ON BOUNDED FUNCTIONS SATISFYING AVERAGING CONDITIONS. II

BY

ROTRAUT GOUBAU CAHILL

ABSTRACT. Let $S(f)$ denote the subspace of $L^\infty(R^n)$ consisting of those real valued functions $f$ for which

$$\lim_{r \to 0} \frac{1}{|B(x, r)|} \int_{B(x, r)} f(y) dy = f(x)$$

for all $x$ in $R^n$ and let $L(f)$ be the subspace of $S(f)$ consisting of the approximately continuous functions. A number of results concerning the existence of functions in $S(f)$ and $L(f)$ with special properties are obtained. The extreme points of the unit balls of both spaces are characterized and it is shown that $L(f)$ is not a dual space. As a preliminary step, it is shown that if $E$ is a $G_\delta$ set of measure 0 in $R^n$, then the complement of $E$ can be decomposed into a collection of closed sets in a particularly useful way.

Introduction. Let $L^\infty(R^n)$ denote the space of all real valued $L^\infty(R^n)$ functions. If $f$ is in $L^\infty(R^n)$ and if $E$ is a measurable subset of $R^n$, let $J(f, E)$ denote $\int_E f$. For each $f$ in $L^\infty(R^n)$ define:

$$L(f) = \left\{ x \in R^n \mid \lim_{r \to 0} (J(|f - f(x)|, B(x, r))/|B(x, r)|) = 0 \right\}$$

where $B(x, r) = \{ y \in R^n \mid |y - x| < r \}$, i.e. $L(f)$ is the Lebesgue set of $f$.

$$S(f) = \left\{ x \in R^n \mid \lim_{r \to 0} (J(f, B(x, r))/|B(x, r)|) = f(x) \right\}.$$

Let $S(n, T)$ be the subspace of $L^\infty(R^n)$ consisting of those functions $f$ for which $S(f) = R^n$, and let $L(n, T)$ be the subspace of $L^\infty(R^n)$ consisting of those functions for which $L(f) = R^n$.

A function $f$ in $L^\infty(R^n)$ is defined to be approximately continuous at $x$ if $x$ is a point of density of $\{ y \mid |f(y) - f(x)| < \epsilon \}$ for every $\epsilon > 0$. It is easy to see that $L(n, T)$ consists precisely of those functions in $L^\infty(R^n)$ which are approximately continuous at each point of $R^n$. An example of a function which is in $S(n, T)$ but not in $L(n, T)$ is the function

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The example shows that $S(n, T)$ is not an algebra, whereas it is readily shown that $L(n, T)$ is an algebra.

In this paper a number of results will be obtained about the existence of functions in $S(n, T)$ and $L(n, T)$ which have special properties. The extreme points of the unit balls of these spaces will also be characterized. In the case of $L(n, T)$ it will be shown that there are only two such extreme points.

The proofs depend primarily on the fact that if $E$ is a $G_6$ subset of measure 0 contained in $\mathbb{R}^n$, then $E'$, the complement of $E$, can be decomposed in a special way into a collection of closed sets $\{\Phi_k\}_{k \geq 1}$ so that the function $\mu$ defined in $\mathbb{R}^n$ by

$$
\mu(x) = \begin{cases} 
0, & x \in E, \\
\frac{1}{\inf_x \{\lambda \mid x \in \Phi_k\}}, & x \notin E,
\end{cases}
$$

is approximately continuous and has a number of other useful properties. It will first be shown how to obtain such a decomposition of $E'$. The procedure used generalizes a method developed by Zygmunt Zahorski for obtaining a decomposition of the complement of a $G_6$ set of measure 0 contained in the open interval $(0, 1)$ [2].

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**Inverse Zahorski functions.**

**Lemma 1.** Let $M_1$ and $M_2$ be two bounded measurable subsets of $\mathbb{R}^n$ with measures $u_1$ and $u_2$ respectively. Suppose that $M_2$ is a closed subset of $M_1$ consisting only of points of density of $M_1$. Then for every positive number $p$, there is a closed set $M_p$ with $M_2 \subset M_p \subset M_1$ satisfying:

1. Every point of $M_2$ is a point of density of $M_p$ and every point of $M_p$ is a point of density of $M_1$.
2. $|M_p| \geq u_2 + (1 - 2^{-1-p})(u_1 - u_2)$.
3. Let $x \in M_2$, and let $\epsilon$ be an arbitrary number in $(0, 1)$. If $r$ is any positive number for which $(|M_1 \cap B(x, r)|/|B(x, r)|) > 1 - \epsilon$, then

$$
(|M_p \cap B(x, r)|/|B(x, r)|) > 1 - \epsilon - 2^{-m-p + c_n}
$$

for every positive integer $m$ for which $r \leq 1/m$, where $c_n$ is a constant which depends only on the dimension $n$.

