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A BANACH ALGEBRA OF FUNCTIONS WITH BOUNDED nTH DIFFERENCES

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ABSTRACT. Several characterizations are given for the Banach algebra of $(n-1)$-times continuously differentiable functions whose $(n-1)$st derivative satisfies a bounded Lipschitz condition. The structure of the closed primary ideals is investigated and spectral synthesis is shown to be satisfied.

0. **Introduction.** The aim of this paper is to study the ideal structure of the subalgebra of $D^{n-1}[0, 1]$ (the algebra of $(n-1)$-times continuously differentiable functions on $[0, 1]$) consisting of functions whose $(n-1)$st derivative satisfies a bounded Lipschitz condition. We present several characterizations of this algebra and then show that every closed ideal I is the intersection of primary ideals determined by I; i.e., that spectral synthesis holds. We show further that the closed primary ideals are determined by the differentiability properties of the algebra.

We let $\mathcal{D}^n$ (the interval $[0, 1]$ is fixed throughout) be the algebra of functions f for which there exists a sequence $\{f_p\}_{p=1}^{\infty}$ of functions $f_p \in D^n$ such that $\|f_p - f\|_{n-1} \to 0$ and $\sup_p \|f_p\|_n < \infty$ where $\|\cdot\|_r = \sum_{i=0}^r \|f^{(i)}\|_\infty / i!$ is the standard Banach algebra norm on D^r. $\mathcal{D}^n$ is a commutative Banach algebra under the norm

$$\|f\| = \inf\left\{\sup_p \|f_p\|_n : f_p \in D^n \text{ and } \|f_p - f\|_{n-1} \to 0\right\}.$$

$\mathcal{D}^n$ is easily seen to be regular since we have $D^n \subset \mathcal{D}^n \subset D^{n-1}$ by Theorem 1.3.

In [7] D. Sherbert studies Banach algebras of Lipschitz functions on metric spaces. His work includes $\mathcal{D}^1$ as a special case although the theorem on spectral synthesis was obtained later by Waelbroeck [8] and independently by G. Glaeser [3]. Henceforth we refer to $\mathcal{D}^n$ as $\mathcal{D}$.

1. **Functions with bounded nth-order differences.** This section is devoted to the proof of Theorem 1.3 which characterizes $\mathcal{D}$ as an algebra of functions

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which satisfy a Lipschitz condition on the $(n-1)$st derivative. We begin with several preliminary definitions and lemmas.

The kth order divided difference of $f(x)$ is defined inductively by setting

$$\Delta_h^0 f(x) = f(x), \qquad \Delta_h^1 f(x) = f(x+h) - f(x),$$

$$\Delta_h^k f(x) = \Delta_h^{k-1} f(x+h) - \Delta_h^{k-1} f(x) \quad \text{for } k = 2, 3, \ldots .$$

An easy induction yields

$$\Delta_h^k f(x) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f(x+hi).$$

If $f^{(k)}(x)$ exists, then

$$\lim_{h \to 0} \frac{\Delta_h^k f(x)}{h^k} = f^{(k)}(x),$$

but not conversely. (See [2, p. 65] or [5].)

LEMMA 1.1. *Let $f(x)$ be a continuous real-valued function defined on (a, b) such that $|\Delta_h^n f(x)| \leqslant Mh^n$, $M < \infty$, whenever $a < x < x + nh < b$. Then $|\Delta_{h_1}^1 \cdots \Delta_{h_n}^1 f(x)| \leqslant Mh_1 \cdots h_n$ for any choice of x and n positive numbers h_i which satisfy $a < x < x + h_1 + \cdots + h_n < b$.*

PROOF. We set $g(x) = Mx^n/n!$ and observe that $\Delta_h^n(g(x) \pm f(x)) \geqslant 0$ since $\Delta_h^n g(x) = Mh^n$. By [1, Lemma 1, p. 497] we have $\Delta_{h_1}^1 \cdots \Delta_{h_n}^1 (g(x) \pm f(x)) \geqslant 0$. The desired inequality is immediate upon the substitution of the identity $\Delta_{h_1}^1 \cdots \Delta_{h_n}^1 g(x) = Mh_1 \cdots h_n$. □

Suppose now that $\lim_{h\to 0}(\Delta_h^p f(x)/h^p) = f^{(p)}(x)$ exists on (a, b). Then

$$\Delta_\delta^k f^{(p)}(x) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f^{(p)}(x + jk) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \lim_{h \to 0} \left(\frac{\Delta_h^p f(x + jk)}{h^p} \right)$$

$$= \lim_{h \to 0} \left(\Delta_\delta^k \frac{\Delta_h^p f(x)}{h^p} \right).$$

Hence if $|\Delta_h^n f(x)| \leqslant Mh^n$ for $a < x < b$, we have

$$|\Delta_h^1 f^{(n-1)}(x)| = \lim_{\delta \to 0} \left| \left(\frac{\Delta_h^1 \Delta_\delta^{n-1} f(x)}{\delta^{n-1}} \right) \right| \leqslant Mh$$

by the preceding remark and Lemma 1.1. We summarize these results in Lemma 1.2 for later reference.

LEMMA 1.2. *If $f^{(p)}(x)$ exists on (a, b), then*

$$\Delta_\delta^k f^{(p)}(x) = \lim_{h\to 0}\left(\Delta_\delta^k \frac{\Delta_h^p f(x)}{h^p}\right).$$

Moreover if $|\Delta_h^n f(x)/h^n| \leq M$ on (a, b), then $|(\Delta_h^1/h)f^{(n-1)}(x)| \leq M$ on (a, b).

The next theorem contains several characterizations of the functions in $\mathcal{D}$. The proof presented here relies strongly on results obtained by Boas and Widder [1]. The authors thank James Case for pointing out this reference.

THEOREM 1.3. *Let f be a continuous complex-valued function defined on* $[0, 1]$. *The following statements are equivalent:*

(a) $f \in D^{n-1}$ *and there exists a sequence $\langle g_p\rangle \subset D^n$ such that $\|g_p - f\|_{n-1} \to 0$ and $\sup_p \|g_p\|_n = M < \infty$.*

(b) *There exists a sequence $\langle g_p\rangle \subset D^n$ such that $\|g_p - f\|_\infty \to 0$ and $\sup_p\|g_p\|_n = M < \infty$.*

(c) *There exists $M < \infty$ such that $|\Delta_h^n f(x)| \leq Mh^n$* $(0 \leq x < x + nh \leq 1)$.

(d) *$f^{(n-1)}$ is absolutely continuous and $|f^{(n)}(x)| \leq M$ a.e. on* $[0, 1]$ *for some constant $M < \infty$.*

PROOF. We can assume that $f(x)$ is real-valued without loss of generality since we can break $f(x)$ up into its real and complex parts. We show that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (a). The implication (a) $\Rightarrow$ (b) is immediate since $\|f\|_\infty \leq \|f\|_{n-1}$ for all f.

