THE DEGREE OF APPROXIMATION
FOR GENERALIZED POLYNOMIALS
WITH INTEGRAL COEFFICIENTS

BY

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ABSTRACT. The classical Müntz theorem and the so-called Jackson-Müntz
theorems concern uniform approximation on \([0, 1]\) by polynomials whose expo-
nents are taken from an increasing sequence of positive real numbers \(\Lambda\). Under
mild restrictions on the exponents, the degree of approximation for \(\Lambda\)-poly-
nomials with real coefficients is compared with the corresponding degree of approxima-
tion when the coefficients are taken from the integers.

Let \(C[0, 1]\) be the space of all continuous real valued functions defined on
the interval \([0, 1]\) and \(\| \cdot \|\) the supremum norm on \([0, 1]\) \((\|f\| = \sup \{ |f(x)|: 0 < x < 1 \})\). It is well known that the ordinary algebraic polyno-
mials with integral coefficients, i.e. integral polynomials, are dense in the subspace

\[ C_0[0, 1] = \{ f \in C[0, 1]: f(0) = f(1) = 0 \}. \]

This seems to be due originally to Kakeya [10], but many other authors have
also studied this or related problems: Pál [17], Okada [16], Bernstein [2],
Fekete [3]. Finally, Hewitt and Zuckerman [9] obtained necessary and suf-
ficient conditions. With every closed real interval of length less than 4, they
associate a certain finite subset \(J\). A continuous real function \(f\) on the interval is
arbitrarily uniformly approximable by integral polynomials if and only if \(f\) is
equal to some integral polynomial on the set \(J\).

In 1931, Kantorovič [11] proved that for any positive integer \(n\) and any
function \(f \in C_0[0, 1]\) there exists an integral polynomial \(p_n(x) = \sum_{k=0}^{n} b_k x^k\)
such that

\[ \| f - p_n \| \leq 2E_n(f) + O(n^{-1}) \text{ for } n \to \infty \]

holds, where

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Gel’fond [6] and Trigub [18] extended this result to differentiable functions $f$ for the intervals $[0, 1]$ and $[a, b]$, $b - a < 4$, respectively, and obtained analogues of Jackson’s and Timan’s theorems.

Many theorems, which are well known for ordinary polynomials, are also valid for the so-called $\Lambda$-polynomials of the form

$$P_s(x) = \sum_{k=1}^{s} a_k x^{\lambda_k}, \quad a_k \text{ real},$$

where $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$ is a positive increasing sequence of real numbers. Müntz [14] proved that the $\Lambda$-polynomials are dense in the subset $\{f \in C[0, 1] : f(0) = 0\}$ of $C[0, 1]$ if and only if $\sum_{k=1}^{\infty} 1/\lambda_k = \infty$.

Recently, Le Baron O. Ferguson and von Golitschek [4] showed that for every sequence $\Lambda$ of distinct positive integers the $\Lambda$-polynomials with integral coefficients are dense in $C_0[0, 1]$ if and only if Müntz’s condition holds. This result is even valid for every sequence $\Lambda$ of distinct positive real numbers.

Combining Müntz’s and Jackson’s theorems, Newman [15], von Golitschek [7], [8], Ganelius and Westlund [5], Leviatan [12], Bak and Newman [1] obtained upper and lower bounds for the degree of approximation when functions $f$ are approximated by the $\Lambda$-polynomials with real coefficients.

The purpose of this paper is to find analogous Jackson-Müntz theorems for $\Lambda$-polynomials with integral coefficients. More precisely, we shall prove the following two theorems which generalize Kantorović’s result (1).

**Theorem 1.** Let the positive increasing sequence $\Lambda$ satisfy

$$\lambda_1 \geq B, \quad \lambda_{2k} \leq C\lambda_k, \quad \lambda_k \leq Bk \quad \text{for} \quad k \geq k_0,$$

where $k_0$ is a positive integer and $B$ and $C$ are positive constants. For any function $f \in C_0[0, 1]$ and any positive integer $s$ there exist integers $b_j, 1 \leq j \leq s$, such that

$$\left\| f(x) - \sum_{j=1}^{s} b_j x^{\lambda_j} \right\| \leq 2E_s(f; \Lambda) + O(s^{-1}),$$

where.