**Proof.** H. Whitney has shown that since $M_2$ is closed, $M_2'$ is a countable union of closed cubes $Q_k$ with disjoint interiors, where these cubes may be chosen so that the following conditions hold:
(1) $\text{diam } Q_k \leq \text{dist}(Q_k, M_2) \leq 4 \text{ diam } Q_k$.

(2) If $Q_k^*$ is the cube with the same center as $Q_k$ and expanded by a factor $1 + \epsilon$ ($0 < \epsilon < \frac{1}{4}$, $\epsilon$ fixed), then $Q_k^*$ is contained in the union of all the cubes which touch $Q_k$.

(3) For each cube $Q_k$ there are at most $N = (12)^n$ cubes which touch $Q_k$ [1, pp. 167–169].

A cube $Q_k$ will be said to be of class $m$, $m$ a positive integer, if either

$$(1/(m + 1)) < \text{diam } Q_k \leq (1/m) \text{ or } m < \text{diam } Q_k < m + 1.$$  

If $Q_k$ is of class $m$ and if $|Q_k \cap M_1| > 0$, let $F_k$ be a closed subset of $Q_k \cap M_1$ consisting only of points of density of $M_1$, with $|F_k| \geq |Q_k \cap M_1|(1 - 2^{-m-\epsilon})$. Set $M_p = M_2 \cup_k F_k$.

It will be shown that $M_p$ satisfies all the required conditions. First, $M_p$ is closed, for if $(q_m)_{m \geq 1}$ is a convergent sequence in $M_p$, say $q_m \rightarrow q$, and if $q \notin M_2$, then $q$ is in some cube $Q_x$ and some neighborhood of $q$ is contained in $Q_x^*$. Since $Q_x^*$ is contained in the union of at most $N$ cubes $Q_k$, this neighborhood is contained in the union of at most $N$ cubes. Thus for $m$ sufficiently large, say $m \geq M$, $(q_m)_{m \geq M}$ is contained in at most $N$ of the sets $F_k$. Since this union is closed, $q$ is in some $F_k$ and hence $M_p$ is closed.

By construction, each point of $M_p$ is a point of density of $M_1$. It will now be shown that each point of $M_2$ is a point of density of $M_p$. The proof will be such that (3) will be proved simultaneously.

Let $x$ be in $M_2$. Let $\epsilon$ be an arbitrary number in $(0, 1)$ and let $m$ be an arbitrary positive integer. Since, by assumption, $x$ is a point of density of $M_1$, there is a $0 < \delta \leq 1/m$ such that for $r < \delta$, $(|B(x, r) \cap M_1|/|B(x, r)|) > 1 - \epsilon$.

Set $d(x, r) = (|M_1 \cap B(x, r)|/|B(x, r)|) - (|M_p \cap B(x, r)|/|B(x, r)|)$.

It will be shown that $d(x, r) \leq 2^{-m-\epsilon+c_n}$. From this it follows that $(|M_p \cap B(x, r)|/|B(x, r)|) > 1 - \epsilon - 2^{-m-\epsilon+c_n}$, which verifies (1) since $\epsilon$ and $m$ were arbitrary.

The proof that $d(x, r) \leq 2^{-m-\epsilon+c_n}$ will depend only on the fact that $m$ is a positive integer for which $r < 1/m$. Thus (3) will also be proved.