Suppose next that $\langle g_p\rangle \subset D^n$ is as in (b). Given x and h with $0 \leq x < x + nh \leq 1$, we have $\Delta_h^n f(x) = \lim_{p\to\infty}\Delta_h^n g_p(x)$ by the uniform convergence of $\langle g_p\rangle$ to f. Thus

$$\left|\frac{\Delta_h^n f(x)}{h^n}\right| = \lim_{p\to\infty}\left|\frac{\Delta_h^n g_p(x)}{h^n}\right| \leq M$$

since $|\Delta_h^n g_p(x)| = |g_p^{(n)}(w_{x,p})h^n| \leq M$ by hypothesis with $w_{x,p}$ determined by the mean value theorem for nth order divided differences [2, p. 65]. Thus (c) holds and (b) $\Rightarrow$ (c) is established.

We show next that (c) $\Rightarrow$ (d). First consider the case $n = 1$. Given $\epsilon > 0$, set $\delta = \epsilon/M$ and let $\{[x_i, y_i] : 1 \leq i \leq r\}$ be a finite collection of disjoint intervals in $[0, 1]$ with $\Sigma_1^r |y_i - x_i| < \delta$. Setting $h_i = y_i - x_i$, we have

$$\sum_1^r |f(y_i) - f(x_i)| = \sum_1^r |\Delta_{h_i}^1 f(x_i)| \leq \sum_1^r Mh_i < M\delta = \epsilon.$$

Hence f is absolutely continuous on $[0, 1]$ and thus $f'(x)$ exists a.e. Moreover $|f'(x)| = |\lim_{h\to 0}(\Delta_h^1 f(x)/h)| \leq M$ wherever $f'(x)$ exists and (d) holds.

We now assume that (c) holds and that $n \geq 2$. Set $g(x) = Mx^n/n!$ where M is the constant in (c). As in the proof of Lemma 1.1, $\Delta_h^n(g \pm f) \geq 0$. By [1, Theorem, p. 497], $(g \pm f)^{(n-2)}$ are convex continuous functions. Consequently the right- and left-hand derivatives $(g \pm f)_R^{(n-1)}$ and $(g \pm f)_L^{(n-1)}$ exist, are nondecreasing, and satisfy $(g \pm f)_L^{(n-1)} \leq (g \pm f)_R^{(n-1)}$ (see [5]). Thus $f^{(n-2)} = \frac{1}{2}[(g+f)^{(n-2)} - (g-f)^{(n-2)}]$ exists and is continuous. Moreover

$$\begin{aligned} f_R^{(n-1)} - f_L^{(n-1)} &= \{(g+f)_R^{(n-1)} - g_R^{(n-1)}\} - \{(g+f)_L^{(n-1)} - g_L^{(n-1)}\} \\ &= (g+f)_R^{(n-1)} - (g+f)_L^{(n-1)} \geq 0 \end{aligned}$$

since $g^{(n-1)}$ exists. Note that the quantities in brackets yield the existence of $f_R^{(n-1)}$ and $f_L^{(n-1)}$. Similarly, writing

$$f_R^{(n-1)} - f_L^{(n-1)} = \{g_R^{(n-1)} - (g-f)_R^{(n-1)}\} - \{g_L^{(n-1)} - (g-f)_L^{(n-1)}\},$$

we obtain $f_R^{(n-1)} - f_L^{(n-1)} \leq 0$ and thus that $f_R^{(n-1)} = f_L^{(n-1)} = f^{(n-1)}$.

We show next that $f^{(n-1)}$ is absolutely continuous. Given $\epsilon > 0$, choose $\delta = \epsilon/2M$. Let $\{[x_i, y_i] : 1 \leq i \leq r\}$ be a finite set of disjoint intervals with $\Sigma_1^r |y_i - x_i| < \delta$ and let $h_i = y_i - x_i$. Then

$$\sum_1^r |f^{(n-1)}(y_i) - f^{(n-1)}(x_i)| = \sum_1^r |\Delta_{h_i}^1 f^{(n-1)}(x_i)|.$$

By Lemma 1.2

$$\Delta_{h_i}^1 f^{(n-1)}(x_i) = \lim_{h\to 0} \Delta_{h_i}^1 \frac{\Delta_h^{n-1} f(x_i)}{h^{n-1}};$$

hence there exists δ_i such that

$$\left| \Delta_{h_i}^1 f^{(n-1)}(x_i) - \frac{\Delta_{h_i}^1 \Delta_h^{n-1} f(x_i)}{h^{n-1}} \right| < \frac{\epsilon}{2r}$$

whenever $0 < h < \delta_i$. This can be written as

$$|\Delta_{h_i}^1 f^{(n-1)}(x_i)| < \frac{\epsilon}{2r} + \left| \Delta_{h_i}^1 \frac{\Delta_h^{n-1} f(x_i)}{h^{n-1}} \right| \quad \text{for } 0 < h < \delta_i.$$

For $0 < h < h_0 = \min\{\delta_1, \delta_2, \ldots, \delta_r\}$ we have

$$\sum_1^r |f^{(n-1)}(y_i) - f^{(n-1)}(x_i)| = \sum_1^r |\Delta_{h_i}^1 f^{(n-1)}(x_i)|$$

$$< \frac{\epsilon}{2} + \sum_1^r \left| \Delta_{h_i}^1 \frac{\Delta_h^{n-1} f(x_i)}{h^{n-1}} \right| \leqslant \frac{\epsilon}{2} + \sum_1^r M h_i < \epsilon$$

by Lemma 1.1. Thus $f^{(n-1)}$ is absolutely continuous and $f^{(n)}$ exists a.e. The inequality $|f^{(n)}(x)| \leqslant M$ follows easily as in the $n = 1$ case since $f^{(n)}(x) = \lim_{h\to 0}(\Delta_h^n f(x)/h^n)$.

Thus it remains only to show that (d) $\Rightarrow$ (a). Suppose f satisfies (d). Since $f^{(n)} \in L^1[0, 1]$ and $f^{(n)}$ is essentially bounded by M, there exists a sequence $\langle p_r \rangle$ of polynomials with $\|p_r\|_\infty \leqslant M$ for all r which converges to $f^{(n)}$ in the L^1 norm [6, p. 68]. We can write the Taylor expansion for f in the form

$$f(x) = \sum_{k=0}^{n-1} f^{(k)}\left(\frac{1}{2}\right) \frac{(x - \frac{1}{2})^k}{k!} + \frac{1}{(n-1)!} \int_{1/2}^x (x - t)^{n-1} f^{(n)}(t)\, dt.$$

Set

$$q_r(x) = \sum_{k=0}^{n-1} f^{(k)}\left(\frac{1}{2}\right) \frac{(x - \frac{1}{2})^k}{k!} + \frac{1}{(n-1)!} \int_{1/2}^x (x - t)^{n-1} p_r(t)\, dt.$$

