifold \( M \), topologised by the \( C^\infty \) topology (see, for example
[5]). Diff\(^{\infty}\)(\( M \)) contains \( Z(f) = \{ g \in \text{Diff}^{\infty}(M) \mid gf = fg \} \) as a closed subgroup.

**Theorem 2.** Suppose \( f \) is a Morse-Smale diffeomorphism such that at
each saddle \( y \),

\[
\lambda_i \neq \prod_{j=1}^{n} \lambda_j^{m_j},
\]

where if \( m \) is the period of \( f \) on \( y \), \( \lambda_q \) are the eigenvalues of \( D(f^m)_y \), \( m_i \) are
nonnegative integers, and there is \( j \neq i \) with \( m_j \neq 0 \). Then as a topological
group \( Z(f) \) is equivalent to a Lie group.

In a subsequent paper we prove that generically \( Z(f) \) is in fact discrete;
such \( f \) cannot, for example, be a time-one map for a \( C^\infty \) vector field.

This paper was developed from part of my Ph.D. thesis at the University
of California at Berkeley; I would like to particularly thank my supervisor
M. Hirsch for his encouragement and help.
1. Preliminaries. We shall adopt some notations of [4], which we briefly recall. $C^\infty(R^n)$ denotes the local ring of $C^\infty$ mapping germs at the origin and $M$ its maximal ideal. Analogously, let $N$ be the ideal in $R[x_1, \ldots, x_n]$ of polynomials without constant term. $N$ is generated as a real vector space by the monomials—we denote the monomial $x_1^{i_1} \cdots x_n^{i_n}$ by $x^I$, where $I = (i_1, \ldots, i_n)$ is an ordered $n$-tuple of nonnegative integers. For such $I$, $I! = i_1! \cdots i_n!$ and $\deg I = i_1 + \cdots + i_n$. We shall need the fact that there is a canonical identification of $N$ with $\bigoplus_{i=1}^\infty O^iV$ (where $O^iV$ denotes the $i$th symmetric power of $V \cong R^n$) under which monomials in $x$ are identified with monomials in the standard basis of $V$.

The Taylor homomorphism $j_k: M \to W$ is defined by

$$j_k(f) = \sum_{I} \frac{\partial^{\deg f}}{\partial x^I}(0)x^I,$$

the sum being over all $I$ with $1 \leq \deg I \leq k$. As is well known, $j_k$ induces an isomorphism $M/M^{k+1} \cong N/N^{k+1}$, this quotient being denoted $J^k(n, 1)$, the ring of $k$-jets of vanishing real $C^\infty$ functions. Having introduced $N$, we may speak of “complex $k$-jets”: these are elements of $N/N^{k+1}$ when $N$ is the ideal of complex polynomials in $n$-variables without constant term. $J^k(n, n) = J^k(n, 1) \otimes R^n$ is the space of $k$-jets of functions $R^n$ to $R^n$ and $L^k(n) \subset J^k(n, n)$ the group of jets invertible for the operation of composition.

There is a right linear action of $L^k(n)$ on $J^k(n, 1)$ defined by $x^I \cdot g = j_k(\phi^I)$ where $g = (g_1, \ldots, g_n)$ is in $L^k(n)$. In the basis of monomials, $g$ has the matrix $G_{IJ} = \text{coefficient of } g^I \text{ on } x^J$. Hence by sending $g$ to the adjoint of the linear map $G_{IJ}$, there is obtained a representation $\rho_k: L^k(n) \to GL(J^k(n, 1))$. The matrix of $\rho_k(g)$, in the basis of monomials, is $G_{IJ} = \text{coefficient of } g^I \text{ on } x^J$. The indices $I, J$ range over all monomials in $n$ variables of degree not greater than $k$.

The following facts are easy to see.

(i) $\rho_k$ is faithful (and presents $L^k(n)$ as an algebraic group).

(ii) If $\deg I = 1 = \deg J$, $(\rho_k(g))_{IJ}$ is the Jacobian matrix of $g$. $\rho_k(g) = Dg$. Moreover, if $1 < r < k$ and $\deg I = r = \deg J$, $(\rho_k(g))_{IJ}$ is the matrix of $Dg \circ \cdots \circ Dg: O^rV \to O^rV$ (in the basis of monomials). This implies that the eigenvalues of $\rho_k(g)$ are all monomials $\lambda^I$ of degree $\leq k$ in the eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ of $Dg$.

