QUASI-SIMILAR MODELS
FOR NILPOTENT OPERATORS (1)

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ABSTRACT. Every nilpotent operator on a complex Hilbert space is shown to be quasi-similar to a canonical Jordan model. Further, the para-reflexive operators are characterized generalizing a result of Deddens and Fillmore.

A familiar result states that each nilpotent operator on a finite dimensional complex Hilbert space is similar to its adjoint. One proof proceeds by showing that both a nilpotent operator and its adjoint have the same canonical form. In this note we show that although this result does not extend to infinite dimensional spaces, the weaker quasi-similarity version of it, together with the proof indicated above, still holds on any Hilbert space. This yields an affirmative answer to a question raised by P. Rosenthal in connection with the content of [3].

The canonical form exhibited provides positive evidence that the theory of Jordan models might be extended to cover operators of class \( C_0 \) of infinite multiplicity and indeed, considerable progress [2] has been made recently in this direction. Although the Jordan model for nilpotent operators on infinite dimensional Hilbert spaces is no longer unique, we single out a "canonical" model. A similar result has been obtained independently by Berkovici [1]. We conclude with an application of our results to extend to infinite dimensional spaces a theorem of Deddens and Fillmore [4] which characterizes reflexive operators on finite dimensional spaces.

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1. In this note, a nilpotent operator \( T \) will be called a Jordan operator if \( T = \bigoplus_{\alpha} T_{\alpha} \), where each \( T_{\alpha} \) operates on some \( C^{l_{\alpha}} \) for \( 0 < l_{\alpha} < \infty \) by the Jordan one-cell matrix.
Recall that an operator $X$ between Hilbert spaces $H$ and $K$ is said to be a quasi-affinity if $\ker X = (0)$ and $\overline{\operatorname{clos}(XH)} = K$. An operator $A$ on $H$ is said to be a quasi-affine transform of an operator $B$ on $K$ if there exists a quasi-affinity $X$ such that $XA = BX$. Finally, two operators $A$ and $B$ are quasi-similar if each is a quasi-affine transform of the other. For further information on these concepts see the monograph [8, Chapter II, No. 3.2], or [7].

Our main result is given by the following

**Theorem 1.** *Every nilpotent operator $T$ is quasi-similar to a Jordan operator $T_0$.***

Since for any Jordan operator $T_0$, the operators $T_0$ and $T_0^*$ are obviously unitarily equivalent, we can infer

**Theorem 2.** *If $T$ is a nilpotent operator, then $T$ and $T^*$ are quasi-similar.*

Before starting the proof of Theorem 1, we give an example to show that quasi-similarity cannot be replaced by similarity.

Let $X$ be any compact quasi-affinity on an infinite dimensional Hilbert space $H$ (for example, the Volterra operator on $L^2(0, 1)$) and consider the operator $T$ defined by

$$
\begin{pmatrix}
0 & 1 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix}
$$

Clearly $T^3 = 0$ and thus $T$ and $T^*$ are quasi-similar by Theorem 2, but $T$ and $T^*$ are not similar.\(^2\) The proof of this is straightforward.

If $S$ were an invertible operator on $H \oplus H \oplus H$ with matrix

\(^2\) The same example was found independently by H. Radjavi (see [3, §6]).
which satisfied $ST^* = TS$, then a simple computation shows that $B_2 = C_2 = C_1 = 0$, $C_0 = XB_1$ and $A_2 = B_1X^*$. Thus the operator

$$S_0 = \begin{pmatrix} A_0 & B_0 & 0 \\ A_1 & B_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is a compact perturbation of $S$, and hence a Fredholm operator, which is contradicted by the fact that $\ker S_0 = (0) \oplus (0) \oplus H$ is not finite dimensional (cf. [5, Chapter 5]).

2. We start the proof of Theorem 1 with the following

**Lemma 1.** If $T_0$ and $T_1$ are two Jordan operators and $T_0$ is a quasi-affine transform of $T_1$, then $T_0$ and $T_1$ are quasi-similar.