$$E_s(f; \Lambda) = \inf \left\{ \left\| f(x) - \sum_{k=1}^{s} a_k x^{\lambda_k} \right\| : a_k \text{ real} \right\}.$$
(4) \[ \lambda_{2k} \leq C\lambda_k, \quad \lambda_k \geq Bk \quad \text{for } k \geq 1. \]

For any function \( f \in C_0[0, 1] \) and any positive integer \( s \) there exist integers \( b_j, 1 \leq j \leq s, \) such that

(5) \[ \left\| f(x) - \sum_{j=1}^{s} b_jx^{\lambda_j} \right\| \leq 2E_s(f; \Lambda) + O(\varphi(s)^{-B}), \]

where \( \varphi(s) = \exp(\sum_{k=1}^{s} 1/\lambda_k). \)

**Remark 1.** The restrictions \( \lambda_{2k} \leq C\lambda_k \) \( (k \geq k_0 \) or \( k \geq 1) \) in Theorem 1 and Theorem 2 are mild. Indeed, many different sequences have this property, for example

\[ \lambda_k = k^\beta \quad (k \geq 1), \quad \beta > 0, \]

\[ \lambda_k = k \log k \quad (k > 1), \]

and even converging sequences \( \Lambda \) with \( \lim_{k \to \infty} \lambda_k = \lambda, \lambda > 0. \)

**Remark 2.** It follows from the theory of width (cf. Lorentz [13, Chapter 9]) that the classes \( \Gamma_w \) of functions,

\[ \Gamma_w = \{ f \in C[0, 1]: w(f; h) \leq w(h) \text{ for } 0 < h < 1 \}, \]

where \( w \) is a given modulus of continuity, have the following property. There exists a positive number \( c \) not depending on \( s \) such that

(6) \[ \sup_{f \in \Gamma_w} E_s(f; \Lambda) \geq cw(s^{-1}). \]

It is easy to see that the classes \( \Gamma_{w0} = \Gamma_w \cap C_0[0, 1] \) satisfy (6). Therefore the summand \( O(s^{-1}) \) in (3) of Theorem 1 does not change the rate of convergence if we consider the whole class \( \Gamma_{w0}. \)

Combining Theorem 2 and the Jackson-Müntz theorem [8, Theorem 3] for \( \Lambda \)-polynomials with real coefficients, we are led to the following.

**Corollary.** If (4) holds, then for any function \( f \in C_0[0, 1] \) and any positive integer \( s \) there exist integers \( b_j, 1 \leq j \leq s, \) such that

(7) \[ \left\| f(x) - \sum_{j=1}^{s} b_jx^{\lambda_j} \right\| = O(w(f; \varphi(s)^{-B^*})) \]

where \( B^* = \min\{B; 2\} \) and \( w(f; h) = \sup\{|f(x + t) - f(x)|: |t| \leq h, x, x + t \in [0, 1], 0 \leq h \leq 1\}, \) denotes the modulus of continuity of \( f. \)

**Remark 3.** For \( B > 2 \) the rate of convergence in (7) is best possible for classes \( \Gamma_{w0}, \) even if we approximate by \( \Lambda \)-polynomials with real coefficients (cf. Bak and Newman [1]).
Proofs of Theorem 1 and Theorem 2. By [4, Lemma 1] there exists for any positive integers $q$ and $s$, $q < s$, a $\Lambda$-polynomial $Q_{qs}(x) = \sum_{i=q+1}^{s} c_{i} x^{i}$ such that

$$A_{qs} = \| x^{q} - Q_{qs}(x) \| \leq 2 \exp \left( -2 \lambda_{q} \sum_{i=q+1}^{s} 1/i \right)$$

and $Q_{qs}(1) = 1$, where the first equality in (8) serves to define $A_{qs}$.

**Lemma 1.** Let $r$ and $s$ be positive integers such that $r \leq s + 1 - C \log s < s$. (a) If the assumptions of Theorem 1 hold then

$$\sum_{q=1}^{r} A_{qs} = O(s^{-1}) \quad \text{for } s \to \infty.$$  

(b) If the assumptions of Theorem 2 hold then

$$\sum_{q=1}^{r} A_{qs} = O(\varphi(s)^{-B}) \quad \text{for } s \to \infty.$$  

**Proof of Lemma 1.** Let $s$ be so large that $s/2 > C \log s > k_{0}$.

(a) We apply the inequalities (2) and (8). Then, for $1 < q < C \log s$ and $q_{0} = \max\{q + 1; k_{0}\}$,