(iii) If $f$ is a diffeomorphism of $R^n$, and if $\deg I = 1, \deg J = r < k$, then $\rho_k(f^I)_{IJ}$ is the matrix of $Df^I: O^rV \to V$ taken in the basis $\{|I|x^I|\deg J = r\}$.

2. Normal forms of diffeomorphisms. 1-jets. Let $A$ be a real linear operator. Then there is a basis $\{z_1, z_1', \ldots, z_r, z_r', w_{2r+1}, \ldots, w_n\}$ of $C^n$ in which
the matrix of $A$ is in Jordan form: if $z_i$ is a (generalised) eigenvector for $\lambda$, so is $\overline{z}_i$ for $\overline{\lambda}$, and $w_r$ are the real eigenvectors. By the real Jordan form of $A$, we mean the matrix which $A$ takes in the basis $\{\text{Re}(z_1), \text{Im}(z_1), \ldots, \text{Im}(z_r), w_{2r+1}, \ldots, w_n\}$.

**Example.**

$$
\begin{pmatrix}
\text{Re}\lambda & \text{Im}\lambda & 1 & 0 \\
-\text{Im}\lambda & \text{Re}\lambda & 0 & 1 \\
0 & 0 & \text{Re}\lambda & \text{Im}\lambda \\
0 & 0 & -\text{Im}\lambda & \text{Re}\lambda \\
\end{pmatrix}
\sim
\begin{pmatrix}
\lambda & 0 & 1 & 0 \\
0 & \overline{\lambda} & 0 & 1 \\
0 & 0 & \lambda & 0 \\
0 & 0 & 0 & \overline{\lambda} \\
\end{pmatrix}.
$$

Notice that there is an involution $z_i \leftrightarrow \overline{z}_i$ of the basis of $\mathbb{C}^n$ which induces a corresponding involution $\sigma$ of the monomials, which we denote by $z^I \leftrightarrow z^{\sigma(I)}$. If $z^I = z_1^{i_1}z_2^{i_2}\ldots z_r^{i_r}w$, $z^{\sigma(I)} = z_1^{j_1}z_2^{j_2}\ldots z_r^{j_r}w$.

**Complex jets.** A complex $k$-jet or formal power series ($\infty$-jet) $F$ is in normal form providing:

(i) $\rho_1(F_1) = \text{Jordan normal form}$,
(ii) $\rho_r(F_r)$ is upper triangular for $k \geq r \geq 1$,
(iii) $(\rho_r(F_r))_{IJ} \neq 0 \Rightarrow \lambda^I = \lambda^J$.

**Real jets.** A real $k$-jet or formal power series $F$ is in normal form providing:

(i) $\rho_1(F_1)$ is in real Jordan normal form, so there is a matrix $A_1$, being a sum of blocks ($\begin{pmatrix}1 & 1 \\ 0 & 0\end{pmatrix}$) so that $A_1 \rho_1(F_1)A_1^{-1} = \text{Jordan form of } \rho_1(F_1)$.
(ii) The complex power series $A_1 F_r A_1^{-1}$ is in normal form for each $k \geq r \geq 1$.

It follows from (ii) that $(A_k)_{IJ} = (A_k)_{\sigma(I)\sigma(J)}$, and conversely, if this condition is satisfied for a complex formal power series or jet $A_k$, the eigenvalues of whose linear part occur in conjugate pairs, then $A_k$ is derived from a real f.p.s. via conjugation by $A_1$ (see [1]).

A diffeomorphism germ is in normal form if its associated formal power series is in real normal form.

**Remarks.** (1) The normal form is a conjugacy invariant of formal power series or jets, or diffeomorphisms, but it need not be unique. For instance, if $\lambda \neq 1$ is real, the 3-jets

$$
F(x, y, z) = (\lambda x, \lambda^2 y + x^2, \lambda^3 z + xy + x^3),
$$

$$
G(x, y, z) = (\lambda x, \lambda^2 y + x^2, \lambda^3 z + xy)
$$

are distinct normal forms which are conjugate in $L^3(3)$, by $I + (0, x^2, 0)$.
The matrix \( \rho_k(F_{\infty}) \) will in general not be in Jordan form for \( k > 1 \). The following lemma is a generalisation of a lemma of Sternberg [7].