It suffices to show that $\|q_r - f\|_{n-1} \longrightarrow 0$ since $|q_r^{(n)}(x)| = |p_r(x)| \leqslant \|p_r\|_\infty \leqslant M$ yields $\sup_r \|q_r\|_\infty \leqslant M < \infty$. We have

$$|f^{(k)}(x) - q_r^{(k)}(x)| = \left| \frac{1}{(n-1-k)!} \int_{1/2}^x (x - t)^{n-1-k} (f^{(n)}(t) - p_r(t))\, dt \right|$$

$$\leqslant \frac{1}{(n-1-k)!} \int_{1/2}^x |f^{(n)}(t) - p_r(t)|\, dt \quad \text{for } 0 \leqslant k \leqslant n-1.$$

But this goes to zero with r since p_r converges to $f^{(n)}$ in L^1. Thus $\|q_r - f\|_{n-1} \longrightarrow 0$ and (a) holds. □

Several interesting observations follow directly from the theorem. First is the fact that a sequence from D^n with uniformly bounded nth derivative which converges in supremum norm to a function in $D^n(\mathcal{D}^n)$ must actually converge in the D^{n-1} norm.

The second observation is the fact that we can characterize $\mathcal{D}$ as the set $\mathcal{D} = \{f \in C[0, 1] : f^{(n-1)}$ is absolutely continuous and $|f^{(n)}(x)| \leqslant M$ for some $M < \infty$ whenever $f^{(n)}(x)$ exists$\}$. Consequently we have $D^n \subset \mathcal{D} \subset D^{n-1}$. Given $f \in \mathcal{D}$ set

$$\|f\|' = \sum_{k=0}^{n-1} \frac{1}{k!} \|f^{(k)}\|_\infty + \frac{1}{n!} |||f|||$$

where $|||f|||$ is the ess $\sup|f^{(n)}(x)|$. It can be verified directly that $\|\cdot\|'$ is a Banach algebra norm on $\mathcal{D}$. Since $\mathcal{D}$ is semisimple, $\|\cdot\|'$ and $\|\cdot\|$ are equivalent [4]. The natural injection Ψ of D^n into $\mathcal{D}$ is an isometry with respect to $\|\cdot\|'$. The next proposition asserts that the same is true for $\|\cdot\|$.

PROPOSITION 1.4. $\|f\|_n = \|\Psi(f)\|$.

PROOF. Suppose $f \in D^n$ and $\langle f_p \rangle \subset D^n$ with $\|f_p - f\|_{n-1} \longrightarrow 0$ and $B = \sup_p |||f_p||| < \infty$. Since $f_p^{(k)} \longrightarrow f^{(k)}$ uniformly for $0 \leqslant k < n$, it suffices to prove that $B \geqslant |||f|||$.

Suppose $B < |||f||| = |f^{(n)}(x_0)|$ for some x_0 and set $2\epsilon = |f^{(n)}(x_0)| - B$. Since $\Delta_h^n f(x_0)/h^n \longrightarrow f^{(n)}(x_0)$, $|\Delta_h^n f(x_0)/h^n - f^{(n)}(x_0)| < \epsilon$ for some h sufficiently small. By the mean value theorem for nth order differences [2, p. 65],

$$|\Delta_h^n f_p(x_0)/h^n| = |f_p^{(n)}(\xi)| \leqslant \|f_p^{(n)}\|_\infty \leqslant B,$$

and thus

$$|\Delta_h^n f(x_0)/h^n| = |\lim(\Delta_h^n f_p(x_0)/h^n)| \leqslant B$$

which is contrary to the choice of ϵ. Thus $B \geqslant |||f|||$ and the proof is complete. □

2. **The structure of $J(K)$ in $\mathcal{D}$.** In this section we introduce several closed ideals which play a fundamental role in our treatment of the ideal structure of $\mathcal{D}$. These ideals are important in any regular Banach algebra (see for instance [4]).

The hull of an ideal I in $\mathcal{D}$ is the set $H(I) = \{x: f(x) = 0, f \in I\}$. Since each $f \in I$ is continuous, $H(I)$ is a closed subset of $[0, 1]$. Given a closed set K in $[0, 1]$, the set $M(K) = \{f \in \mathcal{D}: f = 0 \text{ on } K\}$ is a closed ideal in $\mathcal{D}$ with hull K. Moreover if I is any other ideal in $\mathcal{D}$ with $H(I) = K$ then $I \subset M(K)$.

By a theorem due to Šilov [4] the smallest closed ideal $J(K)$ with hull K (closed) can be specified in a regular Banach algebra. Namely, $J(K)$ is the closure in $\mathcal{D}$ of the set of functions in $\mathcal{D}$ which vanish on an open neighborhood of K. Thus any closed ideal I with hull K must satisfy $J(K) \subset I \subset M(K)$.

In the event that $K = \{x\}$ we will write M_x and J_x for $M(K)$ and $J(K)$ respectively. Here M_x is precisely the maximal ideal of $\mathcal{D}$ at x. The maximal ideal space of $\mathcal{D}$ is identifiable with $[0, 1]$ in the natural way.

We turn now to the problem of characterizing $J(K)$. Our results here are the natural extensions of Sherbert's Theorems 5.1, 5.2 and 5.3 [7]; the proofs make use of these results and utilize similar techniques.

THEOREM 2.1. *Let K be a closed set and $f \in \mathcal{D}$. Then $f \in J(K)$ if and only if f satisfies*

(i) $f^{(p)}(x) = 0$ *for all* $x \in K$ *and* $0 \leqslant p \leqslant n - 1$,

(ii) $\Delta_h^n f(x)/h^n \longrightarrow 0$ *as* $(x, h) \longrightarrow K \times \{0\}$.

Proof. We show first that the set S of $f \in \mathcal{D}$ which satisfy (i) and (ii) of the theorem is closed in $\mathcal{D}$. Let $W = \{(x, h)\colon x \in K,\ h > 0 \text{ and } x + nh \in [0, 1]\}$. Given $f \in \mathcal{D}$, set $\Psi(f) = f^*$ where $f^*(x, h) = (1/n!)\Delta_h^n f(x)/h^n$ for $(x, h) \in W$. Ψ is norm decreasing and thus continuous from $\mathcal{D}$ to $C(W)$, the space of bounded continuous functions on W, since

$$\|f^*\|_W = \frac{1}{n!}\sup_W \left|\frac{\Delta_h^n f(x)}{h^n}\right| \leqslant \frac{1}{n!}\,|||f||| \leqslant \|f\|$$

where $\|\cdot\|_W$ denotes sup norm on W. $S_1 = \{g \in C(W)\colon g(x, h) \longrightarrow 0 \text{ as } (x, h) \longrightarrow K \times \{0\}\}$ is closed in $C(W)$. Thus $S = \Psi^{-1}(S_1)$ is closed in $\mathcal{D}$.

