PRINCIPAL CO-FIBER BUNDLES

BY

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ABSTRACT. Principal co-fiber bundles are defined in the category of topological groups. They are Eckmann-Hilton duals of principal fiber bundles. A classification theorem is provided as well as an example which almost represents the most general case.

1. **Introduction.** The Eckmann-Hilton duality [4] works well as long as we are in the homotopy category. However in the category of topological spaces there are no duals to principal fiber bundles (p.f.b.), since any set with a comultiplication with a strict co-unit reduces to a point. This situation is avoided in [9] by turning to the category of simplicial groups. In this paper we define principal co-fiber bundles (p.c.b.) in the category of topological groups. We face very few surprises in the algebraic part of the problem since this part is similar to the study in [9]. The difficulties are in the topology.

The main result is the existence of two adjoint functors δ and ϵ which correspond p.f.b.'s to p.c.b.'s and under certain conditions are inverses to each other. These functors are applied to classify the p.c.b.'s. The proof of the classification theorem relies heavily on Milnor's work in [7]. We also present a typical example which in fact almost represents the most general case. This class of examples, up to homotopy equivalence, is the loops of a principal cofibration.

In §3 we point out a category of p.f.b.'s. §4 contains the definition of a category of p.c.b.'s. In §§5, 6, and 7, we define the two functors and describe the relations between them. The classification of p.c.b.'s is in §8. The example is in §9. Some proofs are in §10.

All spaces are pointed with base point e, which denotes also the identity of the groups. The disjoint union identifying base points is denoted by $\vee$.

I would like to thank Professor I. Berstein for suggesting the problem and for many discussions with regard to this problem.

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2. Preliminaries.

DEFINITION A. [1] $F(X)$ is a free topological group with basis (X, e), a pointed topological space, if $F(X)$ has the following properties: (i) $F(X)$ is a topological group with subspace X. (ii) $X - e$ generates $F(X)$ freely as a group. (iii) For any continuous map $\phi: X \longrightarrow G$, where G is a topological group, and $\phi(e)$ equals the identity element of G, then ϕ extends to a continuous homomorphism $\Phi: F(X) \longrightarrow G$.

DEFINITION B. (a) A k_W space is a topological space X with compact subsets X_n, such that: (i) $X = \bigcup_{n=1}^{\infty} X_n$. (ii) $X_{n+1} \supset X_n$ for all n. (iii) A subset A of X is closed if and only if $A \cap X_n$ is compact for all n. (b) By a k_W decomposition, $X = \bigcup X_n$, we mean that X_n have properties (i), (ii) and (iii). (c) A k_W group is a topological group which is also a k_W space.

THEOREM A. (a) [1] *For any completely regular space X, the group $F(X)$ exists.* (b) [6] *Let $X = \bigcup_{n=1}^{\infty} X_n$ be a k_W decomposition of X. Then $F(X)$ exists and has the weak topology of $F_n(X)$.* (c) [6] *$F(X)$ also has a k_W decomposition $F(X) = \bigcup_{n=1}^{\infty} F_n(X_n)$. The sets $F_n(X)$ and $F_n(X_n)$ are defined as follows: Let X^{-1} be a copy of X and let us denote $\widetilde{X} = X \vee X^{-1}$ the disjoint union of X and X^{-1}, identifying the base points. $F_n(X)$ is the image of the projection of $(\widetilde{X})^n$ into $F(X)$, where the projection sends a word to its reduced form in $F(X)$. $F_n(X)$ is also given the induced topology of the projection described above. $F_n(X_n)$ is defined similarly.*

DEFINITION C. [2] Let G_1 and G_2 be two topological groups. Then, the topological group $G_1 * G_2$ is said to be a free topological product of G_1 and G_2, if it has the following properties: (i) G_1 and G_2 are subgroups of $G_1 * G_2$, with the subspace topology. (ii) The underlying group of $G_1 * G_2$ is their free product as groups. (iii) If γ_1 and γ_2 are continuous homomorphisms of G_1 and G_2 into a topological group H, then there exists a continuous homomorphism $\Gamma: G_1 * G_2 \longrightarrow H$ such that $\Gamma = \gamma_i$ on G_i, $i = 1, 2$.

THEOREM B. (a) [2] *Let G_1, G_2 be topological groups, then $G_1 * G_2$ exists.* (b) [8] *Let $G_i = \bigcup_j G_i^j$ be k_W decompositions for the topological groups G_1, G_2. Then, $G_1 * G_2 = \bigcup_j h((G_1^j \vee G_2^j)^j)$ is a k_W decomposition where h sends a word to its reduced form in $G_1 * G_2$.*

DEFINITION D. The pair $\langle G, \psi \rangle$ where G is a topological group and ψ a homomorphism $\psi: G \longrightarrow G * G$ is a co-group if the co-multiplication ψ satisfies the following:

(i) $(1_G * \psi)\psi = (\psi * 1_G)\psi$,

(ii) $(C * 1_G)\psi = 1_G$ and $(1_G * C)\psi = 1_G$,

where $C: G \longrightarrow Q$ is the homomorphism into the group of one element, and G is identified with $G * Q$.