Let $K$ be the set of all integers for which $Q_k$ has nonempty intersection with the boundary of $B(x, r)$ and set

$$A = \bigcup_{k \in K} Q_k; \quad A_1 = A \cap B(x, r); \quad A_2 = A \cap B(x, r)^c;$$

$$\xi = |M_p \cap A|/|M_1 \cap A| \text{ if } |M_1 \cap A| > 0, \quad \xi = 1 \text{ if } |M_1 \cap A| = 0;$$

$$\xi_1 = |M_p \cap A_1|/|M_1 \cap A_1| \text{ if } |M_1 \cap A_1| > 0, \quad \xi_1 = 1 \text{ if } |M_1 \cap A_1| = 0;$$
\[ \xi_2 = |M_p \cap A_2|/|M_1 \cap A_2| \text{ if } |M_1 \cap A_2| > 0, \quad \xi_2 = 1 \text{ if } |M_1 \cap A_2| = 0. \]

We have

\[ d(x, r) = \left(1/|B(x, r)|\right) \left\{ \sum_{Q_k \subset B(x, r)} |Q_k \cap M_1| - F_k + |(M_1 - M_p) \cap A_1| \right\}. \]

Since diam \( Q_k \) < dist\( (Q_k, M_2) \) < \( r < 1/m \) for each cube \( Q_k \) which intersects \( B(x, r), \) \( |Q_k \cap M_1| - F_k \) < \( 2^{-m-p} |Q_k \cap M_1|, \) and thus

\[ d(x, r) \leq \left(1/|B(x, r)|\right) \left\{ 2^{-m-p} \sum_{Q_k \subset B(x, r)} |Q_k \cap M_1| + |(M_1 - M_p) \cap A_1| \right\}. \]

Thus if \( |M_1 \cap A_1| = 0, \) \( d(x, r) \leq 2^{-m-p}. \)

Suppose \( |M_1 \cap A_1| > 0. \) Observe that

\[ d(x, r) \leq \left(1/|B(x, r)|\right) \left\{ 2^{-m-p} \sum_{Q_k \subset B(x, r)} |Q_k \cap M_1| + (1 - \xi_2)|M_1 \cap A_1| \right\}. \]

The object of the calculations which follow is to show that \( 1 - \xi_1 \leq 2^{-m-p}\{1 + (|A_2|/|M_1 \cap A_1|)\}. \)

By solving the equation \( \xi |M_1 \cap A_1| = \xi_1 |M_1 \cap A_1| + \xi_2 |M_1 \cap A_2| \) for \( \xi_1 \) and observing that \( |M_1 \cap A_2| = |M_1 \cap A_1| + |M_1 \cap A_2|, \) we obtain

\[ \xi_1 = \xi - (\xi_2 |M_1 \cap A_2| - \xi |M_1 \cap A_1|)/|M_1 \cap A_1|. \]

Since \( |F_k| \geq |Q_k \cap M_1| (1 - 2^{-m-p}) \) for each \( Q_k \) which intersects \( B(x, r), \)
\( |M_p \cap A_1| \geq (1 - 2^{-m-p}) |M_1 \cap A_1| \) and \( \xi > 1 - 2^{-m-p}. \) Thus

\[ \xi_1 > 1 - 2^{-m-p} - \{(\xi_2 - (1 - 2^{-m-p}))/|M_1 \cap A_1|\} |M_1 \cap A_2|. \]

Since \( 0 < \xi_2 < 1 \) and \( |M_1 \cap A_2| \leq |A_2|, \)

\[ \xi_1 > 1 - 2^{-m-p} - 2^{-m-p} |A_2|/|M_1 \cap A_1|, \]

and

\[ 1 - \xi_1 \leq 2^{-m-p}\{1 + (|A_2|/|M_1 \cap A_1|)\}. \]

It follows that

\[ d(x, r) \leq 2^{-m-p}\{1 + \{|M_1 \cap A_1| (1 + (|A_2|/|M_1 \cap A_1|))\}/|B(x, r)|\} \]

\[ \leq 2^{-m-p}\{2 + (|A_2|/|B(x, r)|)\}. \]

Since diam \( Q_k \leq r \) for each \( Q_k \) which intersects \( B(x, r), A_2 \subset B(x, 2r) - B(x, r). \) Thus,

\[ (|A_2|/|B(x, r)|) \leq (1/|B(x, r)|)(|B(x, 2r)| - |B(x, r)|). \]
The number $d_n = (1/|B(x, r)|)(|B(x, 2r)| - |B(x, r)|)$ depends only on $n$ and $d(x, r) \leq 2^{-m-p}(2 + d_n) \leq 2^{-m-p+c_n}$, where $2c_n = 2 + d_n$. Therefore, (1) and (3) both hold.

