SINGULARITIES IN THE NILPOTENT SCHEME
OF A CLASSICAL GROUP

BY

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ABSTRACT. If \((X, x)\) is a pointed scheme over a ring \(k\), we introduce a (generalized) partition \(\text{ord}(x, X/k)\). If \(G\) is a reductive group scheme over \(k\), the existence of a nilpotent subscheme \(N(G)\) of \(\text{Lie}(G)\) is discussed. We prove that \(\text{ord}(x, N(G)/k)\) characterizes the orbits in \(N(G)\), their codimension and their adjacency structure, provided that \(G\) is \(GL_n\), or \(Sp_n\) and \(1/2 \in k\). For \(SO_n\) only partial results are obtained. We give presentations of some singularities of \(N(G)\). Tables for its orbit structure are added.

Introduction. Let \(G\) be a reductive algebraic group over a field of characteristic \(p\). Let \(g\) be its Lie-algebra and \(N(G)\) the closed subset of the nilpotent elements of \(g\), cf. [19]. The \(G\)-orbits in \(N(G)\) are characterized by weighted Dynkin diagrams, cf. [20, III]. Consider the following question. Is it possible to classify the orbits in \(N(G)\) using only the local structure of the variety \(N(G)\)? We prove in (4.3) that the answer is positive if \(G\) is \(GL_n\) or if \(G\) is \(Sp_n\) and \(p \neq 2\).

To this end we introduce a local invariant "ord" for any pointed scheme in §1. We develop the theory of \(N(G)\) over an arbitrary ground ring \(k\) in §2. In §3 we restrict our attention to the classical group schemes. Using a cross section we obtain information about the orbit structure of \(N(G)\). Our main theorem (4.2) relates \(\text{ord}(x, N(G)/k)\) to the Jordan normal form of the nilpotent endomorphism induced by \(x\) in the classical representation.

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Conventions and notations. The cardinality of a set \(V\) is denoted by \(# V\). Any infinite cardinal is represented by \(\infty\). If \(x\) is a real number then \([x]\) is the greatest integer in \(x\). All rings are commutative with 1. Let \(M\) be a module over a ring \(A\). If \(M\) is free the rank of \(M\) is denoted by \(\text{rg}_AM\). An element \(r \in A\) is called \(M\)-regular if \(\alpha: M \rightarrow M\) is injective. Let \(a = (a_1, \ldots, a_r)\) be a...
sequence in $A$. The ideal generated by $a$ is denoted by $(a)$. The sequence is called $M$-regular if $a_i = (M/\langle a_i \rangle)\langle M \rangle$-regular for all $i$, cf. [12, 0IV 15.1].

Unless stated otherwise $k$ is an arbitrary ground ring. General references for schemes and group schemes are [11], [12] and [8]. If we consider a $k$-scheme as a functor from $k$-algebras to sets, cf. [11, p. 17], then the letter $R$ is used to denote an arbitrary $k$-algebra. If $X$ is a $k$-scheme and $R$ is a $k$-algebra then $X \otimes_k R$. If $X$ is an affine scheme then its coordinate ring is denoted by $A(X)$. If $A$ is a local ring its maximal ideal is denoted by $m_A$ and its residue field by $k(A)$. If $X$ is a scheme and $x \in X$ then we write $m_x = m_A$ and $k(x) = k(A)$ where $A = \mathcal{O}_{X,x}$.

1. A near-partition for a local $k$-algebra.

(1.1) A near-partition $\lambda$ is a subset of $\mathbb{N}^2$ such that if $(m, n) \in \lambda$ and $i \leq m$ and $j \leq n$ then $(i, j) \in \lambda$. The set of near-partitions is denoted by $NP$. The duality mapping $D: NP \rightarrow NP$ is induced by $(i, j) \mapsto (j, i)$. The set $NP$ is ordered by $\lambda \preceq \mu$ if and only if $\lambda \subseteq \mu$. We write $|\lambda| := \# \lambda$. A near-partition $\lambda$ is called a partition if $|\lambda| < \infty$. The set of partitions is denoted by $P$.

If $\lambda \in NP$, the nonincreasing sequences $\lambda^*$ and $\lambda^*$ in $\{0\} \cup \mathbb{N} \cup \{\infty\}$ are defined by

$$\lambda^* \geq i \Leftrightarrow (n, 0) \in \lambda \Leftrightarrow \lambda_i \geq n.$$  

Clearly $\lambda_i = (D\lambda)^i = \sup \{n \in \mathbb{N}|\lambda^n \geq i\}$, and dually. A near-partition $\lambda$ is completely determined by its sequence $\lambda^*$ (or $\lambda^*$). We write $\lambda^*_\ast = (\lambda_1, \ldots, \lambda_r)$ if $\lambda_i = 0$ for $i > r$. If $\lambda, \mu \in NP$, we define $\lambda + \mu \in NP$ by $(\lambda + \mu)^x := \lambda^x + \mu^x$, where $x + \infty := \infty + x := \infty$ for all $x$. If $\lambda^*_\ast = (\lambda_1, \ldots, \lambda_r)$ and $\mu^*_\ast = (\mu_1, \ldots, \mu_s)$ then $(\lambda + \mu)^*_\ast$ is the sequence obtained by ordering $(\lambda_1, \ldots, \lambda_r, \mu_1, \ldots, \mu_s)$, see [9, Proposition 6].

(1.2) Definition. A linear extension over a ring $k$ is a surjective morphism $e: E \rightarrow A$ of local $k$-algebras such that $m_E \ker(e) = 0$. Its near-partition $\text{ord}(e)$ is defined by

$$\text{ord}^n(e) := r_{k(E)}(\ker(e) \cap m_E^{n+1}).$$

A linear extension $e: E \rightarrow A$ is called versal over $k$ if for any linear extension $\xi: F \rightarrow B$ over $k$ and any local $k$-morphism $\phi: A \rightarrow B$ there exists a (clearly local) $k$-morphism $\gamma: E \rightarrow F$ with $\xi \circ \gamma = \phi \circ e$, see diagram (i).

(1.3) Proposition. Let diagram (i) be a commutative diagram of $k$-algebras such that $e$ and $\xi$ are linear extensions, that $\phi$ is a flat local morphism and that $m_A B = m_B$. Then we have $\text{ord}(e) \geq \text{ord}(\xi)$. 

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Proof. Let \( n \in \mathbb{N} \). We prove that \( \text{ord}^n(e) \geq \text{ord}^n(\xi) \). It suffices to prove that the ideal \( \ker(\xi) \cap m_F^{n+1} \) is generated by the image of \( \ker(e) \cap m_E^{n+1} \). We may assume that \( \ker(e) \cap m_E^{n+1} = 0 \). Now the mapping \( m_E^{n+1} \to A \) induced by \( e \) is an injection of \( A \)-modules. Since \( B \) is flat over \( A \), it follows that \( m_E^{n+1} \otimes_E B \to B \) is injective and hence that \( \text{Tor}^A_E(E/m_E^{n+1}, B) = 0 \). This implies injectivity of

\[
\ker(\xi) \otimes_E (E/m_E^{n+1}) \to F \otimes_E (E/m_E^{n+1})
\]

so that \( \ker(\xi) \cap m_E^{n+1}F = m_E^{n+1}\ker(\xi) = 0 \). On the other hand \( m_AB = m_B \) implies that \( m_F^{n+1} + \ker(\xi) = m_F \), so that \( m_F^{n+1}F = m_F^{n+1} \). This proves \( \ker(\xi) \cap m_F^{n+1} = 0 \).

(1.4) Let \( A \) be a local \( k \)-algebra. If \( e: E \to A \) is a versal linear extension over \( k \) then (1.3) implies that \( \text{ord}(e) \geq \text{ord}(\xi) \) for any linear extension \( \xi: F \to A \) over \( k \). On the other hand there exists a versal linear extension \( e: E \to A \) over \( k \). In fact, write \( A = R/A \) where \( R \) is some polynomial \( k \)-algebra. Let \( M \) be the ideal in \( R \) such that \( m_A = M \). Then \( R/M \to A \) is a versal linear extension over \( k \), compare [15, p. 37]. Now we can give the following:

Definition. \( \text{ord}(A/k) := \text{ord}(e) \) where \( e: E \to A \) is some (or any) versal linear extension over \( k \).

Example. Let \( k \) be a field. Put \( H := k[T_1, \ldots, T_m] \). Let \( a = (a_1, \ldots, a_r) \) be a sequence in \( H \). Let \( a_i \) be homogeneous of degree \( 1 + \lambda_i \) where \( \lambda \) is a partition with \( \lambda_{r+1} = 0 \). Assume that the ideal \( \langle a \rangle \) is not generated by a strict subsequence of \( a \). Consider the local ring \( A := (H/\langle a \rangle)_p \) where \( p = \langle T_1, \ldots, T_m \rangle \). Then \( \text{ord}(A/k) = \lambda \).

In fact \( e: (H/p, \langle a \rangle)_p \to A \) is a versal linear extension over \( k \) and \( \text{ord}^n(e) = \#\{i | \lambda_i \geq n \} = \lambda^n \).

(1.5) Proposition. Let \( A \) be a local \( k \)-algebra and \( R \) a \( k \)-algebra. Assume that \( A \) or \( R \) is flat over \( k \). Let \( p \in \text{Spec}(A \otimes_k R) \) contract to \( m_A \). Then \( \text{ord}(A/k) \leq \text{ord}((A \otimes_k R)_p/R) \).

Proof. Let \( e: E \to A \) be a versal linear extension over \( k \). Put \( I := \ker(e) \). Let \( q \in \text{Spec}(E \otimes R) \) be the inverse image of \( p \). Since \( A \) or \( R \) is flat over \( k \), \( (I \otimes R)_q \) is an ideal in \( (E \otimes R)_q \). Put \( F := (E \otimes R)_q/(I \otimes R)_q \), so that \( \xi: F \to (A \otimes R)_p \) is a linear extension over \( R \). One verifies that \( I \otimes_k(E)_q \to \ker(\xi) \) is injective and hence that \( \text{ord}(e) \leq \text{ord}(\xi) \). This suffices.
(1.6) **Proposition.** Let $A$ be a local $k$-algebra, $x = (x_1, \ldots, x_m)$ an $A$-regular sequence in $m_A$ and $f$ a nonzero element of $(x)$. Put $B = A/(f)$ and $C = A/(x)$. Let $r \in \mathbb{N}$ and let $\rho$ be a partition with $\rho^*_r = (r - 1)$.

(a) If $f \in m_A^r$ then $\rho + \ord(A/k) \leq \ord(B/k)$.

(b) If $f \notin m_A^{r+1}$ then $\ord(B/k) \leq \rho + \ord(C/k)$.

**Proof.** Let $\varepsilon: E \to A$ be a versal linear extension over $k$. Put $I = \ker(\varepsilon)$. Choose $y_i \in E$ with $\varepsilon(y_i) = x_i$ and $g \in E$ with $\varepsilon(g) = f$. Put $F := E/\text{gm}_E$ and $G := E/m_E(y)$. The linear extensions $\xi: F \to B$ and $\eta: G \to C$ are versal over $k$. Since $x$ is a regular sequence, we have $I \cap (y) = 0$. So the induced mappings $I \to \ker(\xi)$ and $I \to \ker(\eta)$ are injective. This implies that $\ord(\varepsilon) \leq \ord(\xi)$ and $\ord(\varepsilon) \leq \ord(\eta)$.

