WEIERSTRASS NORMAL FORMS AND INVARIANTS OF ELLIPTIC-surfaces

BY

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Abstract. Let \( \pi: S \to B \) be an elliptic surface with a section \( \sigma: B \to S \). Let \( L^1 \to B \) be the normal bundle of \( \sigma(B) \) in \( S \), and let \( W = \mathbb{P}(L^2 \oplus L^3 \oplus 1) \) be a \( \mathbb{P}^2 \)-bundle over \( B \). Let \( S^* \) be the surface obtained from \( S \) by contracting those components of fibres of \( S \) which do not intersect \( \sigma(B) \). Then \( S^* \) may be imbedded in \( W \) and defined by a "Weierstrass equation":

\[
y^2z = x^3 - g_2xz^2 - g_3z^3
\]

where \( g_2 \in H^0(B, \mathcal{O}(L^2)) \) and \( g_3 \in H^0(B, \mathcal{O}(L^3)) \). The only singularities (if any) of \( S^* \) are rational double points. The triples \( (L, g_2, g_3) \) form a set of invariants for elliptic surfaces with sections, and a complete set of invariants is given by \( \{(L, g_2, g_3)\}/G \) where \( G \cong \mathbb{C}^* \times \text{Aut}(B) \).

An elliptic surface is a morphism \( \pi: S \to B \) where \( S \) is a compact complex analytic surface, \( B \) is a compact Riemann surface, and such that for all but finitely many points \( t \in B \), \( C_t = \pi^{-1}(t) \) is a nonsingular elliptic curve in \( S \). Throughout this paper we will assume the existence of a section \( \sigma: B \to S (\pi \circ \sigma = \text{id}_B) \). In this case, it follows that \( S \) is algebraic [3].

\( \pi: S \to B \) will be called a minimal elliptic surface if no fibre of \( S \) contains an exceptional curve of the first kind. It is possible for \( \pi: S \to B \) to be a minimal elliptic surface while \( S \) is not a minimal surface (rational elliptic surface). If \( \pi: S \to B \) and \( \phi: F \to B \) are elliptic surfaces with sections \( \sigma: B \to S \), \( \tau: B \to F \), then a birational mapping is a biholomorphic map \( f: \pi^{-1}(B') \to \phi^{-1}(B') \) where \( B' \subset B \) is a Zariski open set, satisfying: \( \phi \cdot f = \pi \) and \( f \cdot \sigma = \tau \). Then we have the following

Theorem. If \( F \) is a minimal elliptic surface, then \( f \) extends to a holomorphic mapping \( \hat{f}: S \to F \) [3].

It is not hard to prove that if \( \pi: S \to B \) is any elliptic surface (not necessarily containing a section), then the exceptional curves lying in any fibre of \( S \) are disjoint. It follows that there exists a unique minimal model in any birational class of elliptic surfaces [3].
Now let \( \pi: S \to B \) be a minimal elliptic surface. If \( K(S) \) and \( K(B) \) denote the function fields of \( S \) and of \( B \) respectively, then \( K(S) \), as an algebraic function field in one variable over \( K(B) \), is of genus 1, and contains a rational point corresponding to the section. It follows that \( S \) is birationally equivalent to a (possibly singular) elliptic surface \( \pi': S' \to B \) given by a Weierstrass equation. That is, \( S' \subset B \times \mathbb{P}^2 \) is defined by an equation of the form:

\[
y^2z = x^3 - g_2xz^2 - g_3z^3
\]

where \( g_2, g_3 \) in \( K(B) \) are uniquely determined up to the transformation:

\[
(g_2, g_3) \to (h^4g_2, h^6g_3), \quad h \in K(B).
\]

In this paper, we wish to describe a (possibly singular) elliptic surface: \( \pi^*: S^* \to B \) closely related to the Weierstrass surface \( S' \), such that \( S^* \) satisfies:

(i) The only singularities of \( S^* \) are rational double points;

(ii) \( S \) is the minimal resolution of \( S^* \).

Abstractly, \( S^* \) is obtained from \( S \) by contracting those curves in the singular fibres of \( S \) which do not meet the section. We wish to describe a Weierstrass-type equation for \( S^* \).