Finally,

$$|M_\rho| = |M_2| + \sum_k |F_k| \geq u_2 + \sum_k |Q_k \cap M_1| (1 - 2^{-m-p})$$

$$> u_2 + (1 - 2^{-p})(\sum_k |Q_k \cap M_1| = u_2 + (1 - 2^{-p})(u_1 - u_2),$$

so that (2) also holds. Q.E.D.

**Corollary 1.** For each $G_6$ set $E$, of measure 0 in $R^n$, there is an increasing sequence of compact sets $\{F_k\}_{k \geq 1}$ with $|F_k| > k$ such that $E' = \bigcup_k F_k$ and $|B(x, r) \cap F_{k+1}|/|B(x, r)| > 1 - 2^{-m-k+c_n}$ whenever $x \in F_k$ and $r \leq 1/m$, $m$ a positive integer.

**Proof.** Since $E$ is a $G_6$ of measure 0, there exists an increasing sequence of closed sets $\{F_k\}_{k \geq 1}$ with $E' = \bigcup_{k \geq 1} F_k$. Let $\{a_k\}_{k \geq 1}$ be a strictly increasing sequence of positive numbers for which $|B(0, a_k)| > (1/(1 - 2^{-k})) \cdot (k - 2^{-k})$ and for which $a_{k+1} - a_k$ is greater than 1 for all $k$. Let $P_1$ be any closed subset of $E' \cap B(0, a_1)$ for which $|P_1| > 1$ and set $F_1 = P_1 \cup (F_1 \cap B(0, a_1))$.

Since $F_1 \subset E' \cap B(0, a_2)$ and $|F_1| + (1 - 2^{-2})(|B(0, a_2)| - |F_1|) > 2$, the preceding lemma implies that there is a closed set $P_2$ of measure greater than 2 with $F_1 \subset P_2 \subset E' \cap B(0, a_2)$ which satisfies conditions (1), (2) and (3) of the lemma, with $M_2 = F_1$, $M_1 = E' \cap B(0, a_2)$ and $p = 1$.

For each $x$ in $F_1$ and $r < 1$, $B(x, r) \subset B(0, a_2)$ since $a_2 - a_1 > 1$. Thus

$$|E' \cap B(0, a_2) \cap B(x, r)|/|B(x, r)| = 1$$

and by (3) of Lemma 4.1,

$$|P_2 \cap B(x, r)|/|B(x, r)| > 1 - 2^{-m-1+c_n}$$

for every positive integer $m$ such that $r < 1/m$. Set $F_2 = P_2 \cup (F_2 \cap B(0, a_2))$.

Continue inductively. Having defined $F_k$ for $k \leq s$ so that $F_k \subset F_k \cap B(0, a_k)$, $|F_k| > k$ and $|F_k \cap B(x, r)|/|B(x, r)| > 1 - 2^{-m-1(k-1)+c_n}$ for $x \in F_{k-1}$ and $r < 1/m$, let $P_{s+1}$ be a closed set of measure greater than $s + 1$ for which $F_s \subset P_{s+1} \subset E' \cap B(0, a_{s+1})$ and for which (1), (2) and (3) of Lemma 4.1 hold with $M_2 = F_s$, $p = s$ and $M_1 = E' \cap B(0, a_{s+1})$. Since

$$|E' \cap B(0, a_{s+1}) \cap B(x, r)|/|B(x, r)|$$

equals 1 for each $x$ in $F_s$ and $r < 1, (3)$ of the lemma implies that $|P_{s+1} \cap B(x, r)|/|B(x, r)| > 1 - 2^{-s-m+c_n}$ for each positive integer $m$ for which $r < 1/m$. Set $F_{s+1} = P_{s+1} \cup (F_{s+1} \cap B(0, a_{s+1}))$.

The sequence $\{F_k\}_{k \geq 1}$ satisfies the conditions of the theorem. Q.E.D.

We observe that by suitable choice of $a_1$ and $P_1$, $F_1$ can be made to contain a specified compact subset of $E'$. 

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If $E$ is a $G_δ$ set of measure 0 in $\mathbb{R}^n$, an increasing sequence of compact subsets of $\mathbb{R}^n$ satisfying the conditions of Corollary 1 will be called a Zahorski sequence for $E$.