**Proof.** The fact that $T_0$ is a quasi-affine transform of $T_1$ means that there exists a quasi-affinity $X$ such that $XT_0 = T_1X$ and thus $T_0^*X^* = X^*T_1^*$. Since $T_i$ is a Jordan operator, there exists a unitary operator $U_i$ such that $T_i^* = U_i^*T_iU_i$ ($i = 0, 1$) Therefore, $T_0(U_0X^*U_1^*) = (U_0X^*U_1^*)T_1$, where $U_0X^*U_1^*$ is a quasi-affinity, and $T_1$ is also a quasi-affine transform of $T_0$. Consequently, $T_0$ and $T_1$ are quasi-similar.

**Lemma 2.** Any nilpotent operator $T$ has a quasi-affine transform $T_0$ which is a Jordan operator.

**Proof.** Suppose that $T^n = 0, T^{n-1} \neq 0$ for some $n \geq 1$. If we set

$$X_j = \ker T^j \oplus \ker T^{j-1} \quad \text{for } j = 1, 2, \ldots, n,$$

$$Y_n = X_n, \quad Y_{n-1} = X_{n-1} \cap (TY_n)^1, \ldots, \quad Y_1 = X_1 \cap (T^{n-1}Y_n + \cdots + TY_2)^1$$

and

$$H_0 = (Y_n \oplus \cdots \oplus Y_n) \oplus (Y_{n-1} \oplus \cdots \oplus Y_{n-1}) \oplus \cdots \oplus (Y_2 \oplus Y_2) \oplus Y_1 \quad \text{n times} \quad \quad \quad \quad \quad \text{(n – 1) times}$$

we can define the bounded operators $T_0$ on $H_0$ and $A: H_0 \rightarrow H$ by the equations
\[ T_0(y^1_n \oplus \cdots \oplus y^n_n \oplus \cdots \oplus y^1_2 \oplus y^2_n \oplus y^1_1) = 0 \oplus y^1_n \oplus \cdots \oplus y^{n-1}_n \oplus \cdots \oplus 0 \oplus y^1_2 \oplus 0, \]

and

\[ A(y^1_n \oplus \cdots \oplus y^n_n \oplus \cdots \oplus y^1_2 \oplus y^2_n \oplus y^1_1) = y^1_n + \cdots + T^{n-1}y^n_n + \cdots + y^1_2 + Ty^2_n + y^1_n. \]

It is easy to see that \( T_0 \) is a Jordan operator and that \( AT_0 = T_A \). Using the fact that

\[ (T + \cdots + T^{n-1}y^n_n + y^2_n + \cdots + T^{n-2}y^{n-1}_n + \cdots + T_0v_n + \cdots + T^{n-k}y^k_n) = 0, \]

which is proved by induction on \( k \), we conclude that \( \text{clos}(A_0) = H \). To complete the proof we must show that \( A \) is injective.

If \( A \) is not injective, there must exist \( y^j_k \) in \( V \), \( 1 \leq k \leq n \), \( 1 \leq k \leq j \), such that

\[ y^j_k \neq 0 \quad \text{but} \quad \sum_{k=1}^n \sum_{i=1}^j T^{k-1}y^k_{n-j+k} = 0. \]

Let \( m \) be the smallest integer such that \( \sum_{k=1}^m ||y^k_{n-m+k}|| \neq 0 \) and let \( p \) be the smallest integer such that \( y^p_{n-m+p} \neq 0 \). Because we have

\[ \sum_{k=p}^m T^{k-1}y^k_{n-m+k} = \sum_{k=m+1}^n \sum_{i=1}^j T^{k-1}y^k_{n-j+k} \quad \text{in ker } T^{n-m}, \]

it follows that \( y^p_{n-m+p} + \cdots + T^{m-p}y^m_n \) is in ker \( T^{n-m+p-1} \). If we let \( P \) denote the orthogonal projection of \( H \) onto \( X_{n-m+p} \), then

\[ y^p_{n-m+p} + P(Ty^p_{n-m+p+1} + \cdots + T^{m-p}y^m_n) = P(y^p_{n-m+p} + \cdots + T^{m-p}y^m_n) = 0 \]

since \( X_{n-m+p} \) is orthogonal to ker \( T^{n-m+p-1} \) and \( y^p_{n-m+p} \) is in \( X_{n-m+p} \). Moreover, since \( y^p_{n-m+p} \) is orthogonal to \( T^{m-p}y^m_n + T^{m-p-1}y^m_{n-1} + \cdots + T^{m-p}y^m_{n-m+p} \), it follows that

\[ y^p_{n-m+p} \perp P(Ty^p_{n-m+p+1} + \cdots + T^{m-p}y^m_n) \]

and hence that \( y^p_{n-m+p} = 0 \) which is a contradiction.