$$A_{qs} \leq 2 \exp \left( -2 \lambda_{q} \sum_{i=q_{0}}^{s} 1/i \right) \leq 2(q_{0}/s)^{q_{0}/s} \leq 2^q(s)^{2}$$

for $C \log s < q < s/2$,

$$A_{qs} \leq 2 \exp \left( -2 \lambda_{q} \sum_{i=q+1}^{2q} 1/i \right) \leq 2e^{-2q/C} \leq 2s^{-2},$$

and for $s/2 < q < s + 1 - C \log s$,

$$A_{qs} \leq 2 \exp(-2\lambda_{q}(s - q)/\lambda_{q}) \leq 2e^{2/Cs^{-2}}.$$

Combining (11) through (13) we obtain (9).

(b) We apply the inequalities (4) and (8). Let $1 < q < s/2$. Since

$$B \sum_{i=1}^{q} 1/i \leq \sum_{i=1}^{q} 1/i \leq 1 + \log q$$

and

$$\exp \left( -\lambda_{q} \sum_{i=q+1}^{s} 1/i \right) \leq \exp \left( -\lambda_{q} \sum_{i=q+1}^{2q} 1/i \right) \leq e^{-q/C},$$

we obtain

$$A_{qs} \leq 2e^{-q/C} \exp \left( -B \sum_{i=q+1}^{s} 1/i \right) \leq 2eqe^{-q/C} \varphi(s)^{-B}.$$
Again (13) is valid for \( s/2 < q < s + 1 - C \log s \). This together with (14) completes the proof of (10) if we take into consideration that \( \varphi(s)^{-B} \geq (es)^{-1} \).

**Lemma 2.** Let \( n \) and \( m \) be positive integers and \( \alpha_j, n \leq j \leq m, \) be real numbers such that

\[
\sum_{j=n}^{m} \alpha_j = 0 \pmod{1}.
\]

Then there exist integers \( b_j, n \leq j \leq m, \) for which

\[
\left\| \sum_{j=n}^{m} (\alpha_j - b_j) x^{\lambda_j} \right\| \leq 6 \left( \frac{\lambda_m - \lambda_n}{\lambda_n} \right)^2 + \frac{\lambda_m - \lambda_n}{(m - n) \lambda_n}.
\]

**Proof of Lemma 2.** We may assume that \( \lambda_{n+1} - \lambda_n \geq 4 \), because the substitution \( x = y^\beta \) does not change the supremum norm in (16) for any positive number \( \beta \). Applying the method in [4, Lemma 4] we define, recursively, for \( j = m, m - 1, \ldots, n, \) \( d_m = \alpha_m - \lfloor \alpha_m \rfloor \) and

\[
d_j = \begin{cases} 
\alpha_j - \lfloor \alpha_j \rfloor & \text{if } \sum_{i=j+1}^{m} d_i \leq 0 \\
\alpha_j - \lfloor \alpha_j \rfloor - 1 & \text{if } \sum_{i=j+1}^{m} d_i > 0
\end{cases}, \quad m - 1 \geq j \geq n,
\]

where \( \lfloor \alpha \rfloor \) denotes the largest integer less or equal to \( \alpha \). Then the numbers \( \delta_j = \sum_{i=j}^{m} d_i \) have the properties \( \delta_n = 0, |\delta_j| < 1 \) for \( n < j \leq m, \) and

\[
p(x) = \sum_{i=n}^{m} d_i x^{\lambda_i} = \sum_{j=n+1}^{m} \delta_j (x^{\lambda_j} - x^{\lambda_{j-1}})
\]

where the first equality serves to define \( p \).

Let \( q \) be a given integer, \( n < q \leq m \). Let \( a^*, \beta_j, \) and \( T \) be defined by

\[
a^* = p'(1)/(\lambda_q - \lambda_{q-1}), \quad \beta_j = \lambda_{j} - \lambda_n, \quad n \leq j \leq m,
\]

and

\[
T(x) = x^{-\lambda_n}(p(x) - a^*(x^{\lambda_q} - x^{\lambda_q+1})))
\]

\[
= \sum_{j=n+1}^{m} \gamma_j (x^{\beta_j} - x^{\beta_{j-1}})
\]

where \( p' \) denotes the first derivative of \( p \). The numbers \( \gamma_j \) are defined by \( \gamma_j = \delta_j \) for \( j \neq q \) and \( \gamma_q = \delta_q - a^* \) and satisfy \( |\gamma_j| < 1 \) for \( j \neq q \) and \( |\gamma_q| < 1 + |a^*| \). Since \( \beta_n = 0 \) and \( \beta_1 \geq 4 \) it is easy to see that