(a) Now it suffices to prove:

\[ (*) \text{ If } n < r \text{ then } 1 + \ord^n(e) = \ord^n(\xi). \]

We may assume that $g \in m_E^{n+1}$. The cokernel of the injection $I \cap m_E^{n+1} \to \ker(\xi) \cap m_F^{n+1}$ is isomorphic to $(g)/\text{gm}_E$; this proves $(*)$.

(b) By $(*)$ it suffices to prove: If $f \notin m_A^{n+1}$ then $\ord^n(\xi) \leq \ord^n(\eta)$. We may assume $g \in (y)$. Since $f \notin m_A^{n+1}$ we have $g \notin m_E^{n+1}$. Using that $I \cap (y) = 0$, one shows that the mapping $\ker(\xi) \cap m_F^{n+1} \to \ker(\eta) \cap m_G^{n+1}$ is injective.

**Remark.** Usually (1.6) (a) is applied in the situation where $f$ itself is $A$-regular, $m = 1$ and $x_1 = f$.

(1.7) If $X$ is a $k$-scheme and $x \in X$ then $(X, x)$ is called a pointed $k$-scheme. We define $\ord(x, X/k) := \ord(0_{X, x}/k)$. Pointed $k$-schemes $(X, x)$ and $(Y, y)$ are called *smoothly equivalent* if there are smooth $k$-morphisms $f: Z \to X, g: Z \to Y$ and a point $z \in Z$ with $f(z) = x, g(z) = y$. This is an equivalence relation on the class of pointed $k$-schemes, to be denoted by $(X, x) \sim (Y, y)$. See [12, IV 17] for the definition and the basic properties of smooth morphisms.

**Theorem.** If $(X, x) \sim (Y, y)$ then $\ord(x, X/k) = \ord(y, Y/k)$.

**Proof.** We may assume that there is a smooth $k$-morphism $f: X \to Y$ with $f(x) = y$.

Using the regularity of the noetherian local ring $0_{X,x}/m_y 0_{X,x}$ and the arguments of the proof of [12, IV 19.2.9], we construct a subscheme $Z$ of $X$ containing $x$ such that $0_{Z,x} = 0_{X,x}/(a)$ where $a$ is an $0_{X,x}$-regular sequence, that $0_{Y,y} \to 0_{Z,x}$ is flat and that $m_y 0_{Z,x}$ is the maximal ideal of $0_{Z,x}$. By (1.6) (a) we have $\ord(x, X/k) \leq \ord(y, Z/k)$. Using (1.3) one proves that $\ord(x, Z/k) \leq \ord(y, Y/k)$.

We may assume that $Y = \text{Spec} A$ and $y = m_A$ where $A$ is a local $k$-algebra. Choose a versal linear extension $\varepsilon: E \to A$ over $k$. By [12, IV 18.1.1] there is a smooth $E$-algebra $R$ such that $\text{Spec}(A \otimes_E R)$ is isomorphic to an open neigh-
bourhood of $x$ in $X$. So $O_{X,x} \cong (A \otimes_{{\mathbb k}} R)_p$ for some $p \in \text{Spec}(A \otimes_{{\mathbb k}} R)$ contracting to $m_A$. By (1.5) we have $\text{ord}(A|E) \leq \text{ord}((A \otimes_{{\mathbb k}} R)_p/R)$. It is easy to see that this implies $\text{ord}(y, Y/k) \leq \text{ord}(x, X/k)$.

(1.8) The following remark will not be used in the sequel. For proofs and details we refer to [13].

Remark. Let $A$ be a noetherian local $k$-algebra. Then $\text{ord}(A/k)$ is a partition. It is equal to $\text{ord}(\hat{A}/k)$ where $\hat{A}$ is the completion of $A$. If $k$ is noetherian regular and $A$ is of essentially finite type over $k$, then $\text{ord}(A/k) = \text{ord}(A/Z)$. $A$ is regular if and only if $\text{ord}(A/Z) = 0$. If $A = R/I$ where $I$ is an ideal in a noetherian regular local ring $R$, then $\text{ord}(A/Z)$ is determined by the sequence $\nu(I)$, cf. [14, p. 209].

2. The nilpotent scheme.

(2.1) Consider an action $h$ of an affine group scheme $G$ on an affine scheme $X$ over $k$. We have the morphisms $h, \text{pr}_2 : G \times_{{\mathbb k}} X \rightarrow X$. The orbit $Gx$ of $x \in X$ is defined as the subset $h(\text{pr}_1^{-1}(x))$ of $X$. Let $V$ be a subscheme of $X$. Let $U$ be the open set where the induced morphism $h^V : G \times_{{\mathbb k}} V \rightarrow X$ is smooth. $V$ is called a cross section at $x$ if $x \in V$ and $e(V)(x) \subseteq U$. Here $e(V) : V \rightarrow G \times_{{\mathbb k}} V$ is induced by the unit $e \in G(k)$. The subscheme $V$ is called a global cross section if $U \rightarrow \text{Spec}(k)$ is surjective. $V$ is called an invariant subscheme if the morphism $h^V$ factorizes over $V$.

Let $A(X)^G$ be the equalizer of the comorphisms $A(X) \rightrightarrows A(G) \otimes_{{\mathbb k}} A(X)$. If $Y$ is an affine $k$-scheme, a $G$-invariant $k$-morphism $f : X \rightarrow Y$ corresponds to a comorphism $A(Y) \rightarrow A(X)$ factorizing over $A(X)^G$. We define the affine quotient of the action by $[X/G] : = \text{Spec}(A(X)^G)$. It is called universal if the induced morphism $[X(R)/G(R)] \rightarrow [X/G]_{(R)}$ is an isomorphism for any $k$-algebra $R$.

Remarks. (a) Let $G$ be smooth over $k$. Then $\text{pr}_2$ and $h$ are smooth morphisms. If $x' \in Gx$ then $(X, x) \sim (X, x')$, cf. (1.7). If $V$ is a cross section at $x$ then $(X, x) \sim (V, x)$.

(b) The condition, that the affine quotient $[X/G]$ is universal, is a local condition on $\text{Spec}(k)$ for the topology $(f, p, q, c)$, cf. [8, IV], see [13, p. 38]. If $k$ is a field any affine quotient is universal.

(c) Other types of quotients are discussed in [17, p. 3].

(2.2) Proposition. Assume in (2.1) that the morphism $X \rightarrow \text{Spec}(k)$ is smooth and irreducible cf. [12, IV 4.5.5], and that $V$ is affine and a global cross section.

(a) The morphism $A(X)^G \rightarrow A(V)$ is injective.

(b) If $A(X)^G \rightarrow A(V)$ is bijective then $[X/G]$ is universal.
PROOF. (a) Consider a nonzero \( f \in A(X)^G \). Assume that \( f|V = 0 \). There is a commutative diagram (i), so we have \( f \circ h^V = 0 \).

\[
\begin{array}{ccc}
G \times_k V & \xrightarrow{pr_2} & V \\
\downarrow h^V & & \downarrow f|V \\
X & \xrightarrow{f} & \text{Spec}(k[T])
\end{array}
\]

The morphism \( h^V \) is flat on \( U \), so \( h^V(U) \) is an open subset of \( X \) with \( f|h^V(U) = 0 \). Since \( f \neq 0 \), there is a generic point \( x \) of \( \text{Supp}(f|_U) \). Let \( \mathfrak{p} \in \text{Spec}(k) \) be the image of \( x \). Let \( \xi \) be the unique generic point of \( X \otimes_k k(\mathfrak{p}) \). As \( h^V(U) \to \text{Spec}(k) \) is surjective we have \( \xi \in h^V(U) \) and hence \( x \neq \xi \). Since \( \mathcal{O}_{X,x} \otimes_k k(\mathfrak{p}) \)-regular element \( t \in \mathfrak{m}_x \). By [12, IV 11.3.7], \( t \) is \( \mathcal{O}_{X,x} \)-regular. It is easy to see that this contradicts the choice of \( x \). The argument used here was suggested by P. Deligne.

(b) Let \( R \) be a \( * \)-algebra. We have to prove that \( u : A(X)^G \otimes R \to A(X(R))^{G(R)} \) is bijective. As the assumptions of (a) are stable under base-change, the morphism \( v : A(X(R))^{G(R)} \to A(V) \otimes R \) is injective by (a). So it suffices to observe that \( v \circ u \) is bijective.

(2.3) Let \( G \) be a smooth affine group scheme over \( * \). Recall that the Lie-algebra \( \text{Lie}(G) \) is defined as the group functor such that \( \text{Lie}(G)(R) \) is the (additively written) kernel of the morphism \( G(R[\delta]/(\delta^2)) \to G(R) \) induced by \( \delta \mapsto 0 \) where \( R \) is a \( * \)-algebra. \( \text{Lie}(G) \) is a smooth affine group scheme, in fact a vector bundle. There is a canonical action of \( G \) on \( \text{Lie}(G) \). If \( R \) is a \( k \)-algebra then \( \text{Lie}(G)(R) = \text{Lie}(G_{(R)}) \). See [8, II 4]. Usually we write \( \mathfrak{g} : = \text{Lie}(G) \).

If \( K \) is a field over \( k \), a section \( x \in \mathfrak{g}(K) = \text{Lie}(G_{(K)})(K) \) is nilpotent if and only if its image is a nilpotent endomorphism of \( F \) for some (or any) immersion of \( G_{(K)} \) in a \( K \)-group \( Gl(F) \), cf. [2, p. 151]. A point \( x \in \mathfrak{g} \) is called nilpotent if the corresponding section \( x \in \mathfrak{g}(k(x)) \) is nilpotent.

(2.4) Definition. Let \( G \) be a reductive group scheme over \( k \), cf. [8, XIX 2.7]. If the affine quotient \( [\mathfrak{g}/G] \) is universal, cf. (2.1), then we define the nilpotent scheme \( N(G) : = p^{-1}p(0) \) where \( 0 \in \mathfrak{g}(k) \) is the zero section and \( p : \mathfrak{g} \to [\mathfrak{g}/G] \) is the quotient morphism.

PROPOSITION. Let \( N(G) \) be defined.

(a) \( N(G) \) is a \( G \)-invariant closed subscheme of \( \mathfrak{g} \).

(b) If \( R \) is a \( k \)-algebra then \( N(G_{(R)}) = N(G)(R) \).

(c) A point \( x \in \mathfrak{g} \) is nilpotent if and only if \( x \in N(G) \).

PROOF. (a) is trivial. (b) is a consequence of the assumption that \( [\mathfrak{g}/G] \) is universal. (c) By (b) we may assume that \( k \) is a field and that \( x \in \mathfrak{g}(k) \). Now it is well known. The "if-part" follows from Cayley-Hamilton by an embedding.
of $G$ in some $G_l(F)$. The "only-if-part" is a consequence of the following

**Lemma.** Let $G$ be a reductive $k$-group over a field $k$. If $x \in g(k)$ has the additive Jordan decomposition $x = x_s + x_n$, then $x_s$ is in the closure of the orbit $Gx$.