Functions which vanish in a neighborhood of K satisfy conditions (i) and (ii) and thus are in S. Since these functions are dense in $J(K)$ and S is closed, we have $J(K) \subset S$.

To obtain the opposite inclusion we choose $f \in S$ and show first that $f^{(n-1)} \in J^1(K)$, the corresponding ideal in $\mathcal{D}^1$. We have $f^{(n-1)} \in \mathcal{D}^1$ by Theorem 1.3. Since $f^{(n-1)}$ vanishes on K it suffices to show that $\Delta_h^1 f^{(n-1)}(x)/h \longrightarrow 0$ as $(x, h) \longrightarrow K \times \{0\}$ [7, Theorem 5.1, p. 251]. Given (x, h), set

$$M(x, h) = \sup_{0<\delta\leqslant h} \left|\frac{\Delta_\delta^n f(x)}{\delta^n}\right|.$$

By Lemma 1.1 $|(\Delta_h^1/h)(\Delta_\delta^{n-1} f(x)/\delta^{n-1})| \leqslant M(x, h)$ whenever δ satisfies $h + (n-1)\delta < nh$. By Lemma 1.2

$$\lim_{\delta\to 0}\left|\frac{\Delta_h^1}{h}\left(\frac{\Delta_\delta^{n-1} f(x)}{\delta^{n-1}}\right)\right| = \left|\frac{\Delta_h^1}{h}\left(\lim_{\delta\to 0}\frac{\Delta_\delta^{n-1} f(x)}{\delta^{n-1}}\right)\right|$$
$$= |(\Delta_h^1/h)(f^{(n-1)}(x))| \leqslant M(x, h).$$

Since $M(x, h) \longrightarrow 0$ as $(x, h) \longrightarrow K \times \{0\}$, $\Delta_h^1 f^{(n-1)}(x)/h$ does also and we have $f^{(n-1)} \in J^1(K)$.

Thus there is a sequence $\langle g_p\rangle \subset \mathcal{D}^1$ such that each g_p vanishes on a neighborhood U_p of K and $\|g_p - f^{(n-1)}\|_{\mathcal{D}_1} \longrightarrow 0$. The assumption that f satisfies condition (i) enables us to write Taylor's Theorem in the form

$$f(x) = \frac{1}{(n-2)!}\int_{x_0}^{x} (x - t)^{n-2} f^{(n-1)}(t)\,dt \quad \text{for } x_0 \in K.$$

Set

$$h_p(x) = \frac{1}{(n-2)!}\int_{x_0}^{x} (x - t)^{n-1} g_p(t)\,dt$$

and note that $f_p = h_p'$ vanishes on U_p since h_p is constant on the components of U_p. The convergence of f_p to f in $\mathcal{D}$ follows from the observation that

$$\|f_p^{(i)} - f^{(i)}\|_\infty = \frac{1}{(n-2-i)!}\left|\int_{x_0}^x (x-t)^{n-2-i}(g_p(t) - f^{(n-1)}(t))\,dt\right|$$

$$\leqslant K\|g_p - f^{(n-1)}\|_\infty \leqslant K\|g_p - f^{(n-1)}\|_{\mathcal{D}_1} \longrightarrow 0 \quad \text{for } 0 \leqslant i \leqslant n-2$$

together with the $\mathcal{D}^1$ convergence of g_p to $f^{(n-1)}$. This establishes that $f \in J(K)$ and completes the proof. □

We will denote the pth power of an ideal I by I^p. The elements of the ideal I^p are finite sums of products $\Pi_{j=1}^r f_j^{i_j}$ from I where $i_1 + i_2 + \cdots + i_r = p$, i_j positive. An easy application of Leibnitz' Rule for differentiating products shows that if $f \in M^p(K)$, then $f^{(r)}$ vanishes on K for $0 \leqslant r \leqslant \min(p-1, n-1)$. It is a consequence of Theorem 4.6 on spectral synthesis that $\overline{M^p(K)}$ is precisely the set of $f \in \mathcal{D}$ for which $f^{(r)}$ vanishes on K for $0 \leqslant r < p \leqslant n$. Theorem 2.4 states that $\overline{M^{n+1}(K)} = J(K)$. Since $J(K)$ is the smallest closed ideal with hull K we have $\overline{M^q(K)} = J(K)$ for all $q \geqslant n+1$.

The proof of Theorem 2.4 is based on the following lemmas. We begin by noting that

$$\Delta_h^n fg(x) = \sum_{i=0}^n \binom{n}{i} \Delta_h^i f(x) \Delta_h^{n-i} g(x+ih).$$

This is Leibnitz' Rule for nth order differences; it is easily established by induction.

LEMMA 2.2. *Assume that $f \in M(K)$. Then $\Delta_h^k f^p(x)/h^k \longrightarrow 0$ as $(x, h) \longrightarrow K \times \{0\}$ for all $k < p \leqslant n+1$.*

PROOF. We will proceed by induction on p. The result is clear for $p = 1$ since $\Delta_h^0 f(x) = f(x)$. Assume now that the result holds for $p-1$ and apply Leibnitz' Rule to $f^{p-1}(x) \cdot f(x)$. This yields

$$\left|\frac{\Delta_h^k f^p(x)}{h^k}\right| = \left|\sum_{i=0}^k \binom{k}{i} \frac{\Delta_h^i f^{p-1}(x)}{h^i} \frac{\Delta_h^{k-i} f(x+ih)}{h^{k-1}}\right|$$

$$\leqslant \sum_{i=0}^{k-1} \binom{k}{i} \left|\frac{\Delta_h^i f^{p-1}(x)}{h^i}\right| \cdot \left|\frac{\Delta_h^{k-i} f(x+ih)}{h^{k-i}}\right| + \left|\frac{\Delta_h^k f^{p-1}(x)}{h^k}\right| \cdot |f(x+kh)|$$

$$\leqslant \sum_{i=0}^{k-i} \binom{k}{i} \left|\frac{\Delta_h^i f^{p-1}(x)}{h^i}\right| \cdot \|f\| + \|f^{p-1}\| \cdot |f(x+kh)|.$$

The sum tends to zero by the induction hypothesis while the last term tends to zero since $f \in M(K)$. □

LEMMA 2.3. *Assume that $f_1, f_2, \ldots, f_r \in M(K)$ and that $i_1, \ldots, i_r$ are positive integers with sum p. Then $\Delta_h^k(\Pi_{j=1}^r f_j^{i_j})/h^k \longrightarrow 0$ as $(x, h) \longrightarrow K \times \{0\}$ for each $k < p \leqslant n + 1$.*

PROOF. The previous lemma is the case $r = 1$. Assume now that the result holds for all $s < r$. Leibnitz' Rule applied to the product $(f_1^{i_1})(\Pi_{j=2}^r f_j^{i_j})$ yields

$$\frac{\Delta_h^k(\Pi_{j=1}^r f_j^{i_j})(x)}{h^k} = \sum_{j=1}^{k} \binom{k}{j} \left(\frac{\Delta_h^j f_1^{i_1}(x)}{h^j}\right) \cdot \left(\frac{\Delta^{k-j}(\Pi_{j=2}^r f_j^{i_j})(x + jk)}{h^{k-j}}\right).$$