**Lemma.** For any (real) invertible FPS \( F_{\infty} \), there is a (real) invertible FPS \( G \) with \( GF_{\infty}G^{-1} \) in (real) normal form.

**Proof.** We construct \( G = \lim G_k \) by induction on \( k \). If \( k = 1 \), this is the familiar real Jordan normal form theorem for the Jacobian \( F_1 \). For the inductive step, we observe that if \( \rho_k(F_k) \) is in (real) normal form, so is \( \rho_{k+1}(F_k) \). (Any entry \( \langle f^I, x^J \rangle \) of \( \rho_{k+1}(F_k) \) satisfies \( \langle f^I, x^J \rangle = \sum \langle f^{I'_1}, x^{J'_1} \rangle \langle f^{I'_2}, x^{J'_2} \rangle \) where the sum is over all monomials \( f^{I'_1}f^{I'_2} = f^I \) and \( x^{J'_1}x^{J'_2} = x^J \). By inductive hypothesis, some summand can be nonzero only when \( \lambda^{I'_1} = \lambda^{J'_1} \) and \( \lambda^{I'_2} = \lambda^{J'_2} \), in which event \( \lambda^I = \lambda^J \). As before, \( \lambda^I \) denotes a monomial in the eigenvalues of \( F_1 \).

It follows that if \( G_k \) is chosen so that \( \rho_k(G_kF_kG_k^{-1}) \) is in (real) normal form, then \( \rho_{k+1}(G_kF_{k+1}G_k^{-1}) \) will be in (real) normal form except possibly for entries \( f_{iJ} \) with \( \deg i = 1 \), \( \deg J = k + 1 \). (These entries represent the contribution of \( D^kF : O^kV \rightarrow V \).) Dividing up

\[
J^{k+1}(n, 1) = \frac{M}{M^2} \oplus \sum_{1 < r < k+1} \frac{M^r}{M^{r+1}} \oplus \frac{M^{k+1}}{M^{k+2}}
\]

we see that \( \rho_{k+1}(F_{\infty}) \) has, in this decomposition, the form

\[
X = \begin{bmatrix} A & B & D \\ 0 & C & E \\ 0 & 0 & F \end{bmatrix},
\]

where

\[
\begin{pmatrix} A & B & 0 \\ 0 & C & E \\ 0 & 0 & F \end{pmatrix}
\]

is in normal form. \( A \) is the Jacobian and \( F \) is its symmetric power of degree \( (k + 1) \). Conjugating \( X \) by a matrix

\[
Y = \begin{bmatrix} I & 0 & G \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}
\]

we obtain \( D - AG + GF \) in the top right-hand corner.

To prove the lemma is to solve \( (D - AG + GF)_{iJ} = 0 \) when \( \lambda^I \neq \lambda^J \); i.e., \( D_{iJ} = \Sigma A_{ij}G_{JK} - \Sigma G_{iK}F_{KJ} \) with \( 1 \leq j \leq n \) and \( \deg K = k + 1 \).
Using the assumption that $p_{k+1}(F_k)$ is upper triangular (as a complex matrix), this equation becomes

$D_{ij} = G_{ij}(A_{ii} - F_{jj}) + \sum_{i \neq j} A_{ij} G_{jj} - \sum_{k < J} G_{ik} F_{kJ}.$

($\ast$)

Supposing $G_{jj}$ is known for $j < i$, $G_{ik}$ for $K < J$, we may solve inductively for $G_{ij}$ when $A_{ii} \neq F_{jj}$ (i.e. $\lambda^i \neq \lambda^j$).

To prove the statement for real jets, we must know that the basis change taking $F_1$ into its real Jordan form takes some real jet into $Y$. According to Birkhoff [1], the condition that this be so is that $(y^I, x^J) = (\sigma(I), x^{\sigma(J)})$, where the involution $\sigma$ on monomials is defined as above. It is clear in equation ($\ast$) that if $D, A, F$ satisfy the condition, $G$ will satisfy it by the same induction when $\lambda^I \neq \lambda^J$; and otherwise the choice of $G_{ij} = 0$ also satisfies the condition.