**Proof.** Adapt [22, (4.4)] or [21, p. 92].

(2.5) Let $G$ be a reductive group scheme over $k$. By (2.1)(b) the existence of a nilpotent scheme is a local condition on Spec($k$) for the topology $(f, p, q, c)$. So we assume that $G$ is of constant type (cf. [8, XXII 2.7]) with specified root system $\mathcal{R} = (\mathcal{M}, \mathcal{R}, \rho)$, i.e. a root system $\mathcal{R}$ in a given lattice $\mathcal{M}$ (cf. [7, p. 287]). Let $t$ be the torsion index (cf. [7, p. 294]). Let $f$ be the connection index (cf. [4, p. 167]). Consider the following conditions:

(i) $f^{-1} \in k$ and if $\mathcal{R} \cap 2\mathcal{M} \neq \emptyset$ then $1/2 \in k$, cf. [7, p. 296].

(ii) $f^{-1} 1/2 \in k$.

(iii) If $\mathcal{R}$ has a component of type $A_l$ then $(l + 1)^{-1} \in k$, of type $B_l, C_l, D_l, G_2$ then $1/2 \in k$, of type $E_6, E_7, F_4$ then $1/6 \in k$, of type $E_8$ then $1/30 \in k$.

The conditions (ii) and (iii) are equivalent and imply (i).

(2.6) **Theorem.** Let $G$ be as in (2.5) satisfying condition (i).

(a) The affine quotient $[g/G]$ is universal. The quotient morphisms $p: g \rightarrow [g/G]$ is flat. $N(G)$ is defined and flat over $k$.

(b) Let $T$ be a maximal torus of $G$ with Weyl group $W$, cf. [8, XXII 3]. Put $t := \text{Lie}(T)$. The affine quotient $[t/W]$ is universal. The canonical morphism $[t/W] \rightarrow [g/G]$ is an isomorphism.

(c) Assume that (2.5) (ii) holds. Let $\pi: G \rightarrow \text{ad}(G)$ be the projection onto the adjoint group, cf. [8, XXII 4.3]. Then $N(\text{ad}(G))$ is defined and equal to $N(G)$.

**Proof.** (1) We may assume that $G$ is split with respect to a (resp. the) maximal torus $T$, cf. [8, XXII 2.3]. Now $T = D_S(M)$ and $A(t) = S(M) \otimes k$. The group scheme $W$ is the constant group scheme associated to the abstract Weyl group of $\mathcal{R}$. By (2.5) (i) and [7, pp. 295, 296] the affine quotient $[t/W]$ is universal and the quotient morphism $t \rightarrow [t/W]$ is flat.

(2) By [8, XIII 5.1] and [12, IV 17.8.3] the subscheme $t$ is a global cross section for the action of $G$ on $g$, cf. (2.1). By (2.2) this implies that $A(g)^G \rightarrow A(t)^W$ is injective.

(3) We may assume that $k = Z[1/m]$, cf. [8, XXV 1]. It follows from [20, II 3.17'] and [22, p. 220] that $A(g)^G \otimes_k Q \rightarrow A(t)^W \otimes_k Q$ is bijective. Consider $a \in A(t)^W$. There is $a_1 \in A(g)^G$ and a nonzero $s \in k$ with $a_1|t = sa$. Put $R = k/(s)$. Now $a_1 \otimes 1_R|t(\mathcal{R}) = 0$, so by (2) we have $a_1 \otimes 1_R = 0$ in
$A(\mathfrak{g}) \otimes R$. So there is $a_2 \in A(\mathfrak{g})$ with $a_1 = sa_2$. Since $s$ is $A(G) \otimes A(\mathfrak{g})$-regular we have $a_2 \in A(\mathfrak{g})^G$. Since $s$ is $A(t)$-regular we have $a = a_2 |t$. This proves that $A(\mathfrak{g})^G \to A(t)^W$ is bijective. So we have proved (b).

(4) With (b) and (2) one proves that $[g/G]$ is universal in the same way as in (2.2) (b). Let $U$ be the open subset of $\mathfrak{g}$ where $p$ is flat. Since $t \to [g/G]$ is flat by (1) and (b), and $t \subset g$ is a regular immersion, we have $t \subset U$ by [12, 0TV 15.1.16]. As $U$ is $G$-invariant this implies $U = \mathfrak{g}$ by the lemma in (2.4). The other assertions of (a) follow immediately.

(5) In the notations of [8, XXII], condition (2.5) (ii) implies that the central isogenies $G \to \text{corad}(G) \otimes \text{ss}(G)$ and $\text{ss}(G) \to \text{ad}(G)$ are étale morphisms, by [8, VIII 2.1] and [8, XXI 6.5]. So we have an isomorphism

$$A(\text{Lie}(\text{ad}(G)))^{\text{ad}(G)} \otimes A(\text{Lie}(\text{corad}(G))) \cong A(\mathfrak{g})^G.$$ 

With this isomorphism one proves (c).

**Remarks.** (i) Assume that the order of the Weyl group is invertible in $\mathbb{k}$. By [22, (6.9)] the morphism $p$ is normal cf. [12, IV 6.8.1]. (ii) If $i \geq 2$ there is a semisimple group scheme $G$ of type $D_i$ over $\mathbb{Z}$ such that $[g/G]$ is not universal.

(2.7) **Corollary.** Let $G$ be as in (2.6). Let $d_1, \ldots, d_r$ be the degrees of $\mathbb{R}$. Consider the partition $\lambda$ defined by $\lambda_i = d_{r+1-i} - 1$ if $i < r$ and $\lambda_{r+1} := 0$. Let $x$ be a point of the zero section of $\mathfrak{g}$. Then $\text{ord}(x, N(G)\mathbb{k}) = \lambda$.

**Proof.** By (1.7) we may assume that $G$ is split with maximal torus $T$. Let $A(\mathfrak{g})^G = k[a_1, \ldots, a_r]$ where $a_1, \ldots, a_r$ are algebraically independent and $a_i$ is homogeneous of degree $d_{r+1-i} = 1 + \lambda_i$, cf. [7, Theorem 3]. We have $\partial_{N(G),x} = \partial_{\mathfrak{g},x}(a)$. Since $\partial_{\mathfrak{g},x}$ is flat over $A(\mathfrak{g})^G$ the sequence $a$ is $\partial_{\mathfrak{g},x}$-regular. By (1.6) (a) this implies that $\text{ord}(x, N(G)\mathbb{k}) \geq \lambda$. Let $p \in \text{Spec}(\mathbb{k})$ be the image of $x$. By (1.5) we may replace $\mathbb{k}$ by the residue field $\mathbb{k}(p)$. Now the assertion follows from the example in (1.4).

**Remark.** If $\mathbb{k}$ is noetherian regular the multiplicity of the local ring $\partial_{N(G),x}$ is equal to $\Pi_{i=1}^r d_i$, i.e. the order of the Weyl group. This is proved in [13, p. 55] using the methods of [18]. Compare [16, p. 386].

3. In the classical Lie-algebras.

(3.1) We fix a free $\mathbb{k}$-module $F$ of rank $n$. The scheme $\text{End}(F)$ is defined by $\text{End}(F)(\mathbb{R}) := \text{End}_{\mathbb{R}}(F \otimes_k \mathbb{R})$, cf. [11, 19]. The group scheme $G(\mathbb{F})$ (resp. $\text{SL}(F)$) is the open (resp. closed) subscheme of $\text{End}(F)$ where the function $\text{det} \in A(\text{End}(F))$ is invertible (resp. where $\text{det} = 1$). $G(\mathbb{F})$ and $\text{SL}(F)$ are reductive group schemes over $\mathbb{k}$ of type $A_{n-1}$, cf. [6] and [8]. $\text{End}(F)$ is identified with $\text{Lie}(G(\mathbb{F}))$ by $x \leftrightarrow 1 + 5x$ where $x \in \text{End}(F)(\mathbb{R})$, see (2.3) or [8, II 4]. Now $\text{Lie}(\text{SL}(F))$ consists of the endomorphisms with zero trace.
The nilpotent scheme of a classical group

Assume $1/2 \in k$. Let $e$ be 0 or 1. An $e$-form $\phi$ on $F$ is a nondegenerate bilinear form $\phi: F \times F \to k$ which is symmetric if $e = 0$, alternating if $e = 1$. By "nondegenerate" we mean that the mapping $F \to F$ defined by $f \mapsto \phi(f, -)$ is bijective. Let $\phi$ be an $e$-form. The subgroup functor $G'(F, \phi)$ of $G(F)$ is defined by $x \in G'(F, \phi)(R)$ if and only if
$$\phi(xf, xg) = \phi(f, g) \quad (f, g \in F \otimes R).$$
We define $G(F, \phi) : = G'(F, \phi) \cap SL(F)$. If $e = 0$ then $G(F, \phi)$ is the special orthogonal group scheme. If $e = 1$ then $G(F, \phi) = G'(F, \phi)$; it is the symplectic group scheme. Put $l := \lceil \frac{1}{2} n \rceil$ and $\xi := n - 2l$. So $\xi$ is 0 or 1. If $e = 1$ then $\xi = 0$. Now $G(F, \phi)$ is a semisimple group scheme of type $B_l$ if $e = 0$, $\xi = 1$, of type $C_l$ if $e = 1$, $\xi = 0$, of type $D_l$ if $e = \xi = 0$, cf. [6] and [8]. The common Lie-algebra of $G(F, \phi)$ and $G'(F, \phi)$ is denoted by $g(F, \phi)$. For $x \in \text{End}(F)(R)$ we have $x \in g(F, \phi)(R)$ if and only if
$$\phi(xf, g) + \phi(f, xg) = 0 \quad (f, g \in F \otimes R).$$

Convention. In the rest of this paper we consider two cases.

Case I. $G := G' := G(F, \phi) \cap SL(F)$. Let $e, f$ be 0 or 1. If $e = 0$ then $G(F, \phi)$ is the special orthogonal group scheme. If $e = 1$ then $G(F, \phi) = G'(F, \phi)$; it is the symplectic group scheme. Put $l := \lceil \frac{1}{2} n \rceil$ and $\xi := n - 2l$. So $\xi$ is 0 or 1. If $e = 1$ then $\xi = 0$. Now $G(F, \phi)$ is a semisimple group scheme of type $B_l$ if $e = 0$, $\xi = 1$, of type $C_l$ if $e = 1$, $\xi = 0$, of type $D_l$ if $e = \xi = 0$, cf. [6] and [8]. The common Lie-algebra of $G(F, \phi)$ and $G'(F, \phi)$ is denoted by $g(F, \phi)$. For $x \in \text{End}(F)(R)$ we have $x \in g(F, \phi)(R)$ if and only if
$$\phi(xf, g) + \phi(f, xg) = 0 \quad (f, g \in F \otimes R).$$

(3.2) Lemma. Case II. Let $\phi_1$ be another $e$-form on $F$. Then there is a faithfully flat étale $k$-algebra $R$ such that $\phi_1$ and $\phi$ induce equivalent forms on $F \otimes R$.