Hence

$$\left|\frac{\Delta_h^k(\Pi_{j=1}^r f_j^{i_j})(x)}{h^k}\right| \leqslant \sum_{j=0}^{i_1-1} \binom{k}{j} \left|\frac{\Delta_h^j f_1^{i_1}(x)}{h^j}\right| \cdot \left\|\prod_{j=2}^{r} f_j^{i_j}\right\|$$
$$+ \sum_{j=i_1}^{k} \binom{k}{j} \|f_1^{i_1}\| \cdot \left|\frac{\Delta_h^{k-j}(\Pi_{j=2}^r f_j^{i_j})}{h^{k-j}}\right|.$$

Each sum goes to zero, the first by Lemma 2.2, the second by the induction hypothesis. □

THEOREM 2.4. *Let K be closed. Then $\overline{M^{n+1}(K)} = J(K)$.*

PROOF. Lemma 2.3 and the comments preceding Lemma 2.2 show that products of the form $\Pi_{j=1}^r f_j^{i_j} \in M^{n+1}(K)$ satisfy conditions (i) and (ii) of Theorem 2.1. It follows immediately that $\overline{M^{n+1}(K)} \subset J(K)$. Since $J(K)$ is the smallest closed ideal in $\mathcal{D}$ with hull K, $J(K) = \overline{M^{n+1}(K)}$. □

We now state a result which will be useful in the next section. It is the statement for $\mathcal{D}$ of Sherbert's Theorem 5.3 [7, p. 253]; the proof is the same once Leibnitz' Rule is used to establish condition (ii) of Theorem 2.1.

PROPOSITION 2.5. *Let $K = \{x\}$. A closed linear subspace L of $\mathcal{D}$ which satisfies $J_x \subset L \subset S_x^n$ is an ideal where $S_x^n = \{f \in \mathcal{D}: f^{(i)}(x) = 0, 0 \leqslant i < n\}$.*

3. **Higher order point derivations and primary ideals.** In this section we examine the connection between closed primary ideals in $\mathcal{D}$ and nth order systems of point derivations on $\mathcal{D}$. An ideal I in a Banach algebra is primary at x if its hull is precisely $\{x\}$. Thus a closed ideal I in $\mathcal{D}$ is primary at x if and only if $J_x \subset I \subset M_x$. A set of $n + 1$ linear functionals $\{d_0, d_1, \ldots, d_n\}$ is a system of point derivations of order n at x if for each k, $0 \leqslant k \leqslant n$, d_k satisfies Leibnitz' Rule,

$$d_k(fg) = \sum_{j=0}^{k} \binom{k}{j} d_j(f) d_{k-j}(g),$$

for all $f, g \in \mathcal{D}$ and if $d_0(f) = f(x)$ for all $f \in D$. A linear functional d on $\mathcal{D}$ is an nth order point derivation at x if $\{d_0, d_1, \ldots, d_{n-1}, d\}$ is a system of point derivations of order n at x where $d_k(f) = f^{(k)}(x)$. We consider only continuous point derivations in this paper.

Let I be a closed ideal in $\mathcal{D}$. For each j, $0 < j \leq n$, set $H^j(I) = \{x | f^{(i)}(x) = 0, i = 0, 1, \ldots, j-1, f \in I\}$. Thus $H^1(I) = H(I)$ is the hull of I. Let D_x denote the set of nth order point derivations at x and let $D_x(I) = I^{\perp} \cap D_x = \{d \in D_x\colon df = 0 \text{ for all } f \in I\}$. $D_x(I)$ is a weak-star closed subset of D_x (see [7, p. 263]).

We begin with a slight extension of a result of Singer and Wermer [7, p. 262] which states that $d \in \mathcal{D}^*$ is a point derivation at x if and only if it vanishes on $\overline{M_x^2}$ and the identity $1 \in \mathcal{D}$. Let $S_x^p = \{f \in \mathcal{D}\colon f^{(j)}(x) = 0, 0 \leq j < p\}$ for $0 < p \leq n$. S_x^p is a closed primary ideal at x and $\overline{M_x^p} \subset S_x^p$. Clearly $S_x^1 = M_x$; Proposition 3.5 shows that $S_x^p = \overline{M_x^p}$ for all p.

LEMMA 3.1. *Suppose that $d \in \mathcal{D}^*$ vanishes on $\Sigma_{p=0}^{n-1} \oplus \mathbf{C}(z-x)^p \oplus J_x$ and $d[(z-x)^n] = \lambda \neq 0$. Then $\alpha d \in D_x$ where $\alpha = n!\lambda^{-1}$.*

PROOF. It suffices to verify that αd satisfies Leibnitz' Rule. We can express $f \in \mathcal{D}$ in terms of its Taylor expansion about x in the form

$$f(z) = \sum_{i=0}^{n-1} f^{(i)}(x) \frac{(z-x)^i}{i!} + F(z)$$

where

$$F(z) = \frac{1}{(n-1)!} \int_x^z (z-t)^{n-1} f^{(n)}(t)\, dt.$$

Thus for $f, g \in \mathcal{D}$ we have

$$fg(z) = \sum_{k=0}^{2n-2} \left(\sum_{i+j=k} \frac{f^{(i)}(x) g^{(j)}(x)}{i!j!} \right) (z-x)^k + f(z)G(z) + g(z)F(z) - F(z)G(z).$$

It is easily checked that F and G are in S_x^n and that $(z-x)^p \in S_x^{p-1}$. We obtain

$$\begin{aligned} d(fg) &= \sum_{i+j=n} \frac{f^{(i)}(x) g^{(j)}(x)}{i!j!}\, d(z-x)^n + d(fG + gF - FG) \\ &= \sum_{i+j=n} \frac{f^{(i)}(x) g^{(j)}(x)}{i!j!}\, \lambda + f(x)d(g) + g(x)d(f) \end{aligned}$$

since $d(F) = df$ and $d(G) = d(g)$ by the hypothesis on d. Substituting αd for d, we get

$$(\alpha d)(fg) = f(x)(\alpha d)(g) + \sum_{i=1}^{n-1} \binom{n}{i} f^{(i)}(x) g^{(n-i)}(x) + g(x)(\alpha d)(f)$$

and thus $\alpha d \in D_x$. □

We can now characterize the closed primary ideal J_x in terms of D_x and the vanishing of the first $n-1$ derivatives.

PROPOSITION 3.2. $J_x = D_x^\perp \cap S_x^n$. *That is,* $D_x(J_x) = D_x$.