THEOREM C. [5] *Let X' be a copy of X, and $x' \in X'$ will correspond to $x \in X$. Then the homomorphism $\psi: F(X) \longrightarrow F(X) * F(X')$ such that $\psi(x) = x \cdot x' \in F(X) * F(X')$ is a co-multiplication for $F(X)$. This homomorphism is called the associated co-multiplication with X.*

Let X be a subset of the group H. Then $G_n(X)$ denotes all reduced words in X whose length is not larger than n.

3. **A category of p.f.b's.** In this section we consider p.f.b.'s with their standard terminology [3], but not in the usual approach. In our study we do not keep the fiber fixed, but rather the base. Consequently the morphisms have to be changed. To keep our notation straight, for a given fixed topological space X, we define the objects and morphisms in the category.

DEFINITION 1. [3] Let A be a topological group acting effectively with continuous translation functions on the right of a topological space S. If the space of orbits of S with the identification topology is homeomorphic to X, then the triple $\langle S, p, A \rangle$ is a p.f.b., where p is the map sending $s \in S$ to $p(s) \in X$, which corresponds under the homeomorphism to the orbit of s.

DEFINITION 2. Let $\alpha_i = \langle S_i, p_i, A_i \rangle$, $i = 1, 2$, be p.f.b.'s. A morphism from α_1 to α_2 is a pair $\langle f, f_S \rangle$ where $f_S: S_1 \longrightarrow S_2$ is a map, and $f: A_1 \longrightarrow A_2$ is a homomorphism such that: (i) $f_S(sa) = f_S(s) \cdot f(a)$, (ii) $p_2 f_S(s) = p_1(s)$.
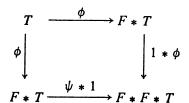
DEFINITION 3. Let $\alpha_1 = \langle S_1, p_1, A_1 \rangle$ be a locally trivial fiber bundle with a system of transition functions $\{g_{ij}\}$ relative to the open cover $\{U_i\}$, and let $f: A_1 \longrightarrow A_2$ be a homomorphism into a topological group A_2. Then the p.f.b. $f_*(\alpha_2) = \langle S_2, p_2, A_2 \rangle$ constructed by the system of transition functions $\{f, g_{ij}\}$ relative to $\{U_i\}$ is called the co-induced bundle of α_1 by f. If $f_S: S_1 \longrightarrow S_2$ is the map induced by f, we get a morphism $\langle f, f_S \rangle: \alpha_1 \longrightarrow \alpha_2$.

PROPOSITION 1. *Let α_1 and α_2 be as in Definition 3. If X, S_1, A_1, A_2 are k_W spaces so is S_2.* (*Proof in* §10.)

We denote the category of locally trivial p.f.b.'s with morphisms as in Definition 2 by $\bar{F}$.

4. **The category of p.c.b.'s.** We start this section with the definition of a p.c.b. Later on we will restrict our interest to a special kind of p.c.b.

DEFINITION 4. The tuple $\langle T, \phi, A, F, \psi \rangle$ is called a p.c.b. with T the total space, ϕ the co-operation, A the co-base, F the co-fiber if the following axioms hold: (i) $\langle F, \psi \rangle$ is a co-group. (ii) The continuous homomorphism ϕ makes the following diagram commutative:

$$T \xrightarrow{\ \phi\ } F * T$$

$$\phi \downarrow \qquad\qquad \downarrow 1 * \phi$$

$$F * T \xrightarrow{\ \psi * 1\ } F * F * T$$

(iii) If r_1 and r_2 are the projections of $F * T$ on F and T respectively then $r_2\phi = 1_T$ and $r_1\phi$ is an epimorphism. (iv) The group A is imbedded by an isomorphism $i\colon A \longrightarrow T$ onto the subgroup, $I = \{t \in T | \phi(t) \in T \subset F * T\}$ of the invariant elements of T relative to ϕ.

A morphism between the p.c.b.'s with the same co-group is defined next:

DEFINITION 5. Let $\beta_i = \langle T_i, \phi_i, A_i, F, \psi \rangle$, $i = 1, 2$, be p.c.b.'s. Then $\langle f, f_T \rangle\colon \beta_1 \longrightarrow \beta_2$ is a morphism if the following diagram of groups and homomorphisms commutes:

$$
\begin{array}{ccc}
A_1 & \xrightarrow{\ f\ } & A_2 \\
{\scriptstyle i_1}\big\downarrow & & \big\downarrow{\scriptstyle i_2} \\
T_1 & \xrightarrow{\ f_T\ } & T_2 \\
{\scriptstyle \phi_1}\big\downarrow & & \big\downarrow{\scriptstyle \phi_2} \\
F * T_1 & \xrightarrow{\ 1_F * f_T\ } & F * T_2
\end{array}
$$

PROPOSITION 2. *Let $\beta_1 = \langle T_1, \phi_1, A_1, F, \psi \rangle$ be a p.c.b., and let $f\colon A_1 \longrightarrow A_2$ be a homomorphism into the topological group A_2. Then there exists a p.c.b. $\beta_2 = \langle T_2, \phi_2, A_2, F, \psi \rangle$, and a morphism $\langle f, f_T \rangle\colon \beta_1 \longrightarrow \beta_2$. The p.c.b. β_2 is called the p.c.b. induced from β_1 by f and will be denoted by $f_*(\beta_1)$.*