Proof. By [15, pp. 34, 35] the scheme $\text{Isom}(\phi_1, \phi)$ is smooth over $k$. If $K$ is an algebraically closed field over $k$ then $\text{Isom}(\phi_1, \phi)(K) \neq \emptyset$. Hence by [12, IV 17.16.3] there is a faithfully flat étale $k$-algebra $R$ with $\text{Isom}(\phi_1, \phi)(R) \neq \emptyset$.

(3.3) Definition. In Case I, $z \in g(R)$ is called a standard nilpotent with base-data $(f, \lambda)$ if $f = (f_1, \ldots, f_r)$ is a sequence in $F \otimes R$ and $\lambda$ is a partition, such that $\lambda^1 = r$, that the set $\{z_i f_i\}$, where $1 \leq i \leq r$ and $0 < a < \lambda_i$, is a basis of $F \otimes R$ and that $z^a f_i = 0$ if $a > \lambda_i$.

In Case II, $z \in g(R)$ is called a standard nilpotent with base-data $(f, \lambda, \beta, \alpha)$ if $z \in g(R)(R)$ is a standard nilpotent with base-data $(f, \lambda)$, $\beta$ is a permutation of $\{1, \ldots, r\}$ where $r = \lambda^1$, and $\alpha: \{1, \ldots, r\} \to R$ is a mapping such that
\begin{align}
(1) \quad \phi(z^a f_i, z^b f_j) &= (-1)^a \alpha(i) \quad \text{if } j = \beta i \text{ and } a + b + 1 = \lambda_i, \\
\phi(z^a f_i, z^b f_j) &= 0 \quad \text{otherwise}.
\end{align}
Remark. Clearly $|\lambda| = n$. In Case II the assumptions imply

\begin{equation}
\alpha(f)^{-1} \in R, \quad \beta^2 = \text{id}, \quad \lambda_{\beta} = \lambda_f, \quad \alpha(\beta) = (-1)^{\lambda_f - 1 + \epsilon} \alpha(f).
\end{equation}

(3.4) The set $P_{\epsilon}$ is defined as the subset of $P$ consisting of the partitions $\lambda$ such that for any $m > 1$ with $m \equiv (2) \mod 2$ the number of indices $i$ with $\lambda_i = m$ is even. These partitions are called orthogonal, resp. symplectic; in [10, p. 556]. We define $P(n)$ as the set of partitions $\lambda$ with $|\lambda| = n$, and $P_{\epsilon}(n) := P_{\epsilon} \cap P(n)$. We write $P_{\epsilon}(\lambda)$ to denote $P$ in Case I and $P_{\epsilon}$ in Case II. So in (3.3) we have $\lambda \in P_{\epsilon}(\lambda)$.

(3.5) If $x \in g$ is nilpotent, cf. (2.3), then the section $x \in g(k(x))$ is a standard nilpotent by [20, IV]. Let $\lambda \in P(n)$. We define $O(\lambda)$ as the set of points $x \in g$ such that the section $x$ is a standard nilpotent with partition $\lambda$. In case II we have $O(\lambda) = O_e(\lambda) \cap g$, and $O(\lambda) \neq \emptyset$ if and only if $\lambda \in P_{\epsilon}(n)$.

Let $k$ be a field and $x \in O(\lambda)$. By [20, IV] we have $O(\lambda) = G'x$, and $O(\lambda) \neq Gx$ if and only if we are in the very-even case: Case II $(0, 0)$ with $\lambda_i$ even for all $i$.

(3.6) Lemma. Case I. If $\lambda \in P(n)$, there is a standard nilpotent element $z \in g(k)$ with partition $\lambda$.

Case II. If $\lambda$, $\beta$ and $\alpha$ satisfy the conditions (3.3)(2), then there is an $e$-form $\phi_1$ on $F$ and a standard nilpotent element $z \in g(F, \phi_1)(k)$ with base-data $(f, \lambda, \beta, \alpha)$ for some sequence $f$ in $F$.

Proof. Case I is trivial.

Case II. Choose a standard nilpotent $z \in g_f(k)$ with base-data $(f, \lambda)$. Let

\[ \phi_1 : F \otimes F \to k \]

be the bilinear form defined by (3.3)(1). One verifies that $\phi_1$ is an $e$-form on $F$ with $z \in g(F, \phi_1)(k)$.

(3.7) The standard cross section. Let $z \in g(k)$ be a standard nilpotent element with base-data $(f, \lambda)$, resp. $(f, \lambda, \beta, \alpha)$. Below we construct a linear subscheme $L \subset g$ such that $g(R) = [g(R), z] \oplus L(R)$ for any $k$-algebra $R$. This implies that the subscheme $z + L \subset g$ is a cross section for the adjoint action of $G$ in all points of the section $z$, cf. (2.1). In fact the tangent morphism of $\text{Ad} : G \times (z + L) \to g$ at the section $(e, z)$ is the surjective morphism $g \oplus L \to g$ given by $(x, y) \mapsto [x, z] + y$. So smoothness of $\text{Ad}$ at $(e, z)$ follows from [12, IV 17.11.1].

Let $\Psi$ be the set of pairs $(i, a)$ with $0 \leq a < \lambda_i$. Put $f(i, a) := z^a f_i$. Then $\{f(i, a) | \psi \in \Psi \}$ is a basis of $F$. Let $\{u(\psi)\}$ be the dual basis of $F^\ast$. This means that $\{u(\psi)\}$ is the basis of $F^\ast = \text{Hom}(F, k)$ with

\[ \langle u(\psi), f(\psi) \rangle = \delta_{\psi, \psi'} \quad (\text{Kronecker delta}). \]

The coordinates $\xi(\psi; \psi')$ on $g_f$ are defined by $\xi(\psi; \psi')(x) = \langle u(\psi), xf(\psi') \rangle$. 

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Clearly \{e(\psi; \psi')|\psi, \psi' \in \Psi\} is a basis of \mathfrak{g}_j(k). Let \{e(\psi; \psi')\} be the dual basis of \mathfrak{g}_j(k). We have
\[ e(\psi; \psi')f(\psi'') = \delta_{\psi'; \psi''} f(\psi), \]
\[ [e(i, a; i, b), z] = e(i, a; i, b - 1) - e(i, a + 1; i, b) \]
where \(e(i, a; i, b) = 0\) if \(a \geq \lambda_i\) or \(b < 0\). In Case I let \(\mathfrak{g}_{ij}\), \(L_{ij}\) and \(L\) be the linear subschemes of \(\mathfrak{g}\) defined by
\[ \mathfrak{g}_{ij}(R) := \sum_{a, b} Re(i, a; j, b), \]
\[ L_{ij}(R) := \sum_{a, i} Re(i, a; \lambda_j - 1), \quad 0 \leq a < \min(\lambda_i, \lambda_j), \]
\[ L(R) := \sum_{i, j} L_{ij}(R). \]
Then we have \(\mathfrak{g}_{ij} = [\mathfrak{g}_{ij}, z] \oplus L_{ij}\) and \(\mathfrak{g} = [\mathfrak{g}, z] \oplus L\).

Case II. The coordinates \(\eta(\psi; \psi')\) on \(\mathfrak{g}\) are defined by \(\eta(\psi; \psi')(x) = \phi(f(\psi), x f(\psi'))\). Since \(\eta(\psi; \psi') = (-1)^1 + e \eta(\psi'; \psi)\) we have a basis of \(\mathfrak{g}(k)'\) consisting of the \(\eta(i, a; j, b)\) with \(i < j\), or \(i = j\) and \(a < b + e\). Let \(\gamma(\psi; \psi')\) be the dual basis of \(\mathfrak{g}(k)\). One shows that
\[ [\gamma(i, a; i, b), z] = \gamma(i, a; j, b - 1) + \gamma(i, a - 1; j, b) \]
if \(i < j\), or \(i = j\) and \(a < b - 1\),
\[ [\gamma(i, a; i, a + 1), z] = \gamma(i, a - 1; i, a + 1) + 2e\gamma(i, a; i, a), \]
\[ [\gamma(i, a; i, a), z] = \gamma(i, a - 1; i, a) \quad \text{if} \quad e = 1, \]
where \(\gamma(\psi; \psi') = 0\) if not yet defined. For \(i \leq j\) let \(\mathfrak{g}_{ij}\), \(L_{ij}\) and \(L\) be the linear subschemes of \(\mathfrak{g}\) defined by
\[ \mathfrak{g}_{ij}(R) := \sum_{a, b} Ry(i, a; j, b), \]
\[ L_{ij}(R) := \sum_{a, i} Ry(i, \lambda_i - 1; j, b) \quad \text{if} \quad i < j, \]
\[ L_{ii}(R) := \sum_{a} Ry(i, \lambda_i - 2 + e - a; i, \lambda_i - 1 - a), \quad 0 \leq a \leq \frac{1}{2}(\lambda_i - 2 + e), \]
\[ L(R) := \sum_{i < j} L_{ij}(R). \]
Then we have \(\mathfrak{g}_{ij} = [\mathfrak{g}_{ij}, z] \oplus L_{ij}\) and \(\mathfrak{g} = [\mathfrak{g}, z] \oplus L\).

Let \(F^*\) be identified with \(F\) in such a way that \(\langle u, f \rangle = \phi(u, f)\). Putting \(|f|_u = (-1)^e \alpha(i)^{-1}\), we get the following glossary:
\[ \|_{a}(i, a) = u(\beta_i, \lambda_i - 1 - a), \]
\[ \|_{a}(i, a; j, b) = \xi(\beta_i, \lambda_i - 1 - a; j, b) |\mathfrak{g}, \]
\[ y(i, a; j, b) = |\|_{a}e(\beta_i, \lambda_i - 1 - a; j, b) - (-1)^{e} |\|_{b}e(\beta_j, \lambda_j - 1 - b; i, a) \]
if \( i < j \), or \( i = j \) and \( a < b \),
\[ y(i, a; i, a) = |\|_{a}e(\beta_i, \lambda_i - 1 - a; i, a) \]
if \( e = 1 \),
\[ y(i, a; j, b) = 0 \]
otherwise.

Remark. In Case I our \( z + L \) is one of the cross sections of Arnold [1].

(3.8) An elementary calculation shows that \( L(k) \) is a free \( \mathbb{k} \)-module of rank \( l + \gamma(e)(\lambda) \) if we write \( \gamma(e)(\lambda) := \gamma(\lambda) \) in Case I and \( \gamma(e)(\lambda) := \gamma_e(\lambda) \) in Case II where

\[ \gamma(\lambda) := 2 \sum (i - 1) \lambda_i \quad \text{if} \quad \lambda \in \mathfrak{p}(n), \]
\[ \gamma_e(\lambda) := \sum (i - 1) \lambda_i + (2e - 1)[\frac{1}{2} \# \{ i | \lambda_i \equiv 1 \pmod{2} \}] \quad \text{if} \quad \lambda \in \mathfrak{p}_e(n). \]

Now the centralizer of \( z \) in \( \mathfrak{g}(k) \) is also a free \( \mathbb{k} \)-module of rank \( l + \gamma(e)(\lambda) \). By [20, I 5.6] we have the following:

Corollary. Assume that \( \mathbb{k} \) is a field.

(a) If \( x \in \Sigma(\lambda) \) then \( \dim(\mathfrak{g}x) = \dim(\mathfrak{g}) - l - \gamma(e)(\lambda) \).