PROOF. We show first that $J = D_x^\perp \cap S_x^n$ is a closed ideal primary at x. Since J is clearly a closed subspace with hull $\{x\}$, it suffices to show that $fg \in J$ for $f \in J$ and $g \in \mathcal{D}$. Given $d \in D_x$, we have

$$d(fg) = \sum_{1}^{n-1} \binom{n}{i} f^{(i)}(x) g^{(n-i)}(x) + f(x)d(g) + d(f)g(x) = 0$$

since $f \in S_x^n$ and $d(f) = 0$. Since $fg \in S_x^n$, J is an ideal.

Since $J_x \subset J$ it remains to prove that $J \subset J_x$. Thus we suppose that $f \notin J_x$. By the Hahn-Banach Theorem we can choose $d \in D^*$ which vanishes on $J_x \bigoplus \Sigma_{p=0}^{n-1} \oplus \mathbf{C}(z-x)^p$ but not on f or $(z-x)^n$. Thus by Lemma 3.1 $\alpha d \in D_x$ for some $\alpha \neq 0$. But then $(\alpha d)(f) \neq 0$ shows that $f \notin J$ and thus that $J = J_x$. □

If I is a closed ideal in $\mathcal{D}$, then for each $x \in H(I)$ the primary component of I at x is the smallest closed primary ideal at x which contains I. This ideal will be denoted by I_x. We turn now to the problem of characterizing I_x. Given a weak-star closed subset H of D_x, we set $I(H) = H^\perp \cap S_x^n$.

LEMMA 3.3. *$I(H)$ is a closed primary ideal at x.*

PROOF. It is easily seen that $I(H)$ is a closed subspace and that $I(H) \subset S_x^n$. Since $H \subset D_x$, $H^\perp \supset D_x^\perp$ and thus $I(H) = H^\perp \cap S_x^n \supset J_x$ by Proposition 3.2. The conclusion now follows from Proposition 2.5. □

PROPOSITION 3.4. *Suppose that I is a closed ideal in $\mathcal{D}$ and that $x \in H^n(I)$. Then $I_x = I(D_x(I))$. In particular $S_x^n = \overline{M_x^n}$.*

PROOF. Lemma 3.3 shows that $I_1 = I(D_x(I))$ is a closed primary ideal at x. Since $I \subset I_1$ it follows that $I_x \subset I_1$. As in the proof of Proposition 3.2 given $f \notin I_x$ we can use the Hahn-Banach Theorem and Lemma 3.1 to find $d \in D_x(I)$ with $d(f) \neq 0$. But then $f \notin I_1$ and thus $I_1 = I_x$. □

This proposition describes I_x in the event that $x \in H^n(I)$. It remains to examine the primary ideals I at x where $x \in H^p(I)$ but not in $H^{p+1}(I)$ where $0 < p \leq n-1$. We note first that $I \subset S_x^p$ and that $IS_x^1 \subset S_x^{p+1}$ where IS_x^1 is the standard product ideal. By hypothesis we can choose $f \in I$ with $f^{(p)}(x) \neq 0$. Let d_x^{p+1} be any $(p+1)$th order point derivation at x ($d_x^{p+1}(f) = f^{(p+1)}(x)$ if $p+1 < n$) and apply it to fg where $g \in S_x^1$ satisfies $g'(x) \neq 0$. We have $d_x^{p+1}(fg)$

$= (p+1)f^{(p)}(x)g'(x) \neq 0$ by Leibnitz' Rule and thus that IS_x^1 is not annihilated by any $(p+1)$th order point derivation at x. Suppose now that $p = n-1$. Proposition 3.4 combined with the last result shows that $S_x^{n-1} \supset I \supset IS_x^1 = S_x^n = \overline{M_x^n}$. The quotient spaces S_x^p/S_x^{p+1} are 1-dimensional since $f + S_x^{p+1} \longrightarrow f^{(p)}(x)$ is a vector space isomorphism from S_x^p/S_x^{p+1} to the complex numbers. Since $f \in I$ has $f^{(p)}(x) \neq 0$, it follows from the above that $I = S_x^{n-1}$.

We repeat this argument with $p = n-2$ and obtain $S_x^{n-2} \supset I \supset IS_x^1 = S_x^{n-1}$ since there are no primary ideals between S_x^{n-1} and S_x^{n-2}. It then follows as before that $I = S_x^{n-2}$. Proceeding in this manner, one step at a time, we show that S_x^p is the only primary ideal at x with $x \in H^p(I)$ and $x \notin H^{p+1}(I)$. We state these results as a proposition.

PROPOSITION 3.5. *Suppose that I is a closed ideal in $\mathcal{D}$. If $x \in H^p(I)$ and $x \notin H^{p+1}(I)$, $1 \leqslant p \leqslant n-1$, then $I_x = \overline{M_x^p} = S_x^p$.*

The primary components of a closed ideal can thus be described in terms of the vanishing of lower order derivatives and sets of nth order point derivations. It also follows from Propositions 3.4 and 3.5 that an ideal I is primary at x if and only if I is of the form $\overline{M_x^p}$, $1 \leqslant p \leqslant n$, or $I = I(D_x(I))$. By Proposition 2.5 I has the latter form if and only if I is a closed linear subspace between $\overline{M_x^n}$ and J_x.

We conclude this section by showing that the sets D_x have essentially the same structure that Sherbert [7] found for the corresponding sets D_x^1 of point derivations at x on Lipschitz algebras. This will follow directly from the next lemma which states that $\overline{M_x^n}/J_x$ is isomorphic to the corresponding quotient M_x^1/J_x^1 in Sherbert's algebra $\mathcal{D}^1$.

LEMMA 3.6. *Let K be a connected set. Then $\overline{M^n(K)}/J(K)$ is isomorphic to $M^1(K)/J^1(K)$.*

PROOF. By Theorem 1.3, $f \in \mathcal{D}$ implies that $f^{(n-1)} \in \mathcal{D}^1$ and thus that Ψ defined by setting $\Psi(f) = f^{(n-1)}$ maps $\mathcal{D}$ to $\mathcal{D}^1$ linearly. Moreover $\Psi(\overline{M^n(K)}) \subset M^1(K)$ and $\Psi(J(K)) \subset J^1(K)$; hence the induced quotient map Ψ maps $\overline{M^n(K)}/J(K)$ to $M^1(K)/J^1(K)$ linearly. The fact that Ψ is multiplicative follows easily since both quotients are algebraically trivial in the sense that all products are zero.