PROOF. Let T_2 be the quotient group with the quotient topology of $T_1 * A_2$ relative to the relations $a^{-1} \cdot f(a) = e$ for all $a \in A_1$. We denote the quotient homomorphism by q. The homomorphism i_2 is the restriction of q to A_2, and the homomorphism f_T is the restriction of q to T_1. The co-operation ϕ_2 is the unique homomorphism making the following diagram commutative:

$$
\begin{array}{ccc}
T_1 * A_2 & \xrightarrow{\ \phi_2 * 1_{A_2}\ } & F * T_1 * A_2 \\
{\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle 1_F * q} \\
T_2 & \xrightarrow{\ \phi_2\ } & F * T_2
\end{array}
$$

PROPOSITION 3. *Let β_1 and β_2 be as in Proposition 2. If A_1, T_1, A_2 are k_W groups so is T_2. (The proof is in §10.)*

We will be interested in p.c.b.'s with a fixed co-group of the following type: $(F(X), \psi)$, where ψ is the co-multiplication associated with X. Of course for the existence of $F(X)$ we require X to be completely regular from now on. Because of this restriction we can abbreviate the notation of a p.c.b. to $\langle T, \phi, A \rangle$.

DEFINITION 6. $\langle T, \phi, A \rangle$ *is called a locally trivial p.c.b. if the following holds:* There exists a system $\{g_\alpha : U_\alpha \longrightarrow T\}$ of continuous maps such that $\{U_\alpha\}$ is an open cover of X, and $r_1 \phi g_\alpha = 1_{U_\alpha}$ for all U_α of the cover (r_1: $F(X) * T \longrightarrow F(X)$ the projection). The $\{g_\alpha\}$ are called the sections of $\langle T, \phi, A \rangle$.

PROPOSITION 4. *Let β_1 and β_2 be as in Proposition 2. Then if β_1 is a locally trivial p.c.b. so is $f_*(\beta_1) = \beta_2$.*

PROOF. If $\{g_\alpha\}$ are the sections of β_1 then $\{f_T g_\alpha\}$ are the sections of β_2. We denote the category of locally trivial p.c.b.'s by C.

5. The restriction functor δ. In this section we reveal the technique by which we will study the structure of p.c.b.'s, via the well-known structure of associated p.f.b.'s.

Let $\beta_i = \langle T_i, \phi_i, A_i \rangle \in C$, $i = 1, 2$. Then, $T_{iS} = \phi^{-1}(X \times T_i)$, and ϕ_{iS}: $T_{iS} \longrightarrow X$ is defined as follows: $\phi_{iS} = r_1 (\phi_i|_{T_{iS}})$, with r_1 as in Definition 6. If $\langle f, f_T \rangle$: $\beta_1 \longrightarrow \beta_2$ is a morphism in C, we denote by f_{TS}: $T_{1S} \longrightarrow T_{2S}$ the map $f_T|T_{1S}$. This construction makes sense because $X \times T_i$ is imbedded in $F(X) * T_i$ [8].
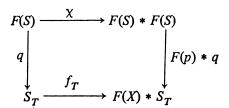
THEOREM 1. *The function δ which assigns to each $\beta = \langle T, \phi, A \rangle \in C$ the triple $\delta(\beta) = \langle T_S, \phi_S, A \rangle$ and to each morphism $\langle f, f_T \rangle$: $\beta_1 \longrightarrow \beta_2 \in C$ the pair $\langle f, f_{TS} \rangle$ is a functor from C to $\bar{F}$. (The proof is straightforward.)*

PROPOSITION 5. *Let T and X be k_W spaces. Then so is T_S. (The proof is in §10.)*
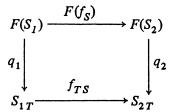
6. The extension functor ϵ. Let $\alpha = \langle S, p, A \rangle \in \bar{F}$. We consider the following relations in $F(S)$:

$$\{s \cdot a \cdot (sa)^{-1} = e \,|\, s \in S, a \in A\}.$$

Denote the quotient group with the quotient topology of $F(S)$ relative to the above relations by S_T, and the quotient homomorphism by q: $F(S) \longrightarrow S_T$. We also denote by p_T the unique homomorphism which makes the following diagram commutative:

$$F(S) \xrightarrow{\quad \chi \quad} F(S) * F(S)$$

$$q \downarrow \qquad\qquad\qquad \downarrow F(p) * q$$

$$S_T \xrightarrow{\quad f_T \quad} F(X) * S_T$$

where χ is the homomorphism induced by the map which sends $s \in S$ to $s_1 \cdot s_2$ $\in F(S) * F(S)$, where s_1, s_2 are copies of s in the two components of the free product. For any $\langle f, f_S \rangle: \alpha_1 \longrightarrow \alpha_2 \in \digamma$ we define $f_{ST}: S_{1T} \longrightarrow S_{2T}$, as the unique homomorphism which makes the following diagram commutative:

$$F(S_1) \xrightarrow{\quad F(f_S) \quad} F(S_2)$$

$$q_1 \downarrow \qquad\qquad\qquad \downarrow q_2$$

$$S_{1T} \xrightarrow{\quad f_{TS} \quad} S_{2T}$$

THEOREM 2. *The function ϵ which assigns to each $\alpha = \langle s, p, A \rangle \in \digamma$ the triple $\epsilon(\alpha) = \langle S_T, p_T, A \rangle$ and to each morphism $\langle f, f_S \rangle: \alpha_1 \longrightarrow \alpha_2 \in \digamma$ the morphism $\epsilon \langle f, f_S \rangle = \langle f, f_{ST} \rangle: \epsilon(\alpha_1) \longrightarrow \epsilon(\alpha_2)$ is a functor from $\digamma$ to C. (The proof is again straightforward.)*

PROPOSITION 6. *If S and A of α, in Theorem 2, are k_W spaces, so is S_T. (The proof is in §10.)*

7. **Relations between ϵ and δ.** We open this section with the remark that ϵ is a left adjoint functor to δ. However for the classification theorem in the next section we need a stronger connection between these two functors. To achieve this we replace C by its subcategory $e(\digamma)$ which we denote by C'. Under this condition we obtain the following result:

THEOREM 3. *The functors ϵ and δ are inverses of each other.*

Theorem 3 is the main tool which enables us to deduce properties of p.c.b.'s from p.f.b.'s.

PROPOSITION 7. *Let $\langle 1, f_T \rangle: \langle T_1, \phi_1, A \rangle \longrightarrow \langle T_2, \phi_2, A \rangle \in C'$. Then $\langle 1, f_T \rangle$ is an equivalence.*

PROOF. Since $\delta \langle 1, f_T \rangle$ is an equivalence in $\digamma$, $\delta \langle 1, f_T \rangle$ is an equivalence in C'.

PROPOSITION 8. *Let $\langle f, f_T \rangle: \beta_1 \longrightarrow \beta_2 \in C'$. Then $f_*(\beta_1)$ is equivalent to β_2.*

PROOF. This follows from the universal property of induced p.c.b.'s (i.e. that any morphism in C' factors through the canonical map into the induced p.c.b.) and Proposition 7.

PROPOSITION 9. *Let $f: A_1 \rightarrow A_2$ be an homomorphism between two topological groups, and let $\beta_1 = \langle T_1, \phi_1, A_1 \rangle \in C$. Then $f_*(\beta_1)$ and $\epsilon(f^*(\delta(\beta_1)))$ are equivalent.*

PROOF. This is an easy consequence of Proposition 7.

At this point we would like to remark that we can replace $\bar{F}$ by a subcategory $\bar{F}'$ of all p.f.b.'s $\langle S, p, A \rangle$, such that S and A are k_W spaces, and C' to be replaced by C'' whose objects $\langle T, \phi, A \rangle$ satisfy: A, T are k_W groups such that T has the weak topology of $F_n(T_S)$. All results of this section hold for these categories with ϵ and δ properly changed.

We close this section with the notion of a trivial p.c.b.

DEFINITION 7. The p.c.b. $\langle T, \phi, A \rangle$ is called trivial if $T = F(X) * A$ and $\phi = \psi * 1_A$.

PROPOSITION 10. *The functors ϵ and δ correspond trivial p.f.b.'s to trivial p.c.b.'s.*

PROOF. It is enough to show that δ maps a trivial p.c.b. to a trivial p.f.b. Since $F(X) \times A$ is embedded in $F(X) * A$, and $(F(X) * A)_T = X \times A$ the result follows.

8. **The classification theorem for p.c.b.'s.** In this section we present and prove a classification theorem for the category C'' of locally trivial p.c.b.'s. For the classification we use the functors δ and ϵ, which reduce the problem to the category $\bar{F}'$. In this category we rely heavily on a result of Milnor's which is also extended in this section.

Let X be a countable simplicial complex. Denote by Z all finite $(n + 1)$-tuples $(x_n, x_{n-1}, \ldots, x_0)$ of points in X such that each pair x_i, x_{i-1} lie in a common simplex of X. We denote by $\widetilde{Z}$ the equivalence classes of Z under the relation: $(x_n, \ldots, x_{i+1}, x_i, x_{i-1}, x_0)$ is equivalent to $(x_n, \ldots, x_{i+1}, x_{i-1}, \ldots, x_0)$ whenever $x_i = x_{i-1}$, and denote the class of $(x_n, \ldots, x_0)$ by $[x_n, \ldots, x_0]$. We also use the following notation: (a) $\widetilde{E}(X) = \{[x_n, \ldots, e] \in \widetilde{Z}\}$; (b) $p: \widetilde{E}(X) \rightarrow X$ such that $p([x_n, \ldots, e]) = x_n$; (c) $\widetilde{G}(X) = \ker \widetilde{p}$.

THEOREM I [7]. $\langle \widetilde{E}(X), \widetilde{p}, \widetilde{G}(X) \rangle \equiv \xi \in \bar{F}'$.

