QUASI-SIMILAR MODELS
FOR NILPOTENT OPERATORS(1)

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ABSTRACT. Every nilpotent operator on a complex Hilbert space is shown to be quasi-similar to a canonical Jordan model. Further, the para-reflexive operators are characterized generalizing a result of Deddens and Fillmore.

A familiar result states that each nilpotent operator on a finite dimensional complex Hilbert space is similar to its adjoint. One proof proceeds by showing that both a nilpotent operator and its adjoint have the same canonical form. In this note we show that although this result does not extend to infinite dimensional spaces, the weaker quasi-similarity version of it, together with the proof indicated above, still holds on any Hilbert space. This yields an affirmative answer to a question raised by P. Rosenthal in connection with the content of [3].

The canonical form exhibited provides positive evidence that the theory of Jordan models might be extended to cover operators of class \( C_0 \) of infinite multiplicity and indeed, considerable progress [2] has been made recently in this direction. Although the Jordan model for nilpotent operators on infinite dimensional Hilbert spaces is no longer unique, we single out a "canonical" model. A similar result has been obtained independently by Berkovici [1]. We conclude with an application of our results to extend to infinite dimensional spaces a theorem of Deddens and Fillmore [4] which characterizes reflexive operators on finite dimensional spaces.

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1. In this note, a nilpotent operator \( T \) will be called a Jordan operator if \( T = \bigoplus_{\alpha} T_{\alpha} \), where each \( T_{\alpha} \) operates on some \( C^{I_\alpha} \) for \( 0 < I_\alpha < \infty \) by the Jordan one-cell matrix

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Recall that an operator $X$ between Hilbert spaces $H$ and $K$ is said to be a quasi-affinity if $\ker X = (0)$ and $\text{clos}(XH) = K$. An operator $A$ on $H$ is said to be a quasi-affine transform of an operator $B$ on $K$ if there exists a quasi-affinity $X$ such that $XA = BX$. Finally, two operators $A$ and $B$ are quasi-similar if each is a quasi-affine transform of the other. For further information on these concepts see the monograph [8, Chapter II, No. 3.2], or [7].

Our main result is given by the following

**Theorem 1.** Every nilpotent operator $T$ is quasi-similar to a Jordan operator $T_0$.

Since for any Jordan operator $T_0$, the operators $T_0$ and $T_0^*$ are obviously unitarily equivalent, we can infer

**Theorem 2.** If $T$ is a nilpotent operator, then $T$ and $T^*$ are quasi-similar.

Before starting the proof of Theorem 1, we give an example to show that quasi-similarity cannot be replaced by similarity.

Let $X$ be any compact quasi-affinity on an infinite dimensional Hilbert space $H$ (for example, the Volterra operator on $L^2(0, 1)$) and consider the operator $T$ defined by

$$
T = \begin{pmatrix}
0 & X & 0 \\
0 & 0 & I \\
0 & 0 & 0
\end{pmatrix}
on H \oplus H \oplus H.
$$

Clearly $T^3 = 0$ and thus $T$ and $T^*$ are quasi-similar by Theorem 2, but $T$ and $T^*$ are not similar. The proof of this is straightforward.

If $S$ were an invertible operator on $H \oplus H \oplus H$ with matrix

$$
(2) \text{ The same example was found independently by H. Radjavi (see [3, §6]).}
$$
which satisfied $ST^* = TS$, then a simple computation shows that $B_2 = C_2 = C_1 = 0$, $C_0 = XB_1$ and $A_2 = B_1X^*$. Thus the operator

$$S_0 = \begin{pmatrix}
A_0 & B_0 & 0 \\
A_1 & B_1 & 0 \\
0 & 0 & 0
\end{pmatrix}$$

is a compact perturbation of $S$, and hence a Fredholm operator, which is contradicted by the fact that $\ker S_0 = (0) \oplus (0) \oplus H$ is not finite dimensional (cf. [5, Chapter 5]).

2. We start the proof of Theorem 1 with the following

**Lemma 1.** If $T_0$ and $T_1$ are two Jordan operators and $T_0$ is a quasi-affine transform of $T_1$, then $T_0$ and $T_1$ are quasi-similar.