\[
\| \beta_j (\beta_j - 1)x^{\beta_j-2} - \beta_{j-1}(\beta_{j-1} - 1)x^{\beta_{j-1}-2} \| \leq \beta_j^2 - \beta_{j-1}^2,
\]
\( n + 1 \leq j \leq m \). Therefore we obtain for the second derivative \( T'' \) of \( T \),

\[
\|T''\| \leq \sum_{j=n+1}^{m} |\gamma_j| (\beta_j^2 - \beta_{j-1}^2) \leq \beta_m^2 + |a^*| (\beta_q^2 - \beta_{q-1}^2).
\]

Since \( |p'(1)| \leq \beta_m \) it follows that \( |a^*| \leq \beta_m/(\lambda_q - \lambda_{q-1}) \) and

\[
(17) \quad \|T''\| \leq 3\beta_m^2.
\]

We notice that \( T(1) = T'(1) = 0 \). Thus, by the mean value theorem,

\[
|T(x)| \leq (x - 1)^2||T''||/2, \quad 0 < x < 1,
\]

and by the inequality \( \|x^\lambda n(x - 1)^2\| \leq 4\lambda_n^{-2} \), we find

\[
(18) \quad |p(x) - a^*(x^\lambda q - x^{\lambda q-1})| = x^\lambda n |T(x)| \leq 6(\lambda_m - \lambda_n)^2\lambda_n^{-2}.
\]

We define the integers \( b_j, n < j < m \), by

\[
(19) \quad b_j = \begin{cases} 
\alpha_j + d_j, & j \neq q, \ j \neq q - 1, \\
\alpha_q + d_q - [a^*], & j = q, \\
\alpha_{q-1} + d_{q-1} + [a^*], & j = q - 1.
\end{cases}
\]

It is evident that

\[
\left\| \sum_{j=n}^{m} (\alpha_j - b_j)x^\lambda j \right\| \leq \|p(x) - a^*(x^{\lambda q} - x^{\lambda q-1})\|
\]

\[
+ (a^* - [a^*]) \|x^{\lambda q} - x^{\lambda q-1}\|.
\]

We choose the integer \( q \) such that \( \lambda_q - \lambda_{q-1} = \inf\{\lambda_j - \lambda_{j-1} : n + 1 \leq j \leq m\} \) and therefore

\[
(20) \quad \lambda_q - \lambda_{q-1} \leq (\lambda_m - \lambda_n)/(m - n).
\]

Applying the inequalities \( \|x^{\lambda q} - x^{\lambda q-1}\| \leq (\lambda_q - \lambda_{q-1})/\lambda_q \) and (18) through (20) we obtain (16) and thus conclude the proof of Lemma 2.

**Remark 4.** In the proof of [4, Lemma 4] we have constructed the \( \Lambda \)-polynomial \( p \) and have proved in [4, Lemma 2] that

\[
(21) \quad \|p\| \leq (\lambda_m - \lambda_n)/\lambda_n.
\]

Setting \( a^* = 0 \) and applying (21) we find integers \( b_j, n < j < m \), such that

\[
(22) \quad \left\| \sum_{j=n}^{m} (\alpha_j - b_j)x^\lambda j \right\| \leq (\lambda_m - \lambda_n)/\lambda_n.
\]

However, the upper bound in (22) is larger than in (16) and not small enough to prove the results of our Theorems 1 and 2. Thus the more complicated construction of our Lemma 2 is necessary.
Let \( f \in C_0[0, 1] \) and \( s \) be sufficiently large. We denote \( t = [(s + 1)/2] \) and consider the set of all pairs of integers \( \Phi = \{(u, v) \colon v = u + [C \log s], t < u < v < s\} \). We choose \((r, m) \in \Phi\) such that \( \lambda_m - \lambda_r = \min \{\lambda_u - \lambda_v ; (u, v) \in \Phi\} \). Then it follows that

\[
\frac{\lambda_m - \lambda_r}{\lambda_r} \leq \frac{(\lambda_s - \lambda_t)C\log s}{\lambda_r(s - t - C\log s)} \leq \frac{C(C - 1)\log s}{s - t - C\log s} \leq 3C(C - 1)s^{-1}\log s.
\]