THEOREM II [7]. *Every $\alpha \in \bar{F}'$ is co-induced from ξ.*

The homomorphism which co-induces α of the theorem depends on a system of slicing functions of α. We call this homomorphism a Milnor homomor-

phism. Next we state the main result:

THEOREM 4. *Denote by $\eta = \epsilon(\xi)$; then:* (a) *Every p.c.b.* $\alpha \in C''$ *is induced by a Milnor homomorphism from* η. (b) *If* $h_0, h_1 \colon \widetilde{G}(X) \longrightarrow A$ *are loop homotopic homomorphisms (i.e. there exists a map* $H \colon \widetilde{G}(X) \times [0, 1] \longrightarrow A$ *such that* $H(\, , t)$ *for a fixed* $t \in [0, 1]$ *is a homomorphism and* $H(\, , 0) = h_0$, $H(\, , 1) = h_1$) *then* $h_{0*}(\eta)$ *is equivalent to* $h_{1*}(\eta)$. (c) *If* k_0 *and* k_1 *are Milnor homomorphisms for two equivalent p.c.b.'s of* C'', *then they are loop homotopic.*

The rest of this section is devoted to the proof of Theorem 4. As for part (a) it is an immediate consequence of Milnor's Theorem II and Proposition 9. The proof of (b) follows also from Proposition 9 and the following:

PROPOSITION 11. *Let* $l_0, l_1 \colon A \longrightarrow A_1$ *be two homomorphisms which are loop homotopic. If* $\alpha = \langle S, p, A \rangle \in F'$, *then* $l_{0*}(\alpha)$ *is equivalent to* $l_{1*}(\alpha)$.

PROOF. If $\{g_{ij}\}$ is a system of transition functions for α with the open cover $\{U_\alpha\}$ of X, then $\{h_0 g_{ij}\}$ and $\{h_1 g_{ij}\}$ are systems of transition functions with the same open cover for $l_{0*}(\alpha)$ and $l_{1*}(\alpha)$ respectively. Let H be the loop homotopy given in the theorem, then $H(g_{ij} \times 1_I)$ is a system of transition functions for the open cover $\{U_\alpha \times I\}$ of $X \times I$. By Theorem 9.8, p. 51 of [3] we conclude that $l_{0*}(\alpha)$ and $l_{1*}(\alpha)$ are equivalent.

The proof of part (c) of Theorem 4 is again a result of Proposition 9 and the following:

PROPOSITION 12. *Let* $k_0, k_1 \colon \widetilde{G}(X) \longrightarrow A$ *be Milnor homomorphisms for equivalent p.f.b.'s in* F'. *Then they are loop homotopic.*

The proof of this is done in several stages:

LEMMA 1. *Theorem I stated above is correct if we replace the simplicial complex X by the complex $X \times I$, whose cells are of the form $\tau \times I$, with τ a simplex of X.*

The proof is the same that Milnor gives for X in [7].

LEMMA 2. *Theorem II holds if we replace X by $X \times I$ as in Lemma 1.*

PROOF. This is again, word for word, the proof of Theorem 5.1 in [7] with the following observation: The proof of Proposition 5.3 which provides a system of slicing functions, depends on the fact that the equivariant cohomology groups $H^k(U \bmod \Delta, B_n, T)$ are all trivial, since simplexes are convex. This fact is also true in our case because $\tau \times I$ are also convex.

LEMMA 3. *Let* $\langle \bar{S}, \bar{p}, A \rangle$ *be a locally trivial p.f.b. over $X \times I$. Let* $\bar{S}|_{X \times \{0\}}$ *and* $\bar{S}|_{X \times \{1\}}$ *have two systems of slicing functions. Then* $\langle \bar{S}, \bar{p}, A \rangle$ *has a system of slicing functions which extend the two already given.*

PROOF. The proof is again similar to the proof of 5.3 of [7], where this time we use the following homotopy: for $(x, r) \times (y, s) \in (\tau \times I) \times (\pi \times I)$

$$
H((x, r), (ys), t)
$$

$$
= \begin{cases}
(x, r + t(s - r)), (y, s + t(r - s)), \quad 0 \leqslant t \leqslant \frac{1}{2}, \\[2mm]
\left(x + \left(t - \frac{1}{2} \right)(y - x), \frac{r+s}{2} + \left(\frac{r+s}{2} - \frac{1}{2} \right)\left(t - \frac{1}{2} \right)(|x - y|) \right), \\[2mm]
\left(y + \left(t - \frac{1}{2} \right)(x - y), \frac{r+s}{2} + \left(\frac{r+s}{2} - \frac{1}{2} \right)\left(t - \frac{1}{2} \right)(|x - y|) \right), \\[2mm]
\hspace{6cm} \frac{1}{2} \leqslant t \leqslant 1,
\end{cases}
$$

such that if for some (x, r), (y, s) there exists a $t_0 \geqslant \frac{1}{2}$ such that $H((x, r), (y, s), t_0)$ is in the diagonal of $X \times I$ or in $(X \times \{0\}) \times (X \times \{0\})$ or in $(X \times \{1\}) \times (X \times \{1\})$ then $H((x, r), (y, s), t) = H((x, r), (y, s), t_0)$ for all $t \geqslant t_0$.