(b) There is a unique nilpotent orbit \( C_{\text{reg}} \) of maximal dimension.

\( C_{\text{reg}} = \Sigma(\nu) \) where \( \nu_* = (n) \) in the Cases I and II (\( e, 1 - e \)) and \( \nu_* = (\eta - 1, 1) \) in Case II \((0, 0)\). We have \( \dim(C_{\text{reg}}) = \dim(\mathfrak{g}) - l \). If \( C \) is another nilpotent orbit in \( \mathfrak{g} \) then \( \dim(C) \leq \dim(\mathfrak{g}) - l - 2 \).

See also [1], [20, IV 2.28] and [21, p. 136].

(3.9) The mapping \( \Sigma: \mathcal{P}(e) \to \mathcal{P} \) is defined by \( (\Sigma \lambda)^m := \Sigma_{i>m} \lambda_i \) \((m \in \mathbb{N})\). As the corresponding propositions in [10, p. 567] are false, we shall prove the following:

Proposition. Let \( \lambda, \mu \in \mathcal{P}(e)(n) \) be such that

\[ \{ \mu \} = \{ \nu \in \mathcal{P}(e)(n) | \Sigma \lambda > \Sigma \nu \geq \Sigma \mu \}. \]

Then there are \( \rho, \sigma, \tau \in \mathcal{P}(e) \) with \( \lambda = \rho + \sigma, \mu = \rho + \tau \) and \( \sigma, \tau \) as described in the following table.
Case  \( \sigma_* \)  \( \tau_* \)  \( \text{Rest}^* \text{ictions} \)
\[ (p, q) \quad (p + 1, q - 1) \quad p > q > 1 \]
\[ (a) \quad (p, p) \quad (p + 1, p - 1) \quad p > 1 \text{ and } p \equiv e (2) \]
\[ (b_1) \quad (p, q) \quad (p + 2, q - 2) \quad p > q > 2 \]
\[ (b_2) \quad (p, p, q) \quad (p + 1, p + 1, q - 2) \quad p > q > 2 \text{ and } p \equiv q \not\equiv e (2) \]
\[ (b_3) \quad (p, q, q) \quad (p + 2, q - 1, q - 1) \quad p > q > 1 \]
\[ (b_4) \quad (p, p, q, q) \quad (p + 1, p + 1, q - 1, q - 1) \quad p > q > 1 \]

\textbf{Proof.} See (1.1) for the addition of partitions. Case I may be left to the reader. Case II. It is easy to see that we may assume disjointness: if \( \lambda_i = \mu_j \) then \( \lambda_i = 0 \). Now we have to prove \( \lambda = \sigma, \mu = \tau \) as in the table.

(a) Assume that there is a minimal \( \lambda \in \mathbb{N} \) with \( \lambda_i \neq 0 \) and \( \lambda_i \equiv e (2) \). There is a maximal \( m \in \mathbb{N} \) with \( \lambda_m = \lambda_i \). Define \( \nu \in P_e(n) \) by \( \nu_i = \lambda_i + 1, \nu_m = \lambda_m - 1 \) and \( \nu_i = \lambda_i \) otherwise. Clearly \( \Sigma \lambda > \Sigma \nu \). Using disjointness one proves \( \Sigma \nu > \Sigma \mu \), so that \( \nu = \mu \) and, again by disjointness, we are in case (a).

(b) Now \( \lambda_i \not\equiv e (2) \) whenever \( \lambda_i > 0 \). By disjointness there is an \( m \in \mathbb{N} \) with \( \mu_m > \lambda_1 > \mu_{m+1} \). It is easy to see that we can define \( \nu \in P_e(n) \) satisfying \( \Sigma \nu > \Sigma \mu \) as follows:

If \( \mu_m \not\equiv e (2) \), then \( \nu_m = \mu_m - 2 \) and \( \nu_i = \mu_i \) if \( i < m \);

If \( \mu_m \equiv e (2) \), then \( \nu_{m-1} = \nu_m = \mu_m - 1 \) and \( \nu_i = \mu_i \) if \( i < m - 1 \);

If \( \mu_{m+1} \not\equiv e (2) \), then \( \nu_{m+1} = \mu_{m+1} + 2 \) and \( \nu_i = \mu_i \) if \( i > m + 1 \);

If \( \mu_{m+1} \equiv e (2) \), then \( \nu_{m+1} = \nu_{m+2} = \mu_{m+1} + 1 \) and \( \nu_i = \mu_i \) if \( i > m + 2 \).

One proves that \( \Sigma \lambda > \Sigma \nu \), so that \( \lambda = \nu \) and we are in one of the four cases (b).

\[ (3.10) \text{Theorem. Let } k \text{ be a field. Consider } z \in \mathfrak{O} (\lambda) \text{ and } x \in \mathfrak{O} (\mu). \text{ We have } z \in Gx - Gx \text{ if and only if } \Sigma \lambda > \Sigma \mu. \]

\textbf{Remark.} This theorem is due to Gerstenhaber, see [9, p. 327] and [10, pp. 567–569]. His proof for Case II is incomplete, see (3.9). Our proof seems to be more explicit.

\textbf{Proof.} We may assume that \( z \) and \( x \) are rational points. So \( z \) is a standard nilpotent in \( \mathfrak{g}(k) \) with partition \( \lambda \). If \( i \in \mathbb{N} \) then the endomorphism \( z^i \) of \( F \) has rank \( (\Sigma D \lambda)^i \), see (1.1) for the definition of \( D \).

Assume that \( z \in Gx - Gx \). The rank of \( z^i \) is less than or equal to the rank of \( x^i \). This implies \( \Sigma D \lambda \leq \Sigma D \mu \) and hence \( \Sigma \lambda > \Sigma \mu \) by [9, p. 327]. As \( \lambda \not\equiv \mu \) it is easy to see that \( \Sigma \lambda > \Sigma \mu \).

Assume that \( \Sigma \lambda > \Sigma \mu \). We have to prove that \( z \in Gx \). We may assume that \( \lambda \) and \( \mu \) are as in (3.9). So \( \lambda \) and \( \mu \) are not both very-even, cf. (3.5), and it suffices to prove that \( z \in \mathfrak{S} (\mu) \). Using the notations of (3.7) we shall construct \( y \in \mathfrak{g}(k) \) and a sequence \( f(t) (t \in k) \) in such a way that \( z(t) = z + ty \in Gx \).
$g(k)$ is a standard nilpotent in $g_*(k)$ with base-data $(f(t), \mu)$ if $t \neq 0$. This will prove $z \in \mathcal{S}(\mu)$. 

Using a direct sum decomposition we may assume $\rho = 0$, $\lambda = \sigma$, $\mu = \tau$; cf. (3.9).

Case I. We have $\lambda_* = (p, q)$ and $\mu_* = (p + 1, q - 1)$. Let $(f_1, f_2, \lambda)$ be base-data for $z$. Put $y := e(1, q - 1; 2, q - 1)$. Put $f_1(t) := f_2$ and, if $q > 1$, $f_2(t) := tf_1 - zf_2$. We have
\[ 0 < a < q - 1 \Rightarrow z(t)^a f_1(t) = z^a f_2, \]
\[ q < a < p \Rightarrow z(t)^a f_1(t) = tz^{a-1} f_1, \]
\[ 0 < a < q - 2 \Rightarrow z(t)^a f_2(t) = tz^a f_1 - z^{a+1} f_2, \]
\[ z(t)^p f_1(t) = 0 \quad \text{and} \quad z(t)^q f_2(t) = 0. \]

This implies that $z(t) \in \mathcal{S}(\mu)$ if $t \neq 0$.

Case II. Of the five possibilities, cf. (3.9), we only treat $(b_3)$ and $(b_4)$ with $q \geq 2$. The other cases are easier, see [13, (4.3.7)], and already settled in [10, pp. 568, 569]. We choose convenient base-data $((f_1, \ldots, f_r), \lambda, \beta, \alpha)$ for $z$. The verifications are left to the reader.

$(b_3)$ $\lambda_* = (p, q, q)$, $p \equiv q \not\equiv \epsilon(2)$, $r = 3$, $\beta = \text{id}$, $\mu_* = (p + 2, q - 1, q - 1)$. Choose
\[ y := y(1, p - 1; 2, 0) + y(1, p - 1; 3, 0) = e(1, 0; 2, 0) + e(1, 0; 3, 0) + e(2, q - 1; 1, p - 1) - e(3, q - 1; 1, p - 1), \]
\[ f_1(t) := f_2, \quad f_2(t) := zf_2 \quad \text{and} \quad f_3(t) := z^{p-q+1}f_1 - tf_2 + tf_3. \]

$(b_4)$ $\lambda_* = (p, p, q, q)$, $p \equiv q \not\equiv \epsilon(2)$, $r = 4$, $\beta = \text{id}$, $\mu_* = (p + 1, p + 1, q - 1, q - 1)$. Choose
\[ y := y(1, p - 1; 3, 0) + y(1, p - 1; 4, 0) = e(1, 0; 2, 0) + e(1, 0; 3, 0) + e(2, q - 1; 1, p - 1) + e(2, q - 1; 1, p - 1) \]
\[ + e(3, q - 1; 1, p - 1) - e(4, q - 1; 1, p - 1) \]
\[ - e(3, q - 1; 2, p - 1) + e(4, q - 1; 2, p - 1), \]
\[ f_1(t) := f_1, \quad f_2(t) := f_3, \quad f_3(t) := zf_3 \quad \text{and} \quad f_4(t) := z^{p-q+1}f_1 - tf_3 + tf_4. \]

4. The classical nilpotent scheme, singularities.

(4.1) The symmetrical polynomials $\sigma_1, \ldots, \sigma_n \in A(\text{End}(F))$ are defined by the equation
\[ \det(x + T \cdot \text{id}) = T^n + \sum_{m=1}^{n} T^{n-m} \sigma_m(x) \]
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in \( R[T] \) where \( R \) is a \( k \)-algebra and \( x \in \text{End}(F)(R) \). They are invariant under the adjoint action of \( GL(F) \) on \( \text{End}(F) \). Let \( X = (x_{ij}) \) be the matrix of \( x \) with respect to some basis \( f_1, \ldots, f_n \) of \( F \). Then

\[
\sigma_m(x) = \sum \det(x_{ij})_{i,j\in I}
\]

where the summation is over all subsets \( I \) of \( \{1, \ldots, n\} \) with \( \# I = m \).

Case II. Clearly \( \sigma_m|_{\mathfrak{g}} \in A(\mathfrak{g})^G \). Let \( \Phi \) be the matrix \( \phi(f_i, f_j) \). We have \( x \in \mathfrak{g}(R) \) if and only if \( X = -\Phi \Phi^{-1} \). This implies that \( \sigma_m|_{\mathfrak{g}} = 0 \) if \( m \) is odd.

Assume \( e = \xi = 0 \). We define \( \tau_i \in A(\mathfrak{g}) \) by \( \tau_i(x) := \text{Pf}(\Phi X) \), where \( \text{Pf} \) denotes the Pfaffian, cf. [3, §5, no. 2]. Using loc. cit. one proves that \( \tau_i^2 = \det(\Phi)\sigma_n \) and that \( \tau_i \in A(\mathfrak{g})^G \).