Remarks. (1) The lemma above constitutes the formal content of the theorem of Chen and Sternberg as in [2], [7] and [8], in the sense that by choosing a diffeomorphism with $G_\infty$ as above, we may conjugate $F$ by $G$ to a diffeomorphism whose jet is in normal form. The analytic content is then to prove that two $C^\infty$ hyperbolic diffeomorphisms with the same normal form are $C^\infty$-conjugate.

(2) If $\lambda^I \neq \lambda^J$ for any monomial $\lambda^I$ of degree $> 1$, the normal form is linear, whereupon $F$ is $C^\infty$ conjugate to a linear map.

(3) If $F$ is a sink (source), there is $k$ such that $\lambda^I = \lambda^J \Rightarrow \deg I \leq k$, so that the normal form of $F$ is a $k$-jet, to which $F$ is $C^\infty$-conjugate.

3. Centralisers. The crucial observation of this section is that if $L: V \rightarrow V$ is a linear map, and if $V = \sum V_i$ is its decomposition into generalised eigenspaces, then if $LM = ML$, each $V_i$ is $M$-invariant. Applying this fact to $p_k(F_k)$ in normal form, if $G_k$ is a commuting jet, then $p_k(G_k)$ also has the property $p_k(G_k)_{ii} \neq 0 \Rightarrow \lambda^I = \lambda^J$; although of course it need not be upper triangular. Furthermore, if $F_k$ is a contraction and $k$ is not less than the maximal degree on the right-hand side of relations $\lambda^I = \lambda^J$, then if $G_\infty$ commutes, $G_\infty = G_k = G_{k+r}$ for all $r$. In particular, $F^{-1}_k$ (or $G^{-1}_k$) coincides with the inverse of $F_k$ in $L^\infty(n)$, and so the polynomial mappings $F_k, F^{-1}_k, G_k, G^{-1}_k$ are globally defined diffeomorphisms of $\mathbb{R}^n$.

The theorem is an application of the following

**Lemma.** Let $F_k$ be a contracting jet in normal form with $k$ as above. Considered as a polynomial diffeomorphism of $\mathbb{R}^n$, for $x, y$ in a compact $K \subset \mathbb{R}^n$, we have

\begin{align*}
(1) & \quad |F^{-m}_k(x) - F^{-m}_k(y)| \leq p(m) \lambda^k m |x - y|, \\
(2) & \quad |F^m_k(x) - F^m_k(y)| \leq q(m) \lambda^k m |x - y|,
\end{align*}
where $p, q$ are polynomials; $\lambda^{-1} =$ smallest eigenvalue of $F_1$, $\mu =$ largest eigenvalue of $F_1$.

**Proof.** We prove (1). Let $\alpha: V \to S^k V$ be defined by $x \mapsto x^I$; i.e., if 
\{e_j\} is a basis of $V$ and the symmetric products of these vectors are denoted $e^I$, then the coefficient of $\alpha(x)$ on $e^I$ is $x^I$. Clearly $\alpha$ has a uniform Lipschitz constant on $K$.

If $\pi$ is the projection $S^k V \to V$, then $F_k^m(x) = \pi \circ \rho(F_k)^m \circ \alpha(x)$ for any $m \in Z$, because $F_k$ is in normal form. Hence

$$|\pi \circ \rho(F_k)^m \circ \alpha(x) - \pi \circ \rho(F_k)^m \circ \alpha(y)| \leq |\pi| |\rho(F_k)^m| |\alpha(x) - \alpha(y)|$$

$$\leq \text{const} |\rho(F_k)^m| |x - y|.$$

We may write $\rho(F_k) = SU$ where these two linear maps commute, $S$ is semisimple—we may suppose diagonal—and $U$ is unipotent. Then $\rho(F_k)^{-m} = S^{-m} U^{-m}$ and

$$|\rho(F_k)^{-m}| \leq |S^{-m}| |U^{-m}| \leq \lambda^m p(m).$$

**Remark.** $(\deg p) + 1 =$ nilpotence degree of $(U - I)$.

**Proof of Theorem 1.** According to the observation at the head of this section, with $k$ as above, if $g$ commutes with $F_k$, then for all $r \leq \infty$, $\rho_{k+r}(g) = \rho_k(g)$. Let $g_k$ be the polynomial diffeomorphism of degree $k$ such that $\rho_k(g_k) = \rho_k(g)$. Then $g_k F_k = F_k g_k$, so that we may write $g g_k^{-1} = I + h$, commuting with $F_k$, and such that $h(x) = O(x^r)$ on a small enough neighbourhood $U_r$ of 0 (for any $r$). In fact, $h \equiv 0$. Since $h(x) = (I + h - I)(x)$, we have