PROOF OF PROPOSITION 12. Since $k_0{}_*(\xi)$ is equivalent to $k_1{}_*(\xi)$, there are two homotopic maps $f_0, f_1 : X \longrightarrow B_A$ (where B_A is the classifying space for fiber bundles with fiber A). Let H be the homotopy from f_0 to f_1 and $\langle \overline{S}, \overline{p}, A \rangle$ the pull back bundle of the universal bundle $(\widetilde{E}_A, \widetilde{p}, B_A)$ via H. By Lemma 3 this bundle has a system of slicing functions extending the original two given. By Lemma 2 the p.f.b. $\langle \overline{S}, \overline{p}, A \rangle$ is induced by a homomorphism $\widetilde{H} : \widetilde{G}(X \times I) \longrightarrow A$. Furthermore there exists a homomorphism $J : \widetilde{G}(X) \times I \longrightarrow \widetilde{G}(X \times I)$ defined as follows: the pair (α, t) is mapped to the tuple which starts on the right with $(e, 0)$, the next coordinates are those of α embedded in $X \times t$, and the last one is again $(e, 0)$. Combining the last two maps we get a loop homotopy $H = \widetilde{H}J : \widetilde{G}(X) \times I \longrightarrow A$ such that $H|\widetilde{G}(X) \times \{0\} = k_0$, and $H|\widetilde{G}(X) \times \{1\}$ is conjugate to k_1. However from the proof of Theorem II it is apparent that the conjugating element is in the path component of the identity of A. Thus $H|\widetilde{G}(X) \times \{1\}$ is loop homotopic to k_1 and we conclude that k_0 and k_1 are loop homotopic.

9. **An example.** In this section we present a p.c.b. which arises in quite a natural way, and show in fact that many p.c.b.'s are of this form. This is done by relating p.c.b.'s to principal co-fibrations.

Let X be a connected subcomplex of a countable connected simplicial complex Y. Let $e \in X \subset Y$ be the base point. We denote by CX the unreduced cone over X, i.e. $CX = X \times I / X \times \{1\}$ and by ΣX the unreduced suspension of X, i.e. $\Sigma X = X \times I / X \times (\{0\}, \{1\})$. We mark the point resulting from the collapsing of $X \times \{0\}$ by v_0 and of $X \times \{1\}$ by v_1. Let Z be the identification

space of $Y \cup CX$ where $(x, 0) \in CX$ is identified with $x \in Y$. CX, ΣX and Z are again countable connected simplicial complexes.

Let $\hat{E}(Z)$ be a subcomplex of $\widetilde{E}(Z)$ (see §8) consisting of all tuples such that no coordinate lies in $(Z - (CX - v_1))$ unless it is the first on the left. Let $\hat{G}(Z)$ denote those tuples of $\hat{E}(Z)$, with first coordinate from the left equal to e. Every element a of $\hat{G}(Z)$ can be represented by a product of special elements $a = a_1 \cdot a_2 \cdots a_n$, with $a_i \in \hat{G}(Z)$, where a_i is one of the following types: $\{[w] \in \widetilde{G}(Y)\}$, $\{[u \cdot x \cdot v_1 \cdot e]|\ u$ is a tuple in Y that starts on the left with e, and $x \in X\}$, $\{[e \cdot v_1 \cdot x \cdot u]|\ u \in \widetilde{E}[Y], x \in X\}$, $\{[e \cdot v_1 \cdot x_1 \cdot u \cdot x_2 \cdot v_1 \cdot e]|\ u$ is a tuple in $Y\}$. Define a homomorphism $\hat{\phi}: \hat{G}(Z) \longrightarrow F(X) * \hat{G}(Z)$ such that:

$$\hat{\phi}(a) = \begin{cases} [w] & \text{for} \quad a = [w], \\ x \cdot [e \cdot v_1 \cdot x \cdot u] & \text{for} \quad a = [e \cdot v_1 \cdot x \cdot u], \\ [u \cdot x \cdot v_1 \cdot e] \cdot x^{-1} & \text{for} \quad a = [u \cdot x \cdot v_1 \cdot e], \\ x_1[e \cdot v_1 \cdot x_1 \cdot u \cdot x_2 \cdot v_1 \cdot e]x_2^{-1} & \text{for} \quad a = [e \cdot v_1 \cdot x_1 \cdot u \cdot x_2 \cdot v_1 \cdot e]. \end{cases}$$

It is easy to check that $\hat{\phi}$ is well defined.

THEOREM 5. $\langle \hat{G}(Z), \hat{\phi}, \widetilde{G}(Y), F(X) \rangle$ *is a locally trivial p.c.b. in* C'' (*see* §7).