**Theorem 1.** Let $E$ be a $G_δ$ set of measure 0 in $\mathbb{R}^n$. There exists a real valued, measurable function $u$ defined on $\mathbb{R}^n$ having the following properties:

1. $0 \leq u \leq 1$.
2. $u$ is 0 precisely on $E$.
3. $u$ is continuous at each point of $E$.
4. For every $x_0$ in $\mathbb{R}^n$ and every $\varepsilon > 0$, there is an $r > 0$ such that $u(x) \leq (1/(1 - \varepsilon))u(x_0)$ whenever $x$ is in $B(x_0, r)$.
5. Every $x$ in $\mathbb{R}^n$ is a Lebesgue point of $u$.

**Proof.** Let $\{\Phi_k\}_{k=1}^\infty$ be a Zahorski sequence for $E$. A closed set $\Phi_r$ will now be defined for each number $r$ of the form $m/2^s$, where $m$ and $s$ are positive integers and $m > 2^s$. These closed sets will satisfy these two conditions:

(a) $\Phi_s \subset \Phi_s'$ if $s > s'$,
(b) $\Phi_s$ consists only of points of density of $\Phi_s$ if $s > s'$.

For each odd integer $k > 2$, $k = 2m + 1$, let $\Phi_{r/2k}$ be a closed set with $\Phi_m \subset \Phi_{k/2} \subset \Phi_{m+1}$ and $|\Phi_{k/2}| > \frac{1}{2}(|\Phi_{m+1}| + |\Phi_m|)$, for which every point of $\Phi_m$ is a point of density of $\Phi_{k/2}$ and every point of $\Phi_{k/2}$ is a point of density of $\Phi_{m+1}$. Such a set exists by Lemma 1. Having defined $\Phi_{r/2k}$ for all $r > 2^k$ and all $k < s$, let $\Phi_{r/2^{s+1}}$, $r > 2^{s+1}$, $r = 2t + 1$, be a closed set with

$\Phi_{t/2^s} \subset \Phi_{r/2^{s+1}} \subset \Phi_{(t+1)/2^s}$ and $|\Phi_{r/2^{s+1}}| > \frac{1}{2}(|\Phi_{(t+1)/2^s}|)$,

for which each point of $\Phi_{t/2^s}$ is a point of density of $\Phi_{r/2^{s+1}}$ and each point of $\Phi_{r/2^{s+1}}$ is a point of density of $\Phi_{(t+1)/2^s}$.

Now let $\lambda$ be any real number greater than or equal to 1 and define $\Phi_\lambda = \bigcap_{m \geq \lambda 2k} \Phi_{m/2^k}$. The collection of closed sets $\{\Phi_\lambda\}_{\lambda \geq 1}$ also satisfies (a) and (b).

Define the function $u$ on $\mathbb{R}^n$ by

$$u(p) = \begin{cases} \inf \{\lambda | p \in \Phi_\lambda \} & \text{if } p \notin E, \\ 0 & \text{if } p \in E. \end{cases}$$

Properties (1) and (2) from the statement of the theorem follow immediately from the definition of $u$. (3)--(5) will now be verified.

Let $p$ be in $E$, and let $\varepsilon > 0$ be arbitrary. If $r$ is less than $\text{dist}(p, \Phi_{1/\varepsilon})$, then $B(p, r) \cap \Phi_{1/\varepsilon}$ is empty and $u(x)$ is less than $\varepsilon$ for $x$ in $B(p, r)$. Thus $u$ is continuous on $E$.

Let $x_0$ be in $E'$, and let $\varepsilon > 0$ be arbitrary. If $r$ is less than $	ext{dist}(x_0, \Phi_{(1-\varepsilon)/u(x_0)})$, then $u(x) \leq u(x_0)/(1 - \varepsilon)$ for all $x$ in $B(x_0, r)$. Thus (4) holds. This property ensures that $u$ is measurable.
Since \( u \) is continuous on \( E \), (5) holds for every \( x \) in \( E \). Let \( x_0 \) be in \( E' \) and let \( \epsilon > 0 \) be arbitrary. Since \( x_0 \) is in \( \Phi_{(1+\epsilon)/u(x_0)} \), \( x_0 \) is a point of density of \( \Phi_{(1+\epsilon)/u(x_0)} \) and thus of \( \{ y : u(y) > u(x_0)/(1+\epsilon) \} \). This, together with (4) and the boundedness of \( u \), yields (5).