We define the sequence \( a = (a_1, \ldots, a_l) \) in \( A(\mathfrak{g}) \) as follows. In Case I we put \( a_i := \sigma_i \). In Case II \((e, 1-e)\) we put \( a_i := \sigma_{2i} \). In Case II \((0, 0)\) we put \( a_i := \sigma_{2i} \) if \( i < l \), and \( a_i := \tau_i \).

**Theorem.** (a) \( A(\mathfrak{g})^G \) is the free polynomial ring \( k[a_1, \ldots, a_l] \).

(b) The sequence \( a \) is \( A(\mathfrak{g}) \)-regular (in any order).

(c) \( N(G) = \text{Spec}(A(\mathfrak{g})/(a)) \), it is flat over \( k \).

(d) \( N(G) \) is smooth over \( k \) in the points of \( \Sigma(v) \) where \( v \) is, cf. (3.8)(b).

(e) If \( k \) is a normal ring then \( N(G) \) is a normal scheme.

**Proof.** (a) Let \( u: k[T_1, \ldots, T_l] \rightarrow A(\mathfrak{g})^G \) be defined by \( T_i \mapsto a_i \). We have to prove that \( u \) is bijective. Replacing \( k \) by a faithfully flat \( k \)-algebra (cf. (3.6) and (3.2)), we may assume the existence of a standard nilpotent \( z \in \mathfrak{g}(k) \) with partition \( \nu \), cf. (3.8)(b). Let \( z + L \) be the cross section of (3.7). By (2.2) the morphism \( v: A(\mathfrak{g})^G \rightarrow A(z + L) \) is injective. Case by case one shows that \( v \circ u \) is bijective, so that \( u \) is bijective.

(b) and (c). By [7], Theorem (2.6) applies. So \( A(\mathfrak{g}) \) is flat over \( A(\mathfrak{g})^G \).

So we have (b) and (c).

(d) We may use the cross section of (a). Now \( (z + L) \cap N(G) \) is a cross section at \( z \) for the action of \( G \) on \( N(G) \), and the assertion follows from \( (z + L) \cap N(G) \cong \text{Spec}(k) \).

(e) By (c) and [12, IV 6.14.1] we may assume that \( k \) is a field. Now \( N(G) \) is nonsingular in codimension one, by (d) and (3.8)(b). So \( N(G) \) is normal by Serre's criterion [12, IV 5.8.6].

**Remarks.** (i) There are other ways to prove the theorem, either avoiding (3.7) or avoiding (2.6) and [7]. (ii) It can be shown that \( N(SL(F)) \) exists and is equal to \( N(GL(F)) \), where \( k \) is arbitrary. Here (2.6) does not apply.

(4.2) If \( \lambda \in \mathcal{P}(n) \), the partition \( \Sigma \lambda \) is defined in (3.9). Case II \((\epsilon, 1-\epsilon)\). If \( \lambda \in \mathcal{P}_\epsilon(n) \) where \( n = 2l + 1 - \epsilon \), then we define the partition \( \Sigma_\epsilon \lambda \) by \( (\Sigma_\epsilon \lambda)_i := (\Sigma \lambda)_{2l-\epsilon} \). Case II \((0, 0)\). If \( \lambda \in \mathcal{P}_0(n) \) where \( n = 2l \), then we define \( \Sigma_0 \lambda \).
\[ \theta + \nu \] where \( \theta, \nu \in \mathbb{P} \) are given by \( \theta_i := (\Sigma \lambda)_2i+1 \) and \( \nu_* := (\frac{1}{2} \lambda^1 - 1) \).

Note: in the last case \( \lambda^1 \) is even and \( (\Sigma \lambda)_1 = \lambda^1 - 1 \). We write \( \Sigma_\epsilon \) to denote \( \Sigma \) in Case I and \( \Sigma_\epsilon \) in Case II. \( \Sigma_{(1)} \) means that \( \epsilon = 0 \) is excluded in Case II.

**Theorem.** Consider \( x \in \Sigma(\lambda) \). Then \( \text{ord}(x, N(G)/k) = \Sigma_{(1)} \lambda \) in Cases I and II (1, 0), and \( \text{ord}(x, N(G)/k) \geq \Sigma_0 \lambda \) in Case II (0, 0).

**Proof.** (1) By (1.7) we may replace \( (N(G), x) \) by a smoothly equivalent pointed scheme. So by (3.6) and (3.2) we may assume the existence of a standard nilpotent \( z \in \mathfrak{g}(k) \) with partition \( \lambda \). By (2.1) (a) we may assume that \( x = z(p) \) for some \( p \in \text{Spec}(k) \). Put \( A := 0_{\mathfrak{g}, x} \) and \( B := 0_{N(G), x} \). We have \( B = A/(a) \) where \( a \) is the \( A \)-regular sequence of (4.1), or rather its image in \( A \).

(2) Let \( J \) be the ideal in \( A(\mathfrak{g}) \) corresponding to the section \( z \). So \( x \) corresponds to the prime ideal \( J + pA(\mathfrak{g}) \). We claim

(a) If \( 1 \leq i \leq n \) and \( m := 1 + (\Sigma \lambda)_n+1-i \) then \( a_i \in J^m \).

(b) In Case II (0, 0) we have \( r_1 \in J^m \) where \( m := \frac{1}{2} \lambda^1 \).

Proof of (a). It suffices to consider Case I. Let \( (f, \lambda) \) be base-data for \( z \).

Using the notation of (3.7) we define

\[ \sigma_P := \det \xi(\psi; \psi')_{\psi, \psi' \in \mathcal{P}} \]

if \( \emptyset \neq P \subset \Psi \). So \( a_i = \Sigma \sigma_P \) where the summation is over all \( P \) with \( \#P = i \). If \( \xi(\psi; \psi') \notin J \) then we have \( \psi' = (j, a) \), \( \psi = (j, a + 1) \) for some \( j \) and \( a \). Consider \( P \) with \( \#P = i \). If \( \pi \) is a permutation of \( P \) then one verifies that

\[ \# \{ (j, a) \in P | \pi(j, a) \neq (j, a + 1) \} \geq 1 + (\Sigma \lambda)_n+1-i = m. \]

This implies \( \sigma_P \in J^m \), proving (a).

Proof of (b). We may assume that \( k \) is reduced. Now the assertion follows from \( r_1^2 = \det(\psi)_{\sigma_n} \in J^2m \), cf. (a).

(3) By (1.6)(a) it follows from (2)(a), (b) that \( \text{ord}(B/k) \geq \Sigma_\epsilon \lambda \). This proves the theorem in Case II (0, \( \xi \)). In the rest of the proof Case II (0, \( \xi \)) is excluded. It suffices to prove

\[ \text{ord}(B/k) \leq \Sigma_{(1)} \lambda. \]

By (1.5) and (4.1)(c) we may replace \( k \) by an algebraic closure of the field \( k(p) \). So henceforth \( k \) is an algebraically closed field. Now \( x \) and \( z \) may be identified.

Let \( (f, \lambda), (f, \lambda, \beta, \alpha) \) be its base-data.

(4) We prove (3) (*) by induction on \( n = |\lambda| \). The cases with \( n \leq 1 \) are trivial. So assume \( n \geq 2 \). Put \( r := \lambda^1 \). Let \( \rho \) be the partition with \( \rho_* = (r - 1) \).

The partition \( \mu \) is defined as follows.

**Case I.** \( \mu_r := \lambda_r - 1, \mu_i := \lambda_i \) if \( i \neq r \).

**Case II.** If \( \lambda_r \) is even then \( \mu_r := \lambda_r - 2 \) and \( \mu_i := \lambda_i \) otherwise. If \( \lambda_r \) is
odd so that \( \lambda_{r-1} = \lambda_r \), then \( \mu_{r-1} := \mu_r := \lambda_r - 1 \) and \( \mu_i := \lambda_i \) otherwise.

One verifies that \( \mu \in P \) and that \( \Sigma(1) \lambda = \rho + \Sigma(1) \mu \).

We have \( F = \Sigma k \psi \), \( \psi \in \Psi \), cf. (3.7). Let \( P \) be the subset of \( \Psi \) containing \( (r, 0) \) and in Case II also \( (\beta_r, \lambda_r - 1) \). Put \( F' = \Sigma k \psi \), \( \psi \in P \), and \( F'' = \Sigma k \psi \), \( \psi \in P \). Clearly \( F = F' \oplus F'' \). In Case II the form \( \phi' := \phi | F' \) is nondegenerate and hence a 1-form on \( F' \). We put \( G' := GL(F') \) in Case I and \( G' := G(F', \phi') \) in Case II. So the convention (3.1) concerning \( G' \) is not applied here. We put \( g' := \text{Lie}(G') \), etc.

Let
\[
\begin{pmatrix}
x' & x_2 \\
x_1 & x_3
\end{pmatrix}
\]
be the matrix of \( x \) with respect to the decomposition \( F = F' \oplus F'' \). Now \( x' \) is a standard nilpotent in \( g'(k) \) with partition \( \mu \). Consider the ring
\[
B' := \mathcal{O}_{N(G'), x'} = \mathcal{O}_{g', x' / \langle \sigma' \rangle_{i < n}}.
\]
By induction we have \( \text{ord}(B'/k) \leq \Sigma(1) \mu \). One verifies that
\[
w \longmapsto \begin{pmatrix} w & x_2 \\ x_1 & x_3 \end{pmatrix}
\]
defines a regular immersion \( u: g' \to g \) such that \( u(x') = x \), \( u^0(\sigma_n) = 0 \) and \( u^0(\sigma_i) = \sigma_i' \) if \( i < n \), where \( u^0: A(g) \to A(g') \) is the comorphism. Put \( R := A / \langle \sigma_i \rangle_{i < n} \) so that \( B = R / f \) where \( f \) is the image of \( \sigma_n \) in \( R \). Now there is an \( R \)-regular sequence \( x \) in \( R \) such that \( B' \cong R / (x) \) and \( f \in (x) \). By (1.6)(b) this implies
\[
\text{ord}(B/k) \leq \rho + \text{ord}(B'/k) \leq \rho + \Sigma(1) \mu = \Sigma(1) \lambda
\]
provided that \( f \notin m^{r+1}_R \). So in order to prove the theorem it suffices to prove that
\[
(\ast) \quad \sigma_{\psi} \notin (\sigma_P)_{P \neq \psi} + m^{r+1}_A
\]
where we have used the notation of (2).

(5) In Case II we normalize the base-data of \( x \) as follows: \( \beta_i \neq i \) if and only if \( \lambda_i \) is odd; \( |\beta_i - i| < 1 \) for all \( i \); if \( i \geq \beta_i \) then \( \alpha(i) = 1 \). Now \( i \leq \beta_i \) implies \( \alpha(i) = (-1)^{\lambda_i} \). With the notation of (3.7) we define a linear subvariety \( M \) of \( g \).

Case I. \( M := \Sigma ke(i, 0; j, \lambda_j - 1) \) (\( 1 < i, j \leq r \)).