Let \( A \) be the unique divisor on \( B \) such that if:

\[
div (g_2) + 4A = \sum_{P \in B} n_P P,
\]

\[
div (g_3) + 6A = \sum_{P \in B} m_P P
\]

then

(i) \( n_p \geq 0, m_p \geq 0 \) for all \( P \in B \),

(ii) \( \min(3n_p, 2m_p) < 12 \) for all \( P \in B \),

i.e., either \( n_p < 4 \) or \( m_p < 6 \). If \( (g_2, g_3) \) is replaced by \( (h^4g_2, h^6g_3) \), then \( A \) is replaced by \( A - \text{div}(h) \). Thus the divisor class of \( A \) is uniquely determined by the elliptic surface \( S \). Let \( L = [A] \) be the line bundle of \( A \), and let \( (l_{ij}) \) be a system of transition functions for \( L \) with respect to some covering \( \{U_i\} \) of \( B \).

The meromorphic functions \( g_2, g_3 \) determine sections:

\[
g_2^* \in H^0(B, \mathcal{O}_B(4L)), \quad g_3^* \in H^0(B, \mathcal{O}_B(6L)).
\]

\( (g_2^*, g_3^*) \) are determined by \( S \) up to the transformation: \( (g_2^*, g_3^*) \to (\lambda^4g_2^*, \lambda^6g_3^*), \lambda \in \mathbb{C}^* \). \( g_2^* \) and \( g_3^* \) may be described by systems of holomorphic functions \( (g_{2i}^*) \) and \( (g_{3i}^*) \) defined on \( U_i \) satisfying

\[
g_{2i}^* = l_{ij}^4 g_{2j}^*, \quad g_{3i}^* = l_{ij}^6 g_{3j}^*,
\]

on \( U_i \cap U_j \).

Let \( W = 2L \oplus 3L \oplus 1 \). Let \( S^* \subset P(W) \) be such that \( S^* \) is defined over each piece \( U_i \) by the equation:

\[
y_i^2z_i = x_i^3 - g_{2i}^*x_i^2z_i - g_{3i}^*z_i^3
\]
where \((x_i:y_j:z_l)\) are fibre homogeneous coordinates for \(P(W)\) over \(U_l\) satisfying
\[x_i = l_{ij}^2 x_j, \quad y_i = l_{ij}^3 y_j, \quad z_i = z_j,\]
over \(U_i \cap U_j\).

If \(B' = B - \text{supp } (A)\), then it is clear that \(S^*|_{B'} \cong S'|_{B'}\) and therefore \(S^*\) is birationally equivalent to \(S\). We will prove that the only singularities of \(S^*\) are rational double points. Notice that we have:
\[
\min(3 \text{ ord}_P (g_2^*), 2 \text{ ord}_P (g_3^*)) < 12
\]
at every point \(P \in B\).

**Lemma 1.** Consider the isolated singularity

\[y^2 = x^3 - \alpha t^n x - \beta t^m, \quad n > 0, m > 1, \]
in \(\mathbb{C}^2(x,y,t)\) where \(\alpha = \alpha(t), \beta = \beta(t), \alpha(0) \neq 0, \beta(0) \neq 0,\) and where we assume that \(\Delta = 4\alpha^3 t^{3n} - 27\beta^2 t^{2m}\) is not identically zero. Then the above singularity at the origin is a rational double point if and only if \(\min(3n, 2m) < 12\).

**Proof.** We can resolve the singularity explicitly as in [2, p. 81]. We give a table here which describes the graph of the minimal resolution in case \(n < 4\) or \(m < 6\). We follow the notation of [1], namely:

\[
\begin{align*}
A_n & : \xrightarrow{-} \cdots \xrightarrow{-} \quad (n \text{ vertices}) \\
D_n & : \xrightarrow{-} \cdots \xrightarrow{-} \quad (n \text{ vertices}) \\
E_n & : \xrightarrow{-} \cdots \xrightarrow{-} \quad (n \text{ vertices}, n = 6, 7, \text{ or } 8)
\end{align*}
\]

here each vertex represents a nonsingular rational curve with self-intersection \(-2\).