Thus \( u \) satisfies all the required conditions. Q.E.D.

If \( E \) is a \( G_6 \) set of measure 0 in \( \mathbb{R}^n \), a collection of closed sets \( \{ \Phi_{\lambda} \}_{\lambda \geq 1} \), constructed in the manner of the first part of the proof of this last theorem, will be called a Zahorski collection for \( E \). The function

\[
 u(x) = \begin{cases} 
 \frac{1}{\inf_{\lambda} \{ \lambda : x \in \Phi_{\lambda} \}}, & x \notin E, \\
 0, & x \in E,
\end{cases}
\]

will be called the corresponding inverse Zahorski function.

Applications to \( S(n, T) \) and \( L(n, T) \). An immediate consequence of Theorem 1 is

**Theorem 2.** If \( E \) is a \( G_6 \) set of measure 0 in \( \mathbb{R}^n \), then there is a function in \( L(n, T) \) of norm 1 which vanishes precisely on \( E \).

**Proof.** Let \( u \) be an inverse Zahorski function for \( E \). \( u \) has norm 1 and vanishes precisely on \( E \). Since, in addition, every point of \( \mathbb{R}^n \) is a Lebesgue point of \( u \), \( u \) satisfies the conditions of the theorem. Q.E.D.

If \( E \) is a \( G_6 \) of measure 0 contained in \( \mathbb{R}^n \) and if \( F \) is a compact subset of \( E' \), then it is possible to find a Zahorski collection \( \{ \Phi_{\lambda} \}_{\lambda \geq 1} \) for \( E \) for which \( F \) is a subset of \( \Phi_{1} \). The corresponding inverse Zahorski function has norm 1, is 0 on \( E \) and identically 1 on \( F \). Since every point of \( \mathbb{R}^n \) is a Lebesgue point of \( u \), \( u \) is in \( L(n, T) \). We therefore also have

**Theorem 3.** If \( E \) is a \( G_6 \) of measure 0 in \( \mathbb{R}^n \) and if \( F \) is a compact subset of \( \mathbb{R}^n \), disjoint from \( E \), then there is a function of norm 1 in \( L(n, T) \) which is 0 at each point of \( E \) and 1 at each point of \( F \).

**Corollary 2.** If \( \{ w_k \}_{k \geq 1} \) is an arbitrary sequence of distinct points in \( \mathbb{R}^n \) and if \( \{ a_k \}_{k \geq 1} \) is an absolutely summable sequence of real numbers, then there is a function \( g \) in \( L(n, T) \) for which \( g(w_k) = a_k \) for all \( k \).

**Proof.** For each \( i \), let \( S_i \) be a \( G_6 \) of measure 0 containing \( \{ w_k \}_{k \geq 1} \) - \( \{ w_i \} \) and not containing \( w_i \). Let \( u_i \) be an inverse Zahorski function for \( S_i \) for which \( u_i(w_i) = 1 \).

Since \( \sum_{k=1}^{\infty} |a_k| < \infty \) and \( \|u_i\|_{\infty} = 1 \) for all \( i \), every point of \( \mathbb{R}^n \) is a Lebesgue point of the function \( g = \sum_{k=1}^{\infty} a_k u_k \). Thus \( g \) is in \( L(n, T) \). Since \( u_i(w_k) = \delta_{ik}, g(w_k) = a_k \) for every \( k \). Q.E.D.

**Corollary 3.** If \( \{ w_k \}_{k \geq 1} \) is a convergent sequence of distinct points of
$R^n$ with limit $w \neq w_k$ any $k$ and if $\{a_k\}_{k \geq 1}$ is an arbitrary sequence of 0's and 1's, then there is a function $g$ in $L(n, T)$, with $\|g\|_\infty = 1$, for which $g(w_k) = a_k$ for all $k$.

The proof is similar to that of Corollary 2.

**Lemma 2.** Let $f$ be in $L^\infty_R(R^n)$ and let $E$ be a $G_\delta$ of measure 0 containing $\{x | x \notin L(f)\}$. If $u$ is an inverse Zahorski function for $E$, then $uf$ is in $L(n, T)$.