This completes the proof of the lemma.

3. Proof of Theorem 1. By applying Lemma 2 to \( T \) and \( T^* \) we obtain quasi-affinities \( X \) and \( X^* \) together with Jordan operators \( T_0 \) and \( T_1 \) such that
TX = XT₀, and T*X* = X*TX. Hence, X*T = T*X* and T is a quasi-affine transform of the Jordan operator T₁*. Thus T₀ is a quasi-affine transform of T₁* and hence T₀ and T₁* are quasi-similar by Lemma 1. Consequently, T is a quasi-affine transform of T₀ and we have established that T and T₀ are quasi-similar.

4. We make several remarks before continuing.

Since there exist quasi-nilpotent operators T such that ker T = (0) ≠ ker T* (for example, take T to be the weighted shift with weights 1, 1/2, 1/3, . . . ), Theorem 2 is not valid for quasi-nilpotent operators.

As a consequence of Theorem 2, observe that Lemma 1 holds for all nilpotent operators, that is, if one nilpotent operator is a quasi-affine transform of another, then the two operators are actually quasi-similar.

Lastly, by using the Dunford-Riesz spectral decomposition Theorem 2 can be shown to hold for algebraic operators with real spectrum.

5. Theorem 1 provides a Jordan model for every nilpotent operator on Hilbert space. However, in contrast with the finite dimensional case, distinct Jordan models may be quasi-similar. Fortunately, the situation is not as complicated as it might first appear. We obtain a canonical choice and hence a complete set of quasi-similarity invariants for nilpotent operators after introducing some terminology.

For each integer m (1 ≤ m < ∞) and each infinite cardinal η, let Jₘ denote the Jordan operator defined by the m x m operator matrix

\[
\begin{pmatrix}
0 & I_{H} & 0 & \cdots & 0 \\
0 & 0 & I_{H} & \cdots & \\
0 & 0 & 0 & I_{H} & \\
0 & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & 0 & 0
\end{pmatrix}
\]

on \( mH = H \oplus \cdots \oplus H \), m times

where H is a Hilbert space of dimension η.

**Theorem 3.** Every nilpotent operator is quasi-similar to a unique Jordan
model of the form $\bigoplus_{j=1}^{n_j} J_{m_j} \oplus N$, where $1 < m_1 < m_2 < \cdots < m_k < \infty$, $n_1 > n_2 > \cdots > n_k$, and $N$ is a finite rank Jordan model $\bigoplus_{j=1}^{n_j} C^j$ on $\bigoplus_{j=1}^{n_j} C^j$ with $m_k < l_j$ for $j = 1, 2, \ldots, n$.

Proof. By Theorem 1 we need only consider Jordan models and easy arguments reduce the result to proving that $J_k \oplus J_{k-1}$ and $J_k$ are quasi-similar for each $1 < k < \infty$ and infinite cardinal $\aleph_1$. Moreover, since $J_k$ and $J_{k-1}$ are unitarily equivalent to $J_k$ and $J_{k-1}$, respectively, it is sufficient to show that $J_k$ is a quasi-affine transform of $J_k \oplus J_{k-1}$. Let $H$ be a Hilbert space of dimension $\aleph_1$ and suppose $A$ and $B$ are operators on $H$ which satisfy

1. $\ker A = (0),$
2. $\text{clos}(AH) = H$, and
3. $\text{clos}\{Ax \oplus Bx : x \in H\} = H \oplus H.$

Then

\[
\begin{pmatrix}
A & 0 \\
0 & A \\
0 & B \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
J_k^N \\
J_{k-1}^N
\end{pmatrix} =
\begin{pmatrix}
J_k^N & 0 \\
0 & J_{k-1}^N
\end{pmatrix}
\begin{pmatrix}
A & 0 \\
0 & A \\
0 & B \\
0 & 0
\end{pmatrix}
\]

would complete the proof since (1), (2) and (3) imply that the matrix

\[
\begin{pmatrix}
A & 0 \\
0 & A \\
0 & B \\
0 & 0
\end{pmatrix}
\]

defines a quasi-affinity from $kH$ to $(2k - 1)H$.