**Proof.** The fact that $T_0$ is a quasi-affine transform of $T_1$ means that there exists a quasi-affinity $X$ such that $XT_0 = T_1X$ and thus $T_0X^* = X^*T_1^*$. Since $T_j$ is a Jordan operator, there exists a unitary operator $U_j$ such that $T_j^* = U_j^*T_jU_j$ ($j = 0, 1$). Therefore, $T_0(U_0X^*U_1^*) = (U_0X^*U_1^*)T_1$, where $U_0X^*U_1^*$ is a quasi-affinity, and $T_1$ is also a quasi-affine transform of $T_0$. Consequently, $T_0$ and $T_1$ are quasi-similar.

**Lemma 2.** Any nilpotent operator $T$ has a quasi-affine transform $T_0$ which is a Jordan operator.

**Proof.** Suppose that $T^n = 0$, $T^{n-1} \neq 0$ for some $n \geq 1$. If we set

\begin{align*}
\chi_j &= \ker T^j \ominus \ker T^{j-1} \quad \text{for } j = 1, 2, \ldots, n, \\
\nu_n &= \chi_n, \quad \nu_{n-1} = \chi_{n-1} \cap (T\nu_n)^1, \ldots, \\
\nu_1 &= \chi_1 \cap (T^{n-1}\nu_n + \cdots + T\nu_2)^1 \text{ and} \\
\mathcal{H}_0 &= (\nu_n \ominus \cdots \ominus \nu_n) \ominus (\nu_{n-1} \ominus \cdots \ominus \nu_{n-1}) \ominus \cdots \ominus (\nu_2 \ominus \nu_2) \ominus \nu_1 \\
&\quad \text{n times} \quad \text{(n - 1) times}
\end{align*}

we can define the bounded operators $T_0$ on $\mathcal{H}_0$ and $A: \mathcal{H}_0 \to H$ by the equations
\[
T_0(y_n^1 \oplus \cdots \oplus y_n^{r_n} \oplus \cdots \oplus y_2^1 \oplus y_2^2 \oplus y_2^4) = 0 \oplus y_n^1 \oplus \cdots \oplus y_{n-1}^1 \oplus \cdots \oplus 0 \oplus y_2^1 \oplus 0,
\]
and
\[
A(y_n^1 \oplus \cdots \oplus y_n^{r_n} \oplus \cdots \oplus y_2^1 \oplus y_2^2 \oplus y_2^4) = y_n^1 + \cdots + T^{n-1}y_n^1 + \cdots + y_2^4 + Ty_2^2 + y_2^1.
\]

It is easy to see that \(T_0\) is a Jordan operator and that \(AT_0 = TA\). Using the fact that
\[
(y_1 + \cdots + T^{n-1}y_n + y_2 + \cdots + T^{n-2}y_n + \cdots + y_k + \cdots + T^{n-k}y_n) - = \ker T^k,
\]
which is proved by induction on \(k\), we conclude that \(\text{clos}(A^*H_0) = H\). To complete the proof we must show that \(A\) is injective.

If \(A\) is not injective, there must exist \(y_k^j\) in \(V_j\), \(1 \leq j \leq n, 1 \leq k \leq j\), such that
\[
\sum_{j=1}^{n} \sum_{k=1}^{j} T^{k-1}y_{n-j+k} = 0 \quad \text{but} \quad \sum_{j=1}^{n} \sum_{k=1}^{j} \|y_{n-j+k}\| \neq 0.
\]

Let \(m\) be the smallest integer such that \(\sum_{k=1}^{m} \|y_{n-m+k}\| \neq 0\) and let \(p\) be the smallest integer such that \(y_{p}^n_{m+p} \neq 0\). Because we have
\[
\sum_{k=p}^{m} T^{k-1}y_{n-m+k} = -\sum_{j=m+1}^{n} \sum_{k=1}^{j} T^{k-1}y_{n-j+k} \quad \text{in ker } T^{n-m},
\]

it follows that \(y_{p}^n_{m+p} + \cdots + T^{m-p}y_{n}^m\) is in \(\ker T^{n-m+p-1}\). If we let \(P\) denote the orthogonal projection of \(H\) onto \(X_{n-m+p}\), then
\[
y_{n-m+p} + P(Ty_{n-m+p+1} + \cdots + T^{m-p}y_{n}^m) = P(y_{n-m+p} + \cdots + T^{m-p}y_{n}^m) = 0
\]
since \(X_{n-m+p}\) is orthogonal to \(T^{n-m+p-1}\) and \(y_{n-m+p}\) is in \(X_{n-m+p}\). Moreover, since \(y_{p}^n_{m+p}\) is orthogonal to \(T^{m-p}y_{n}^m + T^{m-p-1}y_{n-1} + \cdots + Ty_{n-p+m+1}\), it follows that
\[
y_{n-m+p} + P(Ty_{n-m+p+1} + \cdots + T^{m-p}y_{n}^m)
\]
and hence that \(y_{n-m+p} = 0\) which is a contradiction.