PROOF. Let $S = \{a \in \hat{G}(Z)|\hat{\phi}(a) \in X \times \hat{G}(Z)\}$. These are all elements of the form $\{[e \cdot v_1 \cdot x \cdot u]|u \in \widetilde{E}(Y)\}$ which are homeomorphic to $\{a \in \widetilde{E}(Y)$ whose first element on the left is in $X\}$. We denote this set by $\widetilde{E}(Y, X)$. Since $\langle S = \widetilde{E}(Y, X), \phi_S, \widetilde{G}(Y) \rangle$ is the pull back p.f.b. of the p.f.b. $\langle \widetilde{E}(Y), \widetilde{P}, \widetilde{G}(Y) \rangle$ via the inclusion map of X into Y, the pull back bundle is also locally trivial. The result now follows from Theorem 2.

To see the generality of the above example consider the following: Let $f: X \longrightarrow Y$ be any map between spaces of the homotopy type of countable c.w. complexes since up to homotopy type f can be replaced by the inclusion of a countable simplicial complex into another countable simplicial complex. We consider this situation, $i: X \longrightarrow Y$. Let $Y \longrightarrow Y \cup_i CX \longrightarrow \Sigma X$ be the principal co-fibration induced by i from the co-fibration $X \longrightarrow CX \longrightarrow \Sigma X$.
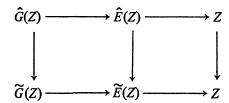
THEOREM 6. *The total space, the co-base, and the co-fiber, of the p.c.b. of Theorem 5, are of the homotopy type of the loops of: the total space, the co-base and the co-fiber of the principal co-fibration* $Y \longrightarrow Y \cup_i CX \longrightarrow \Sigma X$.

PROOF. That $\widetilde{G}(Y)$ has the homotopy type of ΩY follows from [7]. The rest of the proof is split up in the next two propositions:

PROPOSITION 13. $\hat{G}(Z)$ *has the homotopy type of* $\widetilde{G}(Z)$, *thus also of* ΩZ.
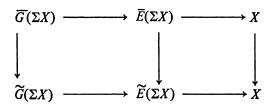
PROPOSITION 14. *$F(X)$ has the homotopy of $\Omega\Sigma X$.*

PROOF OF PROPOSITION 13. We start with the proof that $\langle \hat{E}(Z), \hat{P}, \hat{G}(Z) \rangle$ is a p.f.b. of F'', where $\hat{P}: \hat{E}(Z) \longrightarrow Z$ sends a tuple of $\hat{E}(Z)$ to its first coordinate on the right. It is clear that $\hat{G}(Z)$ operates effectively on $\hat{E}(Z)$. The rest follows by the use of Milnor's arguments in [7] for proving that $\langle \widetilde{E}(X), \widetilde{P}, \widetilde{G}(X) \rangle$ is a universal p.f.b. with the following changes. For the coordinate neighborhood of v_1 there is no change. For the coordinate neighborhoods $st(u)$ where u is a vertex of X, the connection of u with a point in $st(u) \cap X$ is as in the original proof, while the connection with a point $t \in st(u) \cap (CX - X)$ is via an extra coordinate in X, which lies in a straight line under the point in $CX - X$, between t and u. The rest of the proof is with no change. The rest of the proof follows by the five lemma and the homotopy ladder of the following commutative diagram of p.f.b.'s

$$
\begin{array}{ccccc}
\hat{G}(Z) & \longrightarrow & \hat{E}(Z) & \longrightarrow & Z \\
\downarrow & & \downarrow & & \downarrow \\
\widetilde{G}(Z) & \longrightarrow & \widetilde{E}(Z) & \longrightarrow & Z
\end{array}
$$

where the vertical maps are inclusions.

PROOF OF PROPOSITION 14. Let $\overline{G}(\Sigma X)$ be the subgroup of $\widetilde{G}(\Sigma X)$ generated by elements of the form $[e \cdot v_1 \cdot x \cdot v_0 \cdot e]$. $\overline{G}(\Sigma X)$ is homeomorphic to $F(X)$ via the map which sends x to $[e \cdot v_1 \cdot x \cdot v_2 \cdot e]$. To complete the proof it suffices to show that $\overline{G}(\Sigma X)$ is of the homotopy type of $\widetilde{G}(\Sigma X)$. Define $\overline{E}(\Sigma X) \subset \widetilde{E}(\Sigma X)$ which are those of $\overline{G}(\Sigma X)$ with the exception that they do not have to end in e but anywhere in ΣX. If $\overline{P}: \overline{E}(X) \longrightarrow X$ is the projection on the first coordinate from the left then as in the proof of Proposition 13 we get that $\langle \overline{E}(X), \overline{P}, \overline{G}(X) \rangle$ is a p.f.b. The rest of the proof is similar to the proof of Proposition 13, this time with the following diagram:

$$
\begin{array}{ccccc}
\overline{G}(\Sigma X) & \longrightarrow & \overline{E}(\Sigma X) & \longrightarrow & X \\
\downarrow & & \downarrow & & \downarrow \\
\widetilde{G}(\Sigma X) & \longrightarrow & \widetilde{E}(\Sigma X) & \longrightarrow & X
\end{array}
$$

where the vertical maps are inclusions.