There are various ways of exhibiting operators satisfying (1), (2) and (3). For example, let $M_1$ and $M_2$ denote multiplication by the characteristic functions of the first and second quarters $Q_1$ and $Q_2$ of the unit circle respectively, defined from the Hardy space $H^2$ to the $L^2$ spaces $L^2(Q_1)$ and $L^2(Q_2)$ respectively. If $V_1$, $V_2$, and $V_3$ are unitary maps from $H$ onto $H^2 \otimes H$, $L^2(Q_1) \otimes H$, and $L^2(Q_2) \otimes H$, then $A = V_2^*(M_1 \otimes I_H)V_1$ and $B = V_3^*(M_2 \otimes I_H)V_1$ have the desired properties.

This theorem is probably indicative of the kind of uniqueness one can expect for Jordan models for $C_0$-operators of infinite multiplicity.
We conclude this section with a corollary which completes the classification of nilpotents up to quasi-similarity.

**Corollary.** If $T_1$ and $T_2$ are nilpotent operators on the Hilbert spaces $H_1$ and $H_2$, respectively, then $T_1$ and $T_2$ are quasi-similar if and only if
\[ \dim \text{clos} [T_1^l H_1] = \dim \text{clos} [T_2^l H_2] \quad \text{for} \ l = 0, 1, \ldots . \]

**Proof.** If $X$ is a quasi-affinity from $H_1$ to $H_2$ such that $T_2 X = X T_1$, then
\[ \text{clos} [X T_1^l H_1] = \text{clos} [T_2^l X H_1] = \text{clos} [T_2^l H_2] \]
which implies that $\dim \text{clos} [T_1^l H_1] = \dim \text{clos} [T_2^l H_2]$ for $l = 0, 1, 2, \ldots$. Conversely, an easy argument shows that the Jordan model given in the theorem is uniquely determined by these dimensions.

6. The results of this note enable us to extend a characterization of reflexive operator of Deddens and Fillmore [4] to infinite dimensional spaces. Recall that a linear subspace $M$ of the Hilbert space $H$ is said to be para-closed for the operator $T$ on $H$ if $M$ is the range of some bounded operator on $H$. Let us call an operator $T$ on $H$ para-reflexive if any operator $U$ on $H$ leaving invariant the para-closed-invariant spaces of $T$ is an entire function of $T$. The definition is one of the possible natural extensions to infinite dimensional spaces of the concept of reflexive operators on a finite dimensional space.

We begin this section with a result which may have some independent interest.

**Proposition 1.** Para-reflexivity is preserved under quasi-similarity.

**Proof.** If $T$ and $S$ are quasi-similar and $S$ is para-reflexive we must show that $T$ is also para-reflexive. If $T$ is not algebraic, then by virtue of Theorem 2 [6], $T$ is para-reflexive. Thus we can assume that $T$ (and consequently $S$ also) is algebraic. Suppose $T A = A S$, $B T = S B$ where $A$, $B$ are quasi-affinities, and let $Z$ be an operator leaving invariant every finite dimensional subspace invariant for $T$, that is, for every $h$ in $H$ there exists some polynomial $p_h$ such that $Z h = p_h(T) h$. If we set $Z_0 = B Z A$, then
\[ Z_0 h_0 = B Z A h_0 = B p_A h_0 (T) A h_0 = B A p_A (S) h_0 \]
is in $B A H$ for every $h_0$ in $H$. Thus $X = (B A)^{-1} Z_0$ is, by the closed graph theorem, an operator on $H$ such that
\[ X h_0 \]
is in the finite dimensional space $\bigvee_{l>0} S^l h_0$ for every $h_0$ in $H$. It follows that $X$ leaves invariant every finite dimensional
subspace of $H$ invariant under $S$. Thus, since $S$ is para-reflexive, we infer from Corollary 2 [6] that $X = q(S)$, where $q$ is a suitable polynomial. Consequently, $BZ = Z_0 = BA_q(S) = B_q(T)A$ and hence $Z = q(T)$. Using Corollary 2 [6] once again, we conclude that $T$ is para-reflexive.