Case II. \( M := \Sigma k \psi(i, \lambda_i - 1; j, \lambda_j - 1) \) where the summation is over all pairs \( (i, j) \) such that \( i = j \) or \( i \leq \beta_i < j \leq \beta_j \). So in this case \( M \subset M_I \).

The ring \( A(x + M) \) is considered as a graded \( k \)-algebra such that \( x \) corresponds to the augmentation ideal. The functions \( \sigma_P \) and \( M + A \) are homogeneous,
\( \sigma_{\psi} | x + M \) is homogeneous of degree \( r \). So it suffices to prove

\[
(\ast) \quad \sigma_{\psi} | x + M \notin \langle (\sigma_{p}| x + M)_{P \neq \psi} \rangle.
\]

(6) Case I. It is easy to see that \( x + M \) has a subvariety \( x_1 + M_1 \) such that

\( \sigma_{p}| x_1 + M_1 \neq 0 \) if and only if \( P = \Psi \). This proves \((5)(\ast)\) and the theorem.

Case II. Consider the subvariety \( x_1 + M_1 \) of \( x + M \) where

\[
x_1 := x + \sum_{i > \beta_0} y(i, \lambda_i - 1; i, \lambda_i - 1),
\]

\[
M_1 := \sum_{i > \beta_0} k y(i, \lambda_i - 1; i, \lambda_i - 1) \quad (i < \beta_0, i < j < \beta_0).
\]

Now \( x_1 \) is a standard nilpotent in \( g_f(k) \) with base-data \((f', \lambda')\) such that

\[ M_1 = \sum_{i < j} k e'(i, 0; j, \lambda_j' - 1) + e'(j, 0; i, \lambda_i' - 1) \]

with respect to the new base-data. In order to prove \((5)(\ast)\) and hence the theorem, it suffices to show that

\[ \sigma_{\psi} | x_1 + M_1 \notin \langle (\sigma_{p}| x_1 + M_1)_{P \neq \psi} \rangle. \]

This is a consequence of the following:

**Lemma.** Assume \( \text{char}(k) \neq 2 \). Let \( r \in \mathbb{N} \). Consider the ring \( k[T_{ij}] \) where

\[ 1 \leq i \leq j \leq r. \]

Put \( T_{ij} := T_{ji} \) if \( i > j \). Put \( Q := \{1, \ldots, r\} \). If \( 0 \neq P \subset Q \) define \( \sigma_{p} := \det(T_{ij})_{i, j \in P} \). Then \( \sigma_{Q} \notin \langle (\sigma_{p})_{P \neq Q} \rangle \).

**Proof.** We may assume \( r \geq 3 \). Let \( I \) be the ideal generated by all \( T_{ij} \)

such that \( 1 \neq |i - j| \neq r - 1 \), and all \( T_{ij}^2 \). It is easy to see that \( \sigma_{p} \notin I \) if and only if \( P = Q \).

(4.3) The following facts are not proved here, see [13, pp. 11–13].

(i) The mapping \( \Sigma_{(1)}: P_{(1)}(n) \rightarrow P \) is injective.

(ii) If \( \Sigma_{1} \lambda \leq \Sigma_{1} \mu \) where \( \lambda, \mu \in P_{(1)}(n) \) then \( \Sigma \lambda \leq \Sigma \mu \).

(iii) If \( \lambda \in P_{(1)}(n) \) then \( \gamma_{(1)}(\lambda) = 2|\Sigma_{(1)} \lambda| \).

Using (1.7), (2.1)(a), (3.5), (3.8), (3.10), (4.2) we get the following

**Corollary.** Case I and II \((1, 0)\). Let \( x \in \mathcal{O}(\lambda) \) and \( y \in N(G) \).

(a) \( y \in \mathcal{O}(\lambda) \) if and only if \( \text{ord}(y, N(G)/k) = \text{ord}(x, N(G)/k) \).

(b) \( y \in Gx \) if and only if \( (N(G), y) \sim (N(G), x) \), cf. (1.7).

Assume that \( k \) is a field.

(c) \( \text{codim}(Gx, N(G)) = 2|\text{ord}(x, N(G)/k)|. \)

(d) \( y \in Gx \) if and only if \( \text{ord}(y, N(G)/k) \geq \text{ord}(x, N(G)/k) \).

(4.4) **Remark.** In (4.2) Case II \((0, \xi)\), inequality occurs if \( \lambda_1 \) is even and also if \( \lambda_\ast = (3, 3, 2, 2) \), but we have equality if \( \lambda_\ast = (3, 3, 2, 2, 1) \). In the last case we have
codim(Gx, N(G)) = γ₀(λ) > 2|Σ₀λ| = 2|ord(x, N(G)/k)|

if k is a field, compare (4.3)(c) and (4.9) table B₅.

(4.5) The polynomials \( f_a \) are defined by \( f_a := 0 \) if \( a < 0 \), \( f₀ := 1 \) and \( f_a := \sum_{i>1} X_1^a f_a-i \) if \( a > 0 \). They are determined by the generating function

\[
\sum_{a=0}^\infty T^a f_a = \left( 1 - \sum_{i>1} X_i T^i \right)^{-1}.
\]

Clearly \( f_a(X_1) = X_1^a \) if \( a \geq 0 \). One can prove that

\[
f_a(X_1, X_2) = \sum \binom{a - i}{i} X_1^{a-2i} X_2^i \quad (0 \leq i \leq \frac{a}{2}).
\]

Let \( A^m \) denote the affine space over \( k \) of rank \( m \), say with coordinate ring \( k[X_1, \ldots, X_m] \). It is pointed in some point of the origin section. The Kleinian singularities \( A_i \) and \( D_i \) are the pointed subschemes of \( A^3 \) given by one equation:

- \( A_i \), \( l \geq 1 \), by \( X_1^{l+1} + X_2 X_3 = 0 \),
- \( D_i \), \( l \geq 3 \), by \( X_1^{l-1} - X_1 X_2^2 + X_3^2 = 0 \), if \( 1/2 \in k \).

We define the following singularities.

If \( l \geq 3 \), \( AA_i \) in \( A^2 \times A^4 \) by

\[
\begin{align*}
& f_i(X_1, X_2) = Y_1 Y_3 + Y_2 Y_4 = 0, \\
& X_2 f_{i-1}(X_1, X_2) = Y_4(X_1 Y_2 - X_2 Y_1) + Y_2 Y_3 = 0.
\end{align*}
\]

If \( 1/2 \in k \) and \( l \geq 3 \), \( BB_i \) in \( A^2 \times A^4 \) by

\[
\begin{align*}
& f_{i-1}(2X_1, -X_2^2) = 2Y_1 Y_3 + Y_2^2 - Y_4^2 = 0, \\
& X_2 f_{i-2}(2X_1, -X_2^2) = (Y_3 - X_1 Y_1^2 - X_2^2 Y_1^2 - 2Y_4(X_1 Y_4 - X_2 Y_2) = 0.
\end{align*}
\]

If \( 1/2 \in k \) and \( l \geq 2 \), \( CC_i \) in \( A^3 \times A^2 \) by

\[
X_3^2 - X_1 X_2 Y_1 + X_1 Y_2 + 2X_3 Y_1 Y_2 + X_2 Y_2^2 = 0.
\]

If \( 1/2 \in k \) and \( l \geq 5 \), \( CD_i \) in \( A^2 \times A^4 \) by

\[
\begin{align*}
& f_{i-2}(X_1, X_2) = X_1 Y_2^2 - X_2 Y_1^2 - Y_3^2 + 2Y_2 Y_4 = 0, \\
& X_2 f_{i-3}(X_1, X_2) = X_2(Y_1^2 + Y_2^2 - 2Y_1 Y_3) + Y_4^2 = 0.
\end{align*}
\]

If \( 1/2 \in k \) and \( l \geq 3 \), \( DD_i \) in \( A^3 \times A^3 \) by

\[
\begin{align*}
& (x_1^2 + x_2^2 + x_3^2)^{l-1} + y_1^2 + y_2^2 + y_3^2 = 0, \\
& x_1 y_1 + x_2 y_2 + x_3 y_3 = 0.
\end{align*}
\]
Proposition. Assume in Case II that \( l + \epsilon + \xi \geq 3 \). Consider \( \lambda \in P(\epsilon)(n) \) with \( 0 < \gamma(\epsilon)(\lambda) < 6 \), cf. (3.8). If \( x \in \Sigma(\lambda) \) then \((N(G), x)\) is smoothly equivalent (cf. (1.7)) to the singularity (cf. (4.5)) given in the following table.

<table>
<thead>
<tr>
<th>( G )</th>
<th>( \lambda )</th>
<th>Dynkin diagram</th>
<th>( \gamma(\epsilon)(\lambda) )</th>
<th>singularity</th>
</tr>
</thead>
<tbody>
<tr>
<td>( G_{1n} ), ( n \geq 2 )</td>
<td>( (n-1,1) )</td>
<td>( \epsilon )</td>
<td>2</td>
<td>( A_{n-1} )</td>
</tr>
<tr>
<td>( SO_{2\ell+1} ), ( \ell \geq 2 )</td>
<td>( (2\ell-1,1,1) )</td>
<td>( \epsilon )</td>
<td>2</td>
<td>( A_{2\ell-1} )</td>
</tr>
<tr>
<td>( Sp_{2\ell} ), ( \ell \geq 2 )</td>
<td>( (2\ell-2,2) )</td>
<td>( \epsilon )</td>
<td>2</td>
<td>( D_{\ell+1} )</td>
</tr>
<tr>
<td>( SO_{2\ell} ), ( \ell \geq 3 )</td>
<td>( (2\ell-3,3) )</td>
<td>( \epsilon )</td>
<td>2</td>
<td>( D_{\ell} )</td>
</tr>
<tr>
<td>( G_{1n} ), ( n \geq 4 )</td>
<td>( (n-2,2) )</td>
<td>( \epsilon )</td>
<td>4</td>
<td>( AA_{n-1} )</td>
</tr>
<tr>
<td>( SO_{2\ell+1} ), ( \ell \geq 3 )</td>
<td>( (2\ell-3,3,1) )</td>
<td>( \epsilon )</td>
<td>4</td>
<td>( BB_{\ell} )</td>
</tr>
<tr>
<td>( Sp_{2\ell} ), ( \ell \geq 2 )</td>
<td>( (2\ell-2,1,1) )</td>
<td>( \epsilon )</td>
<td>4</td>
<td>( CC_{\ell} )</td>
</tr>
<tr>
<td>( SO_{2\ell} ), ( \ell \geq 3 )</td>
<td>( (3,3) )</td>
<td>( \epsilon )</td>
<td>4</td>
<td>( DD_{\ell} )</td>
</tr>
<tr>
<td>( \ell \geq 4 )</td>
<td>( (2\ell-4,4) )</td>
<td>( \epsilon )</td>
<td>4</td>
<td>( CD_{\ell+1} )</td>
</tr>
<tr>
<td>( SO_{2\ell} ), ( \ell \geq 3 )</td>
<td>( (2\ell-3,1,1,1) )</td>
<td>( \epsilon )</td>
<td>4</td>
<td>( DD_{\ell} )</td>
</tr>
<tr>
<td>( \ell \geq 4 )</td>
<td>( (4,4) )</td>
<td>( \epsilon )</td>
<td>4</td>
<td>( CD_{\ell} )</td>
</tr>
</tbody>
</table>

Remark. We have \( \gamma(\epsilon)(\lambda) = \text{codim}(Gx, N(G)(k(x))) \). For the singularities with \( \gamma(\epsilon)(\lambda) = 2 \), compare [5] and [21, pp. 140–158]. In the table we have added the Dynkin diagram of the section \( x \in g(k(x)) \), cf. [20, III, IV], where \( <---------- \) means a string with numbers 2 attached to the nodes.