Resolution of \(y^2 = x^3 - \alpha t^n x - \beta t^m\)

| \(n\) | \(m\) |  \\
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(\geq 3)</td>
<td>4</td>
<td>(E_6)</td>
</tr>
<tr>
<td>2</td>
<td>(\geq 3)</td>
<td>(D_4)</td>
</tr>
<tr>
<td>3</td>
<td>(\geq 2)</td>
<td>(D_4)</td>
</tr>
<tr>
<td>3</td>
<td>(\geq 5)</td>
<td>(E_7)</td>
</tr>
<tr>
<td>(\geq 4)</td>
<td>5</td>
<td>(E_8)</td>
</tr>
<tr>
<td>(\geq 2)</td>
<td>2</td>
<td>(A_2)</td>
</tr>
<tr>
<td>1</td>
<td>(\geq 2)</td>
<td>(A_1)</td>
</tr>
</tbody>
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As an illustration, we will carry out the resolution of \( y^2 = x^3 - \alpha t^n x - \beta t^5 \), \( n \geq 4 \). After blowing up the origin in the \((x, t)\) plane we get the diagram:

\[
\begin{align*}
(1) \quad \Gamma_1 & \quad r_1^1 = -1 \\
& \quad m_1 = 3 \\
\end{align*}
\]

where \( \Gamma_1 \) is the exceptional curve,

\[
\begin{align*}
is the proper transform of \( x^3 - \alpha t^n x - \beta t^5 = 0 \) which has a simple cusp, and \( m_1 \) is the multiplicity of \( \Gamma_1 \) as a component of the divisor of \( x^3 - \alpha t^n x - \beta t^5 \). After blowing up the cusp, we get the diagram:

\[
\begin{align*}
(2) \quad \Gamma_1 & \quad r_1^2 = -2 \\
& \quad m_2 = 3 \\
& \quad r_2 = -1 \\
& \quad m_3 = 5 \\
\end{align*}
\]

Here \( \Gamma_1 \) represents the proper transform of the \( \Gamma_1 \) of diagram (1). Now, blow up the triple intersection:

\[
\begin{align*}
(3) \quad \Gamma_1 & \quad r_1^3 = -3 \\
& \quad m_1 = 3 \\
& \quad r_2^3 = -2 \\
& \quad m_3 = 5 \\
& \quad r_3^3 = -1 \\
& \quad m_4 = 9 \\
\end{align*}
\]

Again blow up the triple intersection:

\[
\begin{align*}
(4) \quad \Gamma_1 & \quad r_1^4 = -3 \\
& \quad m_1 = 3 \\
& \quad r_2^4 = -3 \\
& \quad m_2 = 5 \\
& \quad r_3^4 = -2 \\
& \quad m_3 = 9 \\
& \quad r_4^4 = -1 \\
& \quad m_4 = 15 \\
\end{align*}
\]

Now blow up each double point. This is to insure that the curves \( \Gamma_i \) with \( m_i \) odd are disjoint from one another and from \( \Gamma \).
The resolution is then the double covering of diagram (5) ramified over the curves $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$, and $\mathcal{R}$. This is clearly the graph $E_8$.

By tautness of rational double points [1], it follows that $y^2 = x^3 - \alpha t^n x - \beta t^m$ is a rational double point if $\min(3n, 2m) < 12$.

Assume now that $n = 4k + n_1$, $m = 6k + m_1$ where $k > 0$, $\min(3n_1, 2m_1) < 12$. Then the graph of the resolution of $y^2 = x^3 - t^n \alpha x - t^m$ is one of the following:

where $\Sigma_{n_1, m_1}$ is taken from the table above;
where $E$ is a rational curve with one cusp, and $E^2 = -1$.

In any case, the singularity is not a rational double point. This completes the proof of Lemma 1.

We now consider the singularities of $S^*$ corresponding to poles of $j$. Thus we consider the surface defined in $C^2 \times \{|r| < \varepsilon\}$ by $y^2 = x^3 - \alpha(t)x - \beta(t)$, with discriminant $\Delta = 4\alpha^3 - 27\beta^2$ and invariant $j = 4\alpha^3/\Delta$. We now assume that $j$ has a pole of order $r > 0$ at $t = 0$. If we set $\alpha(t) = t^n\alpha(t)$, $\beta(t) = t^m\beta(t)$ with $\alpha(0) \neq 0$, and $\beta(0) \neq 0$, then we must have $(n,m) = (2k, 3k)$ for some $k \geq 0$. Notice that after an analytic change of coordinates, the above equation may be transformed to

$$y^2 = (x - t^k)(x^2 - t^{r+2k}y), \quad \gamma = \gamma(t), \gamma(0) \neq 0.$$ 

Here again, this singularity may be resolved explicitly by the methods of [2]. We get the following result:

**Lemma 2.** The singularity $y^2 = (x - t^k)(x^2 - t^{r+2k}y)$ is not a rational double point if $k > 1$. It is a rational double point of type $D_{r+4}$ if $k = 1$, and of type $A_{r-1}$ if $k = 0, r > 1$.