$$|h(x)| \leq |F_k^{-m}(I + h) F_k^{-m} - F_k^{-m} F_k^{-m}(x)| \leq p(m) \lambda^m k |(I + h) F_k^{-m} - F_k^{-m}(x)|$$

$$\leq p(m) \lambda^m |h(F_k^{-m}(x))| \leq p(m) \lambda^m |F_k^{-m}(x)|^r$$

$$\leq \text{const} p(m) \lambda^m q(m) \mu^m r |x|^r.$$

Choosing $r$ and $U_r$ such that $\lambda \mu^r < 1$, and taking the limit of the right-hand side as $m \to \infty$, we see $h(x) \equiv 0$, because the exponential convergence of $(\lambda \mu^r)^m$ dominates the polynomial convergence of $p(m) q(m) \mu^m r |x|^r$.

4. Centralisers of Morse-Smale diffeomorphisms. In this section we prove Theorem 2, stated in the Introduction. If $p$ is the number of points in $\Omega(f)$, since $Z(f)$ acts on $\Omega(f)$, there is a homomorphism from $Z(f)$ to $S_p$ whose kernel contains the identity component $Z(F)_0$: we show $Z(F)_0$ is a Lie group. Moreover, we may suppose $\Omega(f)$ consists of fixed points because $Z(f) \subset Z(f^r)$ for any $r$.

If $\{S_j\}$ are the sources in $\Omega(f)$ and $\{N_j\}$ the sinks, we may choose coordinates for their unstable and stable manifolds, $w^u(S_j)$ and $w^s(N_j)$, so that $f$ is in normal form, and there are Lie groups $G_i = Z(f|w^u(S_j))$, $G_j = Z(f|w^s(N_j))$. Then
if $g \in Z(f)$ acts trivially on $\Omega(f)$, $g$ leaves invariant the $w^u(S_j)$ and $w^s(N_j)$, and so $g|_{w^u(S_j)} \in G_i$, $g|_{w^s(N_j)} \in G_j$. Hence there is defined a homomorphism $R: Z(f)_0 \to \Pi_{i,j} G_i \times G_j$.

By the identity principle this is injective, and it is continuous for the $C^\infty$ topology on $Z(f)_0$. The conclusion will follow from the fact that $R$ is a closed map. In other words, if $g_m$ are $C^\infty$ diffeomorphisms in $Z(f)_0$, such that $g_m \to g$ $C^\infty$-uniformly on compacta in $\bigcup w^u(S_j) \cup w^s(N_j) = M \sim \{saddles\}$, then the convergence is in fact uniform on $M$; in particular, $g$ is $C^\infty$ and hence in the image of $R$.

We restrict attention to a saddle, about which, by assumption (see [8]), coordinates may be chosen making $f$ linear. In this coordinate system, there are unique $f$-invariant linear subspaces $E^u$ and $E^s$ which span $R^n$; the expanding and contracting eigenspaces of the (now linear) map $f$. The diffeomorphisms $g_m$, hence the map $g$, leave these subspaces invariant, whereupon from Theorem 1, $g_m|_{E^u}$ and $g_m|_{E^s}$ are already uniformly convergent on compacta in $E^u \cup E^s$; it follows that $g|_{E^u \cup E^s}$ is $C^\infty$.

Since $f$ is linear, $f = f^u \times f^s: E^u \times E^s \to E^u \times E^s$. This means $f$ commutes with each $g_m^u \times g_m^s$, and therefore with $g_m^u \circ (g_m^u \times g_m^s)^{-1}$. This sequence is the identity along $E^u \cup E^s$, and converges uniformly on all compacta to $g$ if and only if $g_m \to g$ uniformly on all compacta. Hence, we may make the simplifying assumption that $g_m = g = I$ on $E^u \cup E^s$.