Proof. The classification of all possibilities for \( \lambda \) is easy. By the sequence of reductions used in (4.2)(1) we may assume that \( x = z(\mathfrak{p}) \) where \( z \) is a standard nilpotent with partition \( \lambda \) and \( \mathfrak{p} \in \text{Spec}(k) \). In Case II the base-data for \( z \) may be prescribed within the bounds set by (3.3)(2). Let \( z + L \) be the cross section of (3.7). Then \( (z + L) \cap N(G) \) is a cross section at \( z \) for the action of \( G \) on \( N(G) \). So \((N(G), x)\) is smoothly equivalent to \((z + L) \cap N(G), z(\mathfrak{p}))\) by (2.1)(a). The two singularities to be determined for \( GL_n \) will be examples in (4.7) and (4.8). We do not give the tedious calculations needed to settle Case II, see [13, p. 79] for some indications.

(4.7) Case I with \( \lambda_\mathfrak{a} = (p, 1^q) \), i.e. \((p, 1, \ldots, 1)\) with \( q \) times 1. We have \( n = p + q \) and \( r := \lambda^1 = q + 1 \). On \( z + L \) we define the coordinate functions \( \xi_a, \xi_{ij} \) as follows: if \( R \) is a \( k \)-algebra and \( x \in (z + L)(R) \), then
\[ x = z - \sum_{a=1}^{p} \xi_a(x) e(1, p-a; 1, p-1) - \sum_{(i,j) \neq (1,1)} \xi_{ij}(x) e(i, 0; j, \lambda_j - 1). \]

So \( A(z + L) = k[\xi_a, \xi_{ij}] \). Put \( \xi_{11} = 0 \).

If \( a \geq 1 \), let \( s_a, h_a \in k[\xi_{ij}] \) be defined by

\[
\begin{align*}
\left\{ \begin{array}{l}
\quad s_a = \sum \det(\xi_{ij})_{i,j \in I} \\
\quad h_a = \sum \det(\xi_{ij})_{i,j \in \{1\} \cup I}
\end{array} \right.
\]

where in both cases the summation is over the subsets \( I \) of \( \{2, \ldots, r\} \) with \( \#I = a \).

Clearly, if \( a > r \) then \( s_a = h_a = 0 \). The subscheme \( (z + L) \cap N(G) \) of \( z + L \) is defined by the equations \( \sigma_m |(z + L) = 0 \) (\( 1 \leq m \leq n \)). One verifies that

\[
\begin{align*}
\left\{ \begin{array}{l}
(-1)^m \sigma_m |(z + L) = \xi_m + s_m + \sum_{a=1}^{m-1} \xi_a s_{m-a} \quad &\text{if } 1 \leq m \leq p, \\
(-1)^m \sigma_m |(z + L) = h_{m-p} + s_m + \sum_{a=1}^{p} \xi_a s_{m-a} \quad &\text{if } p < m \leq n.
\end{array} \right.
\end{align*}
\]

The first \( p \) equations can be solved inductively. With the notations of (4.5) we obtain \( \xi_m = f_m(-s_1, -s_2, \ldots, -s_q) \) (\( 1 \leq m \leq p \)). So \( (z + L) \cap N(G) \) is isomorphic to the subscheme of \( \Spec k[\xi_{ij}] \) defined by the equations

\[
\begin{align*}
\left\{ \begin{array}{l}
\quad h_{m-p} + \sum_{a=0}^{p} s_{m-a} f_a(-s_1, \ldots, -s_q) = 0 \quad &\text{if } p < m \leq n.
\end{array} \right.
\end{align*}
\]

**EXAMPLES.** (a) \( \lambda_{\ast} = (n - 1, 1) \). Putting \( X_1 = -\xi_{22}, X_2 = \xi_{12}, X_3 = \xi_{21}, \) we get the singularity \( A_n-1, \) cf. (4.5).

(b) \( \lambda_{\ast} = (n - 2, 1, 1) \). The scheme \( (z + L) \cap N(G) \) is isomorphic to the subscheme of \( A^8 = \Spec(k[\xi_{ij}]) \), where \( 1 \leq i, j \leq 3 \leq i + j \), defined by the equations

\[
\begin{align*}
\left\{ \begin{array}{l}
\quad f_{n-1}(-s_1, -s_2) - h_1 = 0, \\
\quad s_2 f_{n-2}(-s_1, -s_2) + h_2 = 0,
\end{array} \right.
\end{align*}
\]

where

\[
\begin{align*}
\quad s_1 &= \xi_{22} + \xi_{33}, \\
\quad s_2 &= \xi_{22} \xi_{33} - \xi_{23} \xi_{32}, \quad \text{and} \quad h_2 = \begin{vmatrix}
0 & \xi_{12} & \xi_{13} \\
\xi_{21} & \xi_{22} & \xi_{23} \\
\xi_{31} & \xi_{32} & \xi_{33}
\end{vmatrix}, \\
\quad h_1 &= \xi_{12} \xi_{21} + \xi_{13} \xi_{31}.
\end{align*}
\]

(4.8) **Case I for arbitrary \( \lambda \).** We use a different cross section, viz. \( z + L'' \) defined by \( L'' = \sum_{i,j} L_{ij} \) where \( L_{ij} \) if \( i \neq 1 \) or \( j = 1 \) and \( L_{1,1}(R) := \sum_{0 \leq b < \lambda_j} \text{Re}(1, 0; j, b) \) if \( j \neq 1 \), see (3.7). Again we have
\((N(G), x) \sim ((z + L'') \cap N(G), z(p)).\)

Put \(p := \lambda_1, q := n - \lambda_1\) and \(\mu_\ast := (p, 1^q)\). Put \(z' := \Sigma_{i=1}^{p-q}(1, a + 1; 1, a)\), so that \(z'\) is a standard nilpotent element in \(\mathfrak{g}(k)\) with partition \(\mu\). In the obvious way we define base-data \((\mathfrak{g}', \mu)\) for \(z'\). The cross section \(z' + L'\) at \(z'\) used in (4.7) contains \(z + L''\). So we can use the elimination in (4.7) of \(\xi_0, 1 \leq a \leq p\), substituting into the matrix \((\xi_{ij})\) at some places the constant functions 0 or \(-1\), cf. (4.7)(1).

**Example.** If \(\lambda_\ast = (n - 2, 2), n \geq 4\), we use the matrix

\[
(\xi_{ij}) = \begin{pmatrix}
0 & Y_3 & Y_4 \\
Y_1 & -X_1 & -1 \\
Y_2 & -X_2 & 0
\end{pmatrix}
\]

and we obtain the equations (4.7)(5) where \(s_1 = -X_1, s_2 = -X_2, h_1 = Y_1 Y_3 + Y_2 Y_4\) and \(h_2 = Y_4(X_1 Y_2 - X_2 Y_1) - Y_2 Y_3\). So \((z + L'') \cap N(G)\) is isomorphic to the singularity \(A_{n-1}\), cf. (4.5).

(4.9) **Tables for the orbits in** \(N(G)\). We give the adjacency structure (cf. (3.10)), the Dynkin diagram (cf. [20, IV]), the codimension of the orbits \(\gamma(x)(\lambda)\) (cf. (3.8)), and the partition \(\text{ord} = \text{ord}(x, N(G)/k)\) (cf. (4.2)). The number of orbits is denoted by \#\. In the cases \(SO_2\) with even \(l\), the partition \(\lambda\) may represent two orbits, cf. (3.5). We give the Dynkin diagram of one of them and indicate how to get the other one by the symbol \(\sim\).

For \(SO_n\) we give \(\Sigma_0 \lambda\), which is a lower bound of \(\text{ord}\), cf. (4.2). Whenever there are reasons to assume \(\text{ord} \neq \Sigma_0 \lambda\), we give a conjectured value of \(\text{ord}\) or a question mark. As \(D_2 = A_1 + A_1, B_2 = C_2\) and \(D_3 = A_3\), the values of \(\text{ord}\) for the cases \(SO_4, SO_5\) and \(SO_6\) are not conjectural.

\[
\begin{array}{cccccc}
A_1 & G_1 & \lambda_\ast & Dy & \gamma(\lambda) & \text{ord}_\ast \\
\# = 2 & & & & & \\
\quad & 2 & 2 & 0 & 0 \\
\quad & 1 & 1 & 0 & 2 & 1 \\
\end{array}
\]

\[
\begin{array}{cccccc}
A_2 & G_1 & \lambda_\ast & Dy & \gamma(\lambda) & \text{ord}_\ast \\
\# = 3 & & & & & \\
\quad & 3 & 2 & 2 & 0 & 0 \\
\quad & 2 & 1 & 1 & 2 & 1 \\
\quad & 1 & 1 & 1 & 0 & 6 & 2 & 1 \\
\end{array}
\]

\[
\begin{array}{cccccc}
D_2 = A_1 + A_1 & SO_\ast & \lambda_\ast & Dy & \gamma_0(\lambda) (\Sigma_0 \lambda)_\ast & \text{ord}_\ast \\
\# = 4 & & & & & \\
\quad & 3 & 1 & 2 & 2 & 0 & 0 \\
\quad & 2 & 2 & 0 & 2 & 2 & 1 & 0 & 1 \\
\quad & 1 & 1 & 0 & 0 & 4 & 1 & 1 \\
\end{array}
\]
### The Nilpotent Scheme of a Classical Group

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THE NILPOTENT SCHEME OF A CLASSICAL GROUP

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# = 14
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\lambda_1 & \quad \gamma(\lambda) \quad \text{ord.} \\
0 & \quad 0 \quad 0 \\
1 & \quad 1 \quad 1 \\
2 & \quad 4 \quad 1 \quad 1 \\
3 & \quad 6 \quad 2 \quad 1 \\
4 & \quad 12 \quad 3 \quad 2 \quad 1 \\
5 & \quad 3 \quad 2 \quad 11 \\
6 & \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\
\end{align*}

\begin{align*}
\lambda_2 & \quad \gamma(\lambda) \quad \text{ord.} \\
0 & \quad 0 \quad 0 \\
1 & \quad 1 \quad 1 \\
2 & \quad 4 \quad 1 \quad 1 \\
3 & \quad 6 \quad 2 \quad 1 \\
4 & \quad 12 \quad 3 \quad 2 \quad 1 \\
5 & \quad 3 \quad 2 \quad 11 \\
6 & \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\
\end{align*}

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### The Nilpotent Scheme of a Classical Group

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THE NILPOTENT SCHEME OF A CLASSICAL GROUP

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6 & & 11 \\
5 & & 111 \\
4 & & 1111 \\
3 & & 11111 \\
2 & & 111111 \\
1 & & 1111111 \\
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\end{align*}

REFERENCES


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