It remains to show that Ψ is a bijection. Suppose $\Psi(f + J(K)) = f^{(n-1)} + J^1(K) = 0$ for some $f \in \overline{M^n(K)}$ and thus that $f^{(n-1)} \in J^1(K)$. An application of Lemma 1.2 shows that $f \in J(K)$ and thus that $f + J(K) = 0$. Hence Ψ is an injection. Finally given $f + J^1(K)$ in $M^1(K)/J^1(K)$, set

$$g(x) = \frac{1}{(n-1)!}\int_{x_0}^{x} (x-t)^{n-1} f(t)\, dt$$

where x_0 is chosen in K. It is easily checked that $g \in \overline{M^n(K)}$ and that $\Psi(g + J(K)) = f + J^1(K)$. Therefore Ψ is also a surjection and thus an algebra isomorphism. □

We now construct 1-1 correspondences which are inverse to each other between D_x and D_x^1. In particular, given $d_x^1 \in D_x^1$, set $d_x(f) = d_x^1(f^{(n-1)})$ for $f \in \mathcal{D}$. Conversely, given $d_x \in D_x$ and $f \in D^1$, set $d_x^1(f) = d_x(\Psi^{-1}(f - f(x) + J_x^1))$ where d_x is defined on $\overline{M_x^n}/J_x$ in the natural way. It is easily checked that d_x and d_x^1 so defined are in D_x and D_x^1 respectively.

We conclude this section by noting that the hypothesis that K be connected in Lemma 3.6 is used only to establish that Ψ is onto. Without this restriction Lemma 3.6 together with the spectral synthesis theorem in $\mathcal{D}^1$ ([3] or [8]) would yield an easy proof of the spectral synthesis theorem in $\mathcal{D}$. The best we can say using 3.6 is that spectral synthesis, namely, $I = \bigcap_{x \in H} I_x$, I closed, holds when $H(I)$ is connected. This result is established in general in the next section.

4. **Spectral synthesis in $\mathcal{D}$.** In this section we verify that $\mathcal{D}$ has spectral synthesis. Our proof is related to a technique used by Glaeser [3] to prove spectral synthesis for the algebra of Lipschitz functions. A more general proof of a different nature was also given by Waelbroeck [8]. We state some preliminary definitions and observations.

DEFINITION 4.1. An $f \in \mathcal{D}$ belongs locally to the ideal I at x if and only if there exists $g_x \in I$ such that $f - g_x \in J_x$.

Let I_x' denote the ideal of functions which belong locally to a closed ideal I at x, and I_x as before denote the smallest closed primary ideal at x which contains I.

LEMMA 4.2. *For each closed ideal I in $\mathcal{D}$, $I_x = \overline{I_x'}$.*

PROOF. $I_x \subseteq \overline{I_x'}$ since $I \subseteq I_x'$ and $\overline{I_x'}$ has hull $\{x\}$. If $f \in I_x'$ then $f - g_x \in J_x \subset I_x$ where $g_x \in I \subset I_x$ so $I_x' \subseteq I_x$. □

We note that if $x \notin H(I)$ then $I_x = \mathcal{D}$ since $\mathcal{D}$ is a regular algebra. The following will be used in the proof of our main theorem and are generalizations of results in [3].

REMARK 4.3. If Φ is a closed interval in $[0, 1]$ and $x \in \Phi$ where $f^{(i)}(x) = 0, i = 0, \ldots, n-1$, then $\|f\|_{n-1} \leq d \cdot K|||f|||$ (norms over Φ) where $K = \Sigma_{i=0}^{n-1} d^{n-i-1}/i!$ and $d = \operatorname{diam}(\Phi)$. For any $y \in \Phi$,

$$|f^{(i)}(y)| = |f^{(i)}(y) - f^{(i)}(x)| \leq |y - x| \, \|f^{(i+1)}\|_\infty, \qquad i = 0, \ldots, n-1,$$

over Φ and $\|f^{(i)}\|_\infty \leq d\|f^{(i+1)}\|_\infty$. Also $|f^{(n-1)}(y)| = |f^{(n-1)}(y) - f^{(n-1)}(x)| \leq d|||f|||$, and thus $\|f\|_{n-1} \leq d \cdot K|||f|||$.

REMARK 4.4. If Φ is a compact set, then for $f, g \in \mathcal{D}$

$$|||f \cdot g||| \leqslant \|f\|_\infty \cdot |||g||| + \|g\|_\infty \cdot |||f||| + C\|f\|_{n-1} \cdot \|g\|_{n-1}, \qquad C > 0,$$

over Φ. This is an application of the mean value theorem and the following:

$$\begin{aligned}
|||fg||| &= \sup_{x \neq y} \left| \frac{(fg)^{(n-1)}(x) - (fg)^{(n-1)}(y)}{x-y} \right| \\
&= \sup_{x \neq y} \left| \sum_{i=0}^{n-1} \binom{n}{i} \frac{f^{(i)}g^{(n-i)}(x) - f^{(i)}g^{(n-i)}(y)}{x-y} \right| \\
&= \sup_{x \neq y} \left| \sum_{i=0}^{n-1} \binom{n}{i} \left\{ \frac{f^{(i)}(x) + f^{(i)}(y)}{2} \left[\frac{g^{(n-i)}(x) - g^{(n-i)}(y)}{x-y} \right] \right.\right. \\
&\qquad \left.\left. + \frac{g^{(n-i)}(x) + g^{(n-i)}(y)}{2} \left[\frac{f^{(i)}(x) - f^{(i)}(y)}{x-y} \right] \right\} \right| \\
&\leqslant \sup_{x \neq y} \left| \sum_{i=1}^{n-2} \binom{n}{i} \left\{ \frac{f^{(i)}(x) + f^{(i)}(y)}{2} \cdot g^{(n-i+1)}(\bar{x}) \right.\right. \\
&\qquad \left.\left. + \frac{g^{(n-i)}(x) + g^{(n-i)}(y)}{2} \cdot f^{(i+1)}(\bar{\bar{x}}) \right\} \right| \\
&\quad + \sup_{x \neq y} \left| \frac{f(x) + f(y)}{2} \left[\frac{g^{(n-1)}(x) - g^{(n-1)}(y)}{x-y} \right] \right. \\
&\qquad \left. + \frac{g(x) + g(y)}{2} \left[\frac{f^{(n-1)}(x) - f^{(n-1)}(y)}{x-y} \right] \right| \\
&\leqslant \sum_{i=1}^{n-2} \binom{n}{i} (\|f^{(i)}\|_\infty \cdot \|g^{(n-i+1)}\|_\infty + \|g^{(n-i)}\|_\infty \cdot \|f^{(i+1)}\|_\infty) \\
&\quad + \|f\|_\infty \cdot |||g||| + \|g\|_\infty \cdot |||f||| \\
&\leqslant K \left[\left(\sum_{i=1}^{n-2} \frac{\|f^{(i)}\|_\infty}{i!} \right) \left(\sum_{i=1}^{n-2} \frac{\|g^{(n-i+1)}\|_\infty}{(n-i+1)!} \right) \right. \\
&\qquad \left. + \left(\sum_{i=1}^{n-1} \frac{\|g^{(n-i)}\|_\infty}{(n-i)!} \right) \left(\sum_{i=1}^{n-2} \frac{\|f^{(i+1)}\|_\infty}{i!} \right) \right] \\
&\quad + \|f\|_\infty \cdot |||g||| + \|g\|_\infty \cdot |||f|||
\end{aligned}$$

and the result follows.