A nilpotent operator on $H$ is said to satisfy the Deddens-Fillmore condition [4], if either $\dim H \leq 1$ or its Jordan model $\bigoplus_{\alpha \in A} J_\alpha$ has the following property: If $n_\alpha$ denotes the order of the matrix of $J_\alpha (\alpha \in A)$ and $\alpha_0$ is chosen in $A$ such that

\[(0) \quad n_{\alpha_0} = \max \{n_\alpha | \alpha \in A\},\]

then

\[(1) \quad \max \{n_\alpha | \alpha \in A \setminus \{\alpha_0\}\} > n_{\alpha_0} - 1.\]

**Proposition 2.** A nilpotent operator $T$ on $H$ is para-reflexive if and only if it satisfies the Deddens-Fillmore condition.

**Proof.** By virtue of Proposition 1 and Theorem 1, it is sufficient to prove the statement in case $T = \bigoplus_{\alpha \in A} J_\alpha$. Exactly as in [4] we can prove that if this $T$ does not fulfill the Deddens-Fillmore condition then $T$ does not have property (A) or (B) of Corollary 2 [6]. Thus, by this corollary, $T$ is not para-reflexive.

Let us now show the sufficiency of the Deddens-Fillmore condition. It is clear that we can assume that

$$T = J_0 \oplus J_1 \oplus \left( \bigoplus_{\alpha \in B} J_\alpha \right),$$

where the order $n_i$ of $J_i$ is the maximum occurring in formula (i) above ($i = 0, 1$); thus the order of any $J_\alpha (\alpha \in B)$ is not greater than $n_1$. Now let $Z$ be an operator leaving invariant all para-closed subspaces invariant for $T$. Then obviously

$$Z = Z_0 \oplus Z_1 \oplus \left( \bigoplus_{\alpha \in B} Z_\alpha \right)$$

and for any $h = h_0 \oplus h_1 \oplus (\bigoplus_{\alpha \in B} h_\alpha)$ there exists a polynomial $p_h$ such that

$$Zh = p_h(T)h, \quad \text{that is,}$$

$$(Z_0 \oplus Z_1 \oplus Z_\alpha)(h_0 \oplus h_1 \oplus h_\alpha) = p_h(J_0 \oplus J_1 \oplus J_\alpha)(h_0 \oplus h_1 \oplus h_\alpha)$$

for every $\alpha$ in $B$. The above relation shows in particular that $Z_0 \oplus Z_1 \oplus Z_\alpha$ leaves invariant every invariant subspace of $J_0 \oplus J_1 \oplus J_\alpha$. By virtue of the Deddens-Fillmore theorem there exists a unique polynomial $p_\alpha$ of degree $\leq n_0$ such that
such that

\[
Z_0 \oplus Z_1 \oplus Z_\alpha = q_\alpha (J_0 \oplus J_1 \oplus J_\alpha)
= q_\alpha (J_0) \oplus q_\alpha (J_1) \oplus q_\alpha (J_\alpha),
\]

for every \( \alpha \) in \( B \). Thus for \( \alpha, \beta \) in \( B \) we have

\[
q_\alpha (J_0) = Z_0 = q_\beta (J_0).
\]

Since \( J_0 \) is of order \( n_0 \) and \( q_\alpha, q_\beta \) are of degree \( \leq n_0 \), (3) implies \( q_\alpha = q_\beta \).

Consequently, there exists a polynomial \( q \) of degree \( \leq n_0 \) such that \( q_\alpha \equiv q \) for every \( \alpha \) in \( B \). From (2) we infer

\[
Z = Z_0 \oplus Z_1 \oplus \left( \bigoplus_{\alpha \in B} Z_\alpha \right) = q(J_0) \oplus q(J_1) \oplus \left( \bigoplus_{\alpha \in B} q(J_\alpha) \right) = q(T)
\]

which finishes our proof.

**Theorem 4.** An operator \( T \) is para-reflexive if and only if either it is nonalgebraic or it is algebraic and the nilpotents corresponding to the points of the spectrum of \( T \) satisfy the Deddens-Fillmore condition.

**Proof.** This follows at once from Proposition 2 above, the Dunford-Riesz spectral decomposition of an algebraic operator, and Corollary 1.

**REFERENCES**


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