For \( r \) and \( s \), we proceed as in the proof of [4, Lemma 4]. There exist real coefficients \( a_{j0} \), \( 1 \leq j \leq s \), such that

\[
\|f(x) - \sum_{i=1}^{s} a_{j0}x^\lambda_i\| \leq 2E_s(f; \Lambda) \quad \text{and} \quad \sum_{i=1}^{s} a_{j0} = 0.
\]

We define the integers \( b_j \), \( 1 \leq j \leq r \), and the real coefficients \( a_{jq} \), \( 1 \leq q \leq r \) and \( q + 1 \leq j \leq s \), by the induction on \( q \) which has been described in the proof of [4, Lemma 4]. Thus we obtain real coefficients \( a_{jr} \), \( r + 1 \leq j \leq s \), for which the inequalities

\[
\|f(x) - \sum_{j=1}^{r} b_jx^\lambda_i - \sum_{j=r+1}^{s} a_{jr}x^\lambda_i\| \leq 2E_s(f; \Lambda) + \sum_{q=1}^{r} A_{qs}
\]

and

\[
\sum_{j=1}^{r} b_j + \sum_{j=r+1}^{s} a_{jr} = 0
\]

are satisfied.

If \( m < s \) we define the integers \( b_j \), \( m + 1 \leq j \leq s \), by \( b_j = \lceil a_{jr} \rceil \). There exist \( \Lambda \)-polynomials \( Q_q(x) = \sum_{i=r+1}^{m} c_{iq}x^\lambda_i \), \( m + 1 \leq q \leq s \), such that

\[
\|x^\lambda_q - Q_q(x)\| \leq 2 \left( \prod_{i=r+1}^{m} \frac{\lambda_q - \lambda_i}{\lambda_q + \lambda_i} \right) \quad \text{and} \quad Q_q(1) = 1.
\]

(Cf. [7, Lemma 2] and [4, Lemma 1].) We apply the inequality \((1 - x)/(1 + x) \leq e^{-2x} \) factorwise for \( x = \lambda_i/\lambda_q \) and obtain from (27),

\[
\|x^\lambda_q - Q_q(x)\| \leq 2 \exp \left( -2 \sum_{i=r+1}^{m} \frac{\lambda_q}{\lambda_i} \right).
\]

Since \( \lambda_q \leq C\lambda_r \) and \( m - r \geq -1 + C\log s \) we get

\[
\|x^\lambda_q - Q_q(x)\| \leq 2 \exp(-2(m - r)\lambda_r/\lambda_q) \leq 2e^{2/C_s^{-2}}.
\]

In the next step we define the real coefficients \( \alpha_j \), \( r + 1 \leq j \leq m \), by

\[
\alpha_j = a_{jr} + \sum_{q=m+1}^{s} (a_{qr} - b_q)c_{jq}
\]
and obtain from (25) and (28)

\[
\left\| f(x) - \sum_{j=1}^{r} b_j x^{\lambda_j} - \sum_{j=m+1}^{s} b_j x^{\lambda_j} - \sum_{j=r+1}^{m} \alpha_j x^{\lambda_j} \right\| \leq 2E_s(f; \Lambda) + \sum_{q=1}^{r} A_{qs} + e^{2/Cs^{-1}}.
\]

(30)

Since \( Q_q(1) = 1 \) for \( m + 1 < q < s \) it follows by (26) and (29) that

\[
\sum_{j=r+1}^{m} \alpha_j = 0 \pmod{1}.
\]

(31)

If \( m = s \) we define the real numbers \( \alpha_j \) by \( \alpha_j = a_{jr}, r + 1 < j < m \). Then, by (26), the equality (31) is also valid. Now we can apply Lemma 2 for \( m \) and \( n = r + 1 \). We find integers \( b_j, r + 1 < j < m \), for which (16) is satisfied. Hence by (30) (if \( m < s \)) or (25) (if \( m = s \)), (16), (23) and the inequality \( m - r \geq C \log s - 1 \) we are led to

\[
\left\| f(x) - \sum_{j=1}^{s} b_j x^{\lambda_j} \right\| \leq 2E_s(f; \Lambda) + \sum_{q=1}^{r} A_{qs} + O(s^{-1}).
\]

(32)

Finally we combine (32) and Lemma 1. This concludes the proofs of Theorem 1 and Theorem 2. For the latter we apply that \( \varphi(s)^{-B} \gg (es)^{-1} \) if (4) holds.

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