This completes the proof of the lemma.

3. Proof of Theorem 1. By applying Lemma 2 to \(T\) and \(T^*\) we obtain quasi-affinities \(X\) and \(X_+\) together with Jordan operators \(T_0\) and \(T_1\) such that
TX = XT₀, and T*X* = X*TX. Hence, X*T = T*X* and T is a quasi-affine transform of the Jordan operator T*₁. Thus T₀ is a quasi-affine transform of T*₁ and hence T₀ and T*₁ are quasi-similar by Lemma 1. Consequently, T is a quasi-affine transform of T₀ and we have established that T and T₀ are quasi-similar.

4. We make several remarks before continuing.

Since there exist quasi-nilpotent operators T such that ker T = (0) ≠ ker T* (for example, take T to be the weighted shift with weights 1, 1/2, 1/3, . . . ), Theorem 2 is not valid for quasi-nilpotent operators.

As a consequence of Theorem 2, observe that Lemma 1 holds for all nilpotent operators, that is, if one nilpotent operator is a quasi-affine transform of another, then the two operators are actually quasi-similar.

Lastly, by using the Dunford-Riesz spectral decomposition Theorem 2 can be shown to hold for algebraic operators with real spectrum.

5. Theorem 1 provides a Jordan model for every nilpotent operator on Hilbert space. However, in contrast with the finite dimensional case, distinct Jordan models may be quasi-similar. Fortunately, the situation is not as complicated as it might first appear. We obtain a canonical choice and hence a complete set of quasi-similarity invariants for nilpotent operators after introducing some terminology.

For each integer m (1 ≤ m < ∞) and each infinite cardinal N, let Jₘ denote the Jordan operator defined by the m x m operator matrix

\[
\begin{pmatrix}
0 & I_N & 0 & \cdots & 0 \\
0 & 0 & I_N & \cdots \\
\vdots & \vdots & \ddots & \ddots \\
\vdots & \vdots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & 0
\end{pmatrix}
\]

on \( mH = H \oplus \cdots \oplus H, \)

\( m \) times

where \( H \) is a Hilbert space of dimension \( N \).

**Theorem 3.** Every nilpotent operator is quasi-similar to a unique Jordan
model of the form $\bigoplus J_{m_1}^N \oplus N$, where $1 \leq m_1 < m_2 < \cdots < m_k < \infty$, $N_1 > N_2 > \cdots > N_k$, and $N$ is a finite rank Jordan model $\bigoplus_{j=1}^n T_j$ on $\bigoplus_{j=1}^n C_j$ with $m_k < I_j$ for $j = 1, 2, \ldots, n$.

**Proof.** By Theorem 1 we need only consider Jordan models and easy arguments reduce the result to proving that $J_k^N \oplus J_{k-1}^N$ and $J_k^N$ are quasi-similar for each $1 \leq k < \infty$ and infinite cardinal $N$. Moreover, since $J_k^N$ and $J_{k-1}^N$ are unitarily equivalent to $J_k^N$ and $J_{k-1}^N$, respectively, it is sufficient to show that $J_k^N$ is a quasi-affine transform of $J_k^N \oplus J_{k-1}^N$. Let $H$ be a Hilbert space of dimension $N$ and suppose $A$ and $B$ are operators on $H$ which satisfy

1. $\ker A = \{0\}$,
2. $\text{clos}(AH) = H$, and
3. $\text{clos}\{Ax \oplus Bx: x \in H\} = H \oplus H$.