**Proof.** It is sufficient to show that $L(uf) = R^n$. If $x$ is in $E$, $u(x) = 0$ and

$$\lim_{r \to 0} \frac{\int (|uf - u(x)f(x)|, B(x, r))/|B(x, r)|}{r} = \lim_{r \to 0} \frac{\int (|u|, B(x, r))/|B(x, r)|}{r} = 0.$$

If $x \notin E$, then $x$ is a Lebesgue point of both $u$ and $f$ and so also for the product. Q.E.D.

Thus every function in $L^\infty_R(R^n)$ can be multiplied by a suitable inverse Zahorski function so that the product is in $L(n, T)$.

**Theorem 4.** If $f$ is in $L^\infty_R(R^n)$ and if $F$ is a compact subset of the Lebesgue points of $f$, then there is a function in $L(n, T)$ whose restriction to $F$ is $f$.

**Proof.** Let $E$ be a $G_\delta$ of measure 0 disjoint from $F$, which contains $\{x \in R^n | x \notin L(f)\}$. Let $\{\phi_k\}_{k \geq 1}$ be a Zahorski collection for $E$ with $F \subset \phi_1$ and let $u$ be the corresponding inverse Zahorski function. $uf$ is the required function. Q.E.D.

Consequently $L(n, T)$ is locally dense in measure in $L^\infty_R(R^n)$, i.e. if $F$ is a compact subset of $R^n$, then there is a sequence of functions in $L(n, T)$ which converges in measure to $f$ on $F$.

Lemma 2 may be applied to characterize the extreme points of the unit ball of $S(n, T)$.

**Theorem 5.** $F$ is an extreme point of the unit ball of $S(n, T)$ if and only if $|F| = 1$ a.e.

**Proof.** If $|F| = 1$ a.e., then $F$ is an extreme point of the unit ball of $L^\infty_R(R^n)$ and hence also of $S(n, T)$. Conversely, suppose $F$ fails to have modulus 1 at each point of some subset of $R^n$ of positive measure. Let $E$ be a $G_\delta$ of measure 0 in $R^n$ containing $\{x \in R^n | x \notin L(1 - |F|)\}$. Let $u$ be an inverse Zahorski function for $E$. By Lemma 5.1, $u(1 - |F|)$ is in $L(n, T)$ and so in $S(n, T)$. Since $u(1 - |F|) \leq 1 - |F|$, $\|u(1 - |F|) - F\|_\infty \leq 1$ and $\|u(1 - |F|) + F\|_\infty \leq 1$ so that $F$ is not extreme. Q.E.D.

It is easy to see that the same result holds for the unit ball of $L(n, T)$, i.e. $F$ is an extreme point of the unit ball of $L(n, T)$ if and only if $|F| = 1$ a.e. If
|F| = 1 a.e., then F is an extreme point of S(n, T) and so also of L(n, T). If F is in L(n, T), then it follows from the inequality
\[ J(|F| - |F(x)|, B(x, r)) < J(|F - F(x)|, B(x, r)) \]
that 1 - |F| is also in L(n, T). Thus if |F| is less than 1 on a set of positive measure, then \( G = 1 - |F| \) is a function in L(n, T) which satisfies \( |F - G|_\infty < 1 \) and \( \|F + G\|_\infty < 1 \) so that F is not extreme.

**Theorem 6.** L(n, T) is not the dual of a Banach space.

**Proof.** It is sufficient to show that the only extreme points of the unit ball of L(n, T) are the constant functions 1 and -1. That this is so is a consequence of the following lemma:

**Lemma 3.** If f is a function in L(n, T) which assumes the value 0 or 1 a.e., then f is constant.

**Proof.** Let \( g(x) = f(x)(1 - f(x)) \). Since
\[ g(x) = \lim_{r \to 0} J(g, B(x, r))/|B(x, r)| = 0 \]
for each \( x \) in \( \mathbb{R}^n \), f actually assumes the values 0 or 1 everywhere.

Let \( K = \{ x \in \mathbb{R}^n | f \text{ is discontinuous at } x \} \). It is sufficient to show that \( K \) is empty.

Suppose \( K \) is not empty.

Claim. If \( x_0 \in K \), then every neighborhood of \( x_0 \) contains some \( x \) in \( K \) for which \( f(x) \neq f(x_0) \).