10. Proofs.

PROOF OF PROPOSITION 1. The total space S_2 is homeomorphic to the

total space of the fiber bundle $\alpha_1[A_2]$ [3, p. 44], which is a quotient space of the k_W space $S_1 \times A_2$.

PROOF OF PROPOSITION 3. The only fact not proved in Proposition 2 is that T_2 is Hausdorff. This follows from Propositions 1 and 9.

PROOF OF PROPOSITION 5. Claim $F(X) \times T$ is closed in $F(X) * T$. Since X is closed in $F(X)$ [1], $X \times T$ is closed in $F(X) * T$. Thus T_S being the inverse image of a closed set is closed and therefore also k_W. Next we prove the claim. Let $F(X) = \bigcup X_i$ and $T = \bigcup Y_i$ be k_W decompositions, then $F(X) * T = \bigcup h((X_i \vee Y_i)^i)$ is also a k_W decomposition, where h sends a word to its reduced form. If we denote X_n and Y_n by K_n^0 and K_n^1 respectively then $h((X_n \vee Y_n)^n)$ can be replaced by a finite union of the following compact sets: $h(\Pi_{j=1}^n K_n^{i_j})$ where i_j takes values 0 and 1. Now it is clear that

$$F(X) \times T \cap h((X_n \vee Y_n)^n) = \bigcup_{\substack{k+l=n \\ k,l \geqslant 0}} h((X_n)^k) \times h((T_n)^l).$$

Since the right hand side of the last equality is compact the claim follows. (Note that the claim and the proof generalize trivially to a finite number of groups.)

PROOF OF PROPOSITION 6. Let $S = \bigcup_{k=1}^{\infty} S_k$ be a k_w decomposition. Define the following sets:

$$A_k = \tau(S_k \times S_k), \quad A_k' = A_k \cup (A_k^{-1}), \quad S_k' = \mu'(S_k \times A_k' \times \cdots \times A_k'),$$

where τ is the translation function of $\langle S, P, A \rangle$ (Definition 1), μ' is defined on $S_k \times (A_k')^k$ and is induced by μ, the action of A on S, and (A_k^{-1}) are the inverses of A_k in the group A. We also use the following notation: $T_k = \{s \cdot a \cdot (sa)^{-1}, (sa)^{-1} \cdot s \cdot a, s \cdot a^{-1} \cdot (s(a^{-1}))^{-1}, (s(a^{-1}))^{-1} \cdot s \cdot a^{-1} | s \in S_k', a \in A_k'\}$, $H_k' = \{u^{-1} \cdot v \cdot u | u \in F_k(S_k'), v \in T_k^{\pm 1}\}$, $H' = \bigcup_{k=1}^{\infty} H_k'$, $H_k'' = F_k(H_k')$, $H'' = \bigcup_{k=1}^{\infty} H_k''$, $H_k = h(H_k'')$ and $H = \bigcup_{k=1}^{\infty} H_k$, where h sends a word to its reduced form in elements of S. It is obvious that $H = \ker q$, and that each H_k is compact. Our problem is now reduced to showing that $H \cap F_k(S_k) \subset H_k$. This is the outcome of the following claims which actually amount to the fact that an element in H, not in H_k, cannot be in $F_k(S_k)$.

For a word w in $F(S)$ we denote by $|W|$ its length in elements of S. We say that the words $e_1, e_2 \in F(S)$ have a link of the first, second, third or fourth kind if one of the following holds respectively: $|e_1 \cdot e_2| > \max\{|e_1|, |e_2|\}$, $|e_1 \cdot e_2| = \max\{|e_1|, |e_2|\}$, $|e_1 \cdot e_2| = \max\{|e_1|, |e_2|\} - 1$, $|e_1 \cdot e_2| < \max\{|e_1|, |e_2|\} - 1$. Let $W = c_1 \cdot c_2 \cdots c_k$ be a representation in elements of H' of a word in $F(S)$, such that c_i cannot be represented by $c_i = d_i^1 \cdots d_i^{l_i}$, $d_i^j \in H'$, with $\max_{1 \leqslant j \leqslant l} |d_i^j| < |c_i|$ for all $1 \leqslant i \leqslant k$. Such a representation will be

called proper. Every word of $F(S)$ has a proper representation.

Claim 1. Let $w = c_1 \cdot c_2 \cdots c_k$ be a proper representation. Then between any two pairs of consecutive elements of $\{c_i\}$ with links of the third kind, there must be a pair of consecutive elements, with link of the first kind. There is no link of the fourth kind.

This claim is achieved by observing that in each of the many possibilities, if the claim is not satisfied, the representation cannot be proper. As an easy consequence we get:

Claim 2. If $w = c_1 \cdots c_k$ is a proper representation, then $|w| \geq [|c_i|/2]$ for $1 \leq i \leq k$, and $|w| \geq k$. ($[x]$ = the largest integer $\leq x$.)

Claim 3. If $w \in F(S_k) \cap H$, then w has a representation in elements of $F(S'_k) \cap H'$. This follows from the definitions of $\{S'_k\}$, $\{A'_k\}$ and the operations needed in the proof of Claim 2.

The proof of Proposition 6 follows now at once.

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