Then the identity

$$
\begin{pmatrix}
A & 0 \\
0 & A \\
B & 0 \\
0 & B
\end{pmatrix}
= 
\begin{pmatrix}
J_k^N & 0 \\
0 & J_{k-1}^N
\end{pmatrix}
\begin{pmatrix}
A & 0 \\
0 & A \\
B & 0 \\
0 & B
\end{pmatrix}
$$

would complete the proof since (1), (2) and (3) imply that the matrix

$$
\begin{pmatrix}
A & 0 \\
0 & A \\
B & 0 \\
0 & B
\end{pmatrix}
$$

defines a quasi-affinity from $kH$ to $(2k-1)H$.

There are various ways of exhibiting operators satisfying (1), (2) and (3). For example, let $M_1$ and $M_2$ denote multiplication by the characteristic functions of the first and second quarters $Q_1$ and $Q_2$ of the unit circle respectively, defined from the Hardy space $H^2$ to the $L^2$ spaces $L^2(Q_1)$ and $L^2(Q_2)$ respectively. If $V_1$, $V_2$, and $V_3$ are unitary maps from $H$ onto $H^2 \otimes H$, $L^2(Q_1) \otimes H$, and $L^2(Q_2) \otimes H$, then $A = V_2^*(M_1 \otimes I_H)V_1$ and $B = V_3(M_2 \otimes I_H)V_1$ have the desired properties.

This theorem is probably indicative of the kind of uniqueness one can expect for Jordan models for $C_0$-operators of infinite multiplicity.
We conclude this section with a corollary which completes the classification of nilpotents up to quasi-similarity.

**Corollary.** If $T_1$ and $T_2$ are nilpotent operators on the Hilbert spaces $H_1$ and $H_2$, respectively, then $T_1$ and $T_2$ are quasi-similar if and only if
\[ \dim \operatorname{clos} [T_1^l H_1] = \dim \operatorname{clos} [T_2^l H_2] \text{ for } l = 0, 1, \ldots. \]

**Proof.** If $X$ is a quasi-affinity from $H_1$ to $H_2$ such that $T_2 X = X T_1$, then
\[ \operatorname{clos} [X T_1^l H_1] = \operatorname{clos} [T_2^l X H_1] = \operatorname{clos} [T_2^l H_2] \]
which implies that $\dim \operatorname{clos} [T_1^l H_1] = \dim \operatorname{clos} [T_2^l H_2]$ for $l = 0, 1, 2, \ldots$. Conversely, an easy argument shows that the Jordan model given in the theorem is uniquely determined by these dimensions.

6. The results of this note enable us to extend a characterization of reflexive operator of Deddens and Fillmore [4] to infinite dimensional spaces. Recall that a linear subspace $M$ of the Hilbert space $H$ is said to be para-closed for the operator $T$ on $H$ if $M$ is the range of some bounded operator on $H$. Let us call an operator $T$ on $H$ para-reflexive if any operator $U$ on $H$ leaving invariant the para-closed-invariant spaces of $T$ is an entire function of $T$. The definition is one of the possible natural extensions to infinite dimensional spaces of the concept of reflexive operators on a finite dimensional space.

We begin this section with a result which may have some independent interest.

**Proposition 1.** Para-reflexivity is preserved under quasi-similarity.

**Proof.** If $T$ and $S$ are quasi-similar and $S$ is para-reflexive we must show that $T$ is also para-reflexive. If $T$ is not algebraic, then by virtue of Theorem 2 [6], $T$ is para-reflexive. Thus we can assume that $T$ (and consequently $S$ also) is algebraic. Suppose $TA = AS$, $BT = SB$ where $A$, $B$ are quasi-affinities, and let $Z$ be an operator leaving invariant every finite dimensional subspace invariant for $T$, that is, for every $h$ in $H$ there exists some polynomial $p_h$ such that $Zh = p_h(T)h$. If we set $Z_0 = BZA$, then
\[ Z_0 h_0 = BZA h_0 = B p_{A h_0} (T) A h_0 = B A p_{A h} (S) h_0 \]
is in $BAH$ for every $h_0$ in $H$. Thus $X = (BA)^{-1} Z_0$ is, by the closed graph theorem, an operator on $H$ such that $X h_0$ is in the finite dimensional space $\bigvee_{l > 0} S^l h_0$ for every $h_0$ in $H$. It follows that $X$ leaves invariant every finite dimensional
subspace of $H$ invariant under $S$. Thus, since $S$ is para-reflexive, we infer from Corollary 2 [6] that $X = q(S)$, where $q$ is a suitable polynomial. Consequently, $BZA = Z_0 = BAq(S) = Bq(T)A$ and hence $Z = q(T)$. Using Corollary 2 [6] once again, we conclude that $T$ is para-reflexive.