**Proof of Claim.** Let \( x_0 \) be in \( K \) and suppose, without loss of generality, that \( f(x_0) = 1 \). Let \( B(x_0, r) \) be an arbitrary ball in \( \mathbb{R}^n \) with center at \( x_0 \) and having radius \( r \). Let \( s \) be any number in \((0, r/2)\). Since f is discontinuous at \( x_0 \), there is some \( a \) in \( B(x_0, s) \) for which \( f(a) = 0 \). If \( a \) is in \( K \), we are done. If \( a \) is not in \( K \), f is continuous at \( a \) and so vanishes in a neighborhood of \( a \). Set \( t_a = \sup \{ t > 0 | f \text{ is identically 0 in } B(a, t) \} \). B(a, ta) is a subset of B(x0, r) and is not tangent to B(x0, r) at any point. (Otherwise we would have \( x_0 \) in B(a, ta) but \( f(x_0) = 1 \).) Let \( x \) be an arbitrary point on the boundary of B(a, ta). We have
\[ f(x) = \lim_{r \to 0} J(f, B(x, r))/|B(x, r)| = \lim_{r \to 0} J(f, B(x, r) \cap B(a, t_a))/|B(x, r)| \]
\[ \leq \lim_{r \to 0} |B(x, r) \cap B(a, t_a)|/|B(x, r)| < 1. \]
Thus \( f(x) = 0 \) and \( f \) vanishes on the boundary of B(a, ta). By choice of \( t_a \) and
compactness of the boundary, \( f \) must have at least one discontinuity \( x' \) on the boundary of \( B(a, t_0) \). Since \( x' \) is in \( K \cap B(x_0, r) \) and \( f(x') \neq f(x_0) \), the proof of the claim is complete.

Now let \( x_1 \) be in \( K \) with \( f(x_1) = 1 \) and let \( 0 < r_1 < \frac{1}{2} \) be such that for \( 0 < r < r_1 \), 
\[
\frac{|J(f, B(x_1, r))|}{|B(x_1, r)|} > 1 - \frac{1}{2}.
\]

Let \( x_2 \) be any point in \( K \cap B(x_1, r_1) \) for which \( f(x_2) = 0 \), and let \( 0 < r_2 < \frac{1}{2} \) be such that for \( 0 < r < r_2 \), 
\[
\frac{|J(f, B(x_2, r))|}{|B(x_2, r)|} < \frac{1}{2}^2 \quad \text{and} \quad B(x_2, r_2) \subseteq B(x_1, r_1).
\]

Continue defining \( x_k \) and \( r_k \) inductively as follows: If \( k \) is odd, let \( x_k \) be any point in \( K \cap B(x_{k-1}, r_{k-1}) \) for which \( f(x_k) = 1 \) and let \( 0 < r_k < \frac{1}{2^k} \) be such that for \( 0 < r < r_k \), 
\[
B(x_k, r) \subseteq B(x_{k-1}, r_{k-1}) \quad \text{and} \quad \frac{|J(f, B(x_k, r))|}{|B(x_k, r)|} > 1 - \frac{1}{2^k}.
\]

If \( k \) is even choose \( x_k \) and \( r_k \) in a similar way except that \( f(x_k) = 0 \) and 
\[
\frac{|J(f, B(x_k, r))|}{|B(x_k, r)|} < \frac{1}{2^k} \quad \text{for} \quad 0 < r < r_k.
\]

Let \( x \) be in the intersection of the \( B(x_k, r_k) \). Then
\[
\lim_{k \to \infty} \frac{|J(f-f(x), B(x_k, r_k))|}{|B(x_k, r_k)|} \leq \left\{ \frac{|B(x, 2r_k)|}{|B(x_k, r_k)|} \right\} \times \lim_{k \to \infty} \frac{|J(f-f(x), B(x, 2r_k))|}{|B(x, 2r_k)|} = 0.
\]

But this implies that \( f(x) \) must be both 0 and 1 which is impossible. Q.E.D.

The example
\[
f(x) = \begin{cases} 
1 & \text{if } x > 0, \\
0 & \text{if } x = 0, \\
-1 & \text{if } x < 0,
\end{cases}
\]

shows that there are nonconstant extreme points of the unit ball of \( S(n, T) \).

REFERENCES


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, WASHINGTON COUNTY CENTER, WEST BEND, WISCONSIN 53095