The first step is to show uniform $C^0$-convergence. Let $g_m = S_m + U_m$ be the coordinate expansion of $g_m$; thus, $S_m(U_m): R^n \to E^u(E^u)$. Then $f^{-r}U_m f^r = U_m$ and $f^t S_m f^{-t} = S_m$ for every $r, t \in \mathbb{Z}$ and every $m$, by the linear and hyperbolic properties of $f$. Consequently, in any norm

$$|g_m(x) - g(x)| \leq |f^{-r}(U_m f^r(x) - U f^r(x))| + |f^t(S_m f^{-t}(x) - S f^{-t}(x))|.$$  

If $W \subset R^n - \{0\}$ is a compact set containing fundamental regions for $f^u$ and $f^s$, then by uniform convergence on $W$, for high $m$, $|U_m - U| + |S_m - S| < \epsilon$ on $W$. But for $x$ in any compact neighborhood of zero, there are numbers $r, t$ so that $f^r(x)$ and $f^{-t}(x)$ are in $W$. The result follows from the fact that $f^{-r}$ acts contractively on $E^u$ and $f^s$ contractively on $E^s$.

The argument for convergence of the higher derivatives is more delicate: we observe that $D^k f = 0$ for $k > 2$ ($f$ is linear). Then by applying the chain rule ($f = Df$ is a constant linear map)

$$\tag{**}(Df)^r (D^k g_m(x) (Df \times \ldots \times Df)^{-r} = D^k g_m(f^r x)$$

for all $k, r, m, x$. This equation is in fact the $r$th iteration of the linear operator $F(A) = Df \circ A \circ (Df \times \ldots \times Df)^{-1}$ operating on $I^k_{sym}$ the space of symmetric...
If $L^k_{sym} = L^u \oplus L^s \oplus L^c$ is the decomposition into eigenspaces for $F$ for the eigenvalues of absolute value $> 1$, $< 1$, and $= 1$ respectively, then the argument used in the $C^0$ case above may be applied to show that the $L^u$ and $L^s$ components of $D^k g_m - D^k G$ tend to zero as $x \to 0$. It remains to settle the component on $L^c$. Note that for $F|_{L^c}$, with a norm on $L^k_{sym}$, $|F^r| \leq p(r)$ for some polynomial $p$.

We claim that $G_m$ has infinite contact with the identity along $E^u \cup E^s$. If $x \in E^s$, we see that $F^r(D^k g_m^c(x)) \to D^k g_m^c(0)$ as $r \to \infty$. For a linear map, this can only happen for a fixed point, and so $D^k g_m^c$ is constant along $E^s$ (similarly, $E^u$, using $F^{-1}$). In fact, for every $k > 1$, $D^k g_m^c(0) = 0$, since, any jet $J_k$ commuting with the linear map $f = Df_0$, satisfies $J_k = J_1$ by the assumption $\lambda_i \neq \prod \lambda_j^m$. But $g_m = I$ along $E^u$ and $E^s$ (by the simplifying assumption above), and this implies that $J_1 = I$: this proves the claim.

We can now show that $(g_m)$ is uniformly convergent on compacta. By (**), for the $L^c$ component, we have

$$|D^k g_m(x) - D^k g(x)| = |F^{-r}(D^k g_m(f^r(x))) - F^{-r}(D^k g(f^r(x)))|$$

$$\leq p(r)|D^k g_m(f^r(x)) - D^k g(f^r(x))|$$

$$\leq p(r)(|D^k (g_m - g)(f^r(x)) - D^k (g_m - g)(z)| + |D^k (g_m - g)(z)|).$$

For any $x$ in a small compact neighborhood of $0$, there is $r$ such that $f^r(x) \in W$, then $z \in E^u$ may be chosen to minimise the distance between $f^r(x)$ and $E^u$. By the preceding, $D^k (g_m - g)(z) = 0$. Now applying the Mean Value Theorem, where $|A|_W$ denotes supremum on $W$,

$$|D^k (g_m - g)(f^r(x)) - D^k (g_m - g)(z)| \leq |D^{k+1} (g_m - g)|_W |f^r(x) - z| \text{ const } \lambda^r$$

for some $0 < \lambda < 1$.

Since exponential convergence still dominates polynomial convergence, this completes the proof. $g_m$ is for each $k$, $C^k$-uniformly convergent on compacta, $R$ is a closed map, and $g$ is a $C^\infty$ diffeomorphism.

REFERENCES


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