A nilpotent operator on $H$ is said to satisfy the Deddens-Fillmore condition [4], if either $\dim H < 1$ or its Jordan model $\bigoplus_{a \in A} J_a$ has the following property: If $n_a$ denotes the order of the matrix of $J_a$ ($a \in A$) and $a_0$ is chosen in $A$ such that

\begin{equation}
\text{n}_{a_0} = \max \{ n_a | a \in A \},
\end{equation}

then

\begin{equation}
\max \{ n_a | a \in A \backslash \{a_0\} \} \geq n_{a_0} - 1.
\end{equation}

**Proposition 2.** A nilpotent operator $T$ on $H$ is para-reflexive if and only if it satisfies the Deddens-Fillmore condition.

**Proof.** By virtue of Proposition 1 and Theorem 1, it is sufficient to prove the statement in case $T = \bigoplus_{a \in A} J_a$. Exactly as in [4] we can prove that if this $T$ does not fulfill the Deddens-Fillmore condition then $T$ does not have property (A) or (B) of Corollary 2 [6]. Thus, by this corollary, $T$ is not para-reflexive.

Let us now show the sufficiency of the Deddens-Fillmore condition. It is clear that we can assume that

\[ T = J_0 \oplus J_1 \oplus \left( \bigoplus_{a \in B} J_a \right), \]

where the order $n_i$ of $J_i$ is the maximum occurring in formula (i) above ($i = 0, 1$); thus the order of any $J_a$ ($a \in B$) is not greater than $n_1$. Now let $Z$ be an operator leaving invariant all para-closed subspaces invariant for $T$. Then obviously

\[ Z = Z_0 \oplus Z_1 \oplus \left( \bigoplus_{a \in B} Z_a \right) \]

and for any $h = h_0 \oplus h_1 \oplus (\bigoplus_{a \in B} h_a)$ there exists a polynomial $p_h$ such that

\[ Z h = p_h(T) h, \]

that is,

\[ (Z_0 \oplus Z_1 \oplus Z_a)(h_0 \oplus h_1 \oplus h_a) = p_h(J_0 \oplus J_1 \oplus J_a)(h_0 \oplus h_1 \oplus h_a) \]

for every $a$ in $B$. The above relation shows in particular that $Z_0 \oplus Z_1 \oplus Z_a$ leaves invariant every invariant subspace of $J_0 \oplus J_1 \oplus J_a$. By virtue of the Deddens-Fillmore theorem there exists a unique polynomial $q_a$ of degree $\leq n_0$ such that
such that
\[ Z_0 \oplus Z_1 \oplus Z_\alpha = q_\alpha(J_0 \oplus J_1 \oplus J_\alpha) = q_\alpha(J_0) \oplus q_\alpha(J_1) \oplus q_\alpha(J_\alpha), \]
for every \( \alpha \in B \). Thus for \( \alpha, \beta \in B \) we have
\[ q_\alpha(J_0) = Z_0 = q_\beta(J_0). \]
Since \( J_0 \) is of order \( n_0 \) and \( q_\alpha, q_\beta \) are of degree \( \leq n_0 \), (3) implies \( q_\alpha = q_\beta \).
Consequently, there exists a polynomial \( q \) of degree \( \leq n_0 \) such that \( q_\alpha \equiv q \) for every \( \alpha \in B \). From (2) we infer
\[ Z = Z_0 \oplus Z_1 \oplus \left( \bigoplus_{\alpha \in B} Z_\alpha \right) = q(J_0) \oplus q(J_1) \oplus \left( \bigoplus_{\alpha \in B} q(J_\alpha) \right) = q(T) \]
which finishes our proof.

**Theorem 4.** An operator \( T \) is para-reflexive if and only if either it is nonalgebraic or it is algebraic and the nilpotents corresponding to the points of the spectrum of \( T \) satisfy the Deddens-Fillmore condition.

**Proof.** This follows at once from Proposition 2 above, the Dunford-Riesz spectral decomposition of an algebraic operator, and Corollary 1.

**REFERENCES**


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