REMARK 4.5 (PARTITION OF UNITY). Following Glaeser [3] we construct a partition of unity for a covering of R^1 in the following manner. Each open set of the cover is a translate of a fixed open interval $\mathcal{O}$ of a predetermined length. Each associated function is the corresponding translate of a fixed D^n function $\nu(x)$ with support contained in $\mathcal{O}$. The construction can be carried out by defining $g(x)$ to be zero off $[-1, 4]$ and determined on $[-1, 4]$ by

$$h(x) = \begin{cases} \frac{1}{2} + \int_0^x (t^2 - 1)^{2n}\,dt \Big/ \int_0^1 (t^2 - 1)^{2n}\,dt, & x \in [-1, 1], \\ 1, \quad x \in [1, 2], \end{cases}$$

and by symmetry about 3/2 on $[2, 4]$. We next contract $[-1, 4]$ to $\mathcal{O}$ and $g(x)$ to $\nu(x)$. The desired covering is obtained by overlapping translates of $\mathcal{O}$ by 1/3 their length. The following properties are easily verified.

(1) $\|\nu_j\|_n = \|\nu\|_n$ for each j.

(2) On $\mathcal{O}_j \cap \mathcal{O}_{j+1}$, $\nu_j(x) + \nu_{j+1}(x) = 1$.

In the proof of our main theorem we use K throughout to denote the various constants which result from our inequalities and Φ denotes $[0, 1]$.

THEOREM 4.6 (SPECTRAL SYNTHESIS). *If I is a closed ideal in $\mathcal{D}$ then $I = \bigcap_{x\in\Phi} I_x$.*

PROOF. Since $I \subseteq \bigcap I_x$ it suffices to prove that $f \in \bigcap I_x$ can be approximated by elements of I. Let $\epsilon > 0$ be given. For each $x \in \Phi$ there exists $h_x \in I'_x$ such that $\|f - h_x\| < \epsilon$. Let $g_x \in I$ where $g_x - h_x \in J_x$. Let $\beta(x, \delta) \subset \{y \in \Phi\colon |\Delta_h^n(g_x - h_x)/h^n| < \epsilon\}$ be a Euclidean neighborhood of x of radius $\delta = \delta(x, \epsilon, h_x)$. Extract a finite subcover $\{\beta_i(x_i, \delta_i)\}_{i=1}^n$ of Φ and let $\tau > 0$ denote the Lebesgue number of the subcover. We next replace the covering with a refinement $\{\mathcal{O}_j\}_{j=1}^m$ which satisfies $\operatorname{diam}(\mathcal{O}_j) < \tau/3$ and which has an associated partition of unity $\{v_j\}_{j=1}^m$ as in Remark 4.5. Each $\mathcal{O}_j \subset \beta_i$ for some i and by relabeling we can associate a pair $\langle h_j, g_j\rangle$ with each $\mathcal{O}_j$ where $|||g_j - h_j|||_{\overline{\mathcal{O}}_j} < \epsilon$. We next approximate f by an element of I. Following Glaeser [3] we consider $\mathcal{O}_j$ to be of type I if its closure intersects $H(I)$ in which case in D^{n-1}, $\|h_j - g_j\|_{\overline{\mathcal{O}}_j} < dK\epsilon$ by Remark 4.3, where $\|\cdot\|_{\overline{\mathcal{O}}_j}$ means the norm $\|\cdot\|_{n-1}$ restricted to $\overline{\mathcal{O}}_j$. For $\mathcal{O}_j$ of type II ($\overline{\mathcal{O}}_j \cap H(I) = \varnothing$) we have $v_j \in J(H(I)) \subset I$. Let $g_\epsilon = \Sigma_{\mathrm{I}} v_j g_j + \Sigma_{\mathrm{II}} v_j f$, $h_\epsilon = \Sigma_{\mathrm{I}} v_j h_j + \Sigma_{\mathrm{II}} v_j f$.

Now $g_\epsilon \in I$ and let $\mathcal{O}$ be an element of our covering; then restricted to $\mathcal{O}$, $h_\epsilon - g_\epsilon = \Sigma_{\mathrm{I}} v_j(h_j - g_j)$ is nontrivial on at most 3 $\mathcal{O}_j$'s because of our method of overlapping. Thus

$$|||h_\epsilon - g_\epsilon|||_{\overline{O}} = \left|\left|\left|\sum v_j(h_j - g_j)\right|\right|\right|_{\overline{O}}$$

$$\leqslant 3 \max_j \{C\|v_j\|_{\overline{O}_j} \cdot \|h_j - g_j\|_{\overline{O}_j} + \|v_j\|_\infty \cdot |||h_j - g_j|||_{\overline{O}_j}$$

$$+ |||v_j|||_{\overline{O}_j} \cdot \|h_j - g_j\|_{\overline{O}_j}\}$$

$$< K\epsilon$$

by our previous observations. Also $\|h_\epsilon - g_\epsilon\|_{\overline{O}} \leqslant dK|||h_\epsilon - g_\epsilon|||_{\overline{O}}$ (Remark 4.3) and we have $\|h_\epsilon - g_\epsilon\|_{n-1} < K\epsilon d$ as O varies throughout our cover.

Again for O in our covering, we have $f - h_\epsilon = \Sigma_{\mathrm{I}} v_j(f - h_j)$ restricted to O nontrivial on at most 3 O_j's and so

$$\|f - h_\epsilon\|_{\overline{O}} = \left\|\sum_{\mathrm{I}} v_j(f - h_j)\right\|_{\overline{O}} \leqslant 3\max(\|v_j\| \cdot \|f - h_j\|) < K\epsilon.$$

Thus we obtain $\|f - h_\epsilon\| < K\epsilon$ as O varies throughout the covering. By the previous results we have $|||h_\epsilon - g_\epsilon|||_{\overline{O}} < K\epsilon$ and $\|h_\epsilon - g_\epsilon\|_{n-1} < dK\epsilon$. Now let x, y belong to Φ. In case x, y do not belong to the same O then $|x - y| > k$ where k is the length of our overlap (chosen as 1/3 length of basic interval). Then

$$\left|\frac{(h_\epsilon^{(n-1)}(x) - g_\epsilon^{(n-1)}(x)) - (h_\epsilon^{(n-1)}(y) - g_\epsilon^{(n-1)}(y))}{x - y}\right|$$

$$\leqslant k^{-1} \cdot 2\|h_\epsilon^{(n-1)} - g_\epsilon^{(n-1)}\|_\infty$$

$$\leqslant k^{-1} \cdot d \cdot 2\|h_\epsilon - g_\epsilon\|_{n-1} < K\epsilon$$

since $k^{-1} \cdot d \leqslant 3$. Thus $\|h_\epsilon - g_\epsilon\| < K\epsilon$ and it follows that $f \in I$ by the triangle inequality.

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