To be more specific, if $k > 1$ is even, then the graph of the resolution is:

```
-2 -2 ...
      1
-2 -2 -3 ...
      2
-2 -2 ...
```

If $k > 1$ is odd, the graph is:

```
-2 -2 ...
      1
-2 -2 -3 ...
      2
-2 -2 ...
```

If $k = 0$, the corresponding fibre is of type $I_r$, while if $k = 1$, the fibre is of type $I_r^*$. It is clear from Lemmas 1 and 2 and our construction of the elliptic surface $S^*$ that the only singularities of $S^*$ are rational double points.
To each point $a \in B$, let $C_a^*$ be the fibre of $S^*$ over $a$. If $S$ is the minimal resolution of $S^*$, then the fibre of $S$ over $a$ is of the form:

$$C_a = C_{a0} + \sum_{j \geq 1} n_j C_{aj}$$

where $C_{a0}$ is the proper transform of $C_a^*$ and where $\bigcup_{j \geq 1} C_{aj}$ (if nonempty) form the minimal resolution of a rational double point. Thus we have $C_{aj}^2 = -2$, $K \cdot C_{aj} = 0$ ($j \geq 1$). Since $C_a$ is a fibre of an elliptic surface, $K \cdot C_a = 0$. It follows that $K \cdot C_{a0} = 0$. Thus $C_{a0}$ is not an exceptional curve of the first kind. We may conclude that $S$ is a minimal elliptic surface. We sum up our results.

**Theorem 1.** Let $\pi: S \to B$ be a minimal elliptic surface which admits a section. Then there exists a line bundle $L$ on $B$ and sections $g_2 \in H^0(B, \mathcal{O}_B(4L))$, $g_3 \in H^0(B, \mathcal{O}_B(6L))$ such that $S$ is the minimal resolution of the surface $S^* \subset \mathbb{P}(2L \oplus 3L \oplus 1)$ defined by the “Weierstrass equation”

$$y^2z = x^3 - g_2xz - g_3z^3.$$

The only singularities of $S^*$ are rational double points. $L$ is uniquely determined by the projection $\pi$. In fact $L^{-1}$ is the normal bundle of any section $s(B)$ in $L$, and we have $\deg(L) = p_g - q + 1$. The pair $(g_2, g_3)$ are uniquely determined up to the transformation $(g_2, g_3) \mapsto (\lambda g_2, \lambda^6 g_3)$, $\lambda \in \mathbb{C}^*$. The pair $(g_2, g_3)$ satisfy

(i) $\Delta = 4g_2^3 - 27g_3^2 \neq 0$.

(ii) For every $t \in B$, $\min(3 \ord_t(g_2), 2 \ord_t(g_3)) < 12$.

We remark that if $S$ is not a $K3$ surface, then the projection $\pi: S \to B$ is uniquely determined up to an automorphism of $B$. In fact, if $q > 0$, the Albanese mapping of $S$ factors through the projection $\pi$ and the Jacobian mapping of $B$. If $q = 0$, then the projection $\pi$ is determined by the linear system $|mK|$, where $m \gg 0$ if $S$ is not a rational surface, and $m < 0$ if $S$ is a rational elliptic surface.

Let $Y(n, B) = \{L, g_2, g_3\}$ where $L$ is a line bundle over $B$ with $\deg(L) = n$, $g_2 \in H^0(B, \mathcal{O}_B(4L))$, $g_3 \in H^0(B, \mathcal{O}_B(6L))$ and satisfying:

(i) $\Delta = 4g_2^3 - 27g_3^2 \neq 0$.

(ii) For every $t \in B$, $\min(3 \ord_t(g_2), 2 \ord_t(g_3)) < 12$.

Let $X(n, B) = Y(n, B)/\mathbb{C}^*$ where $\mathbb{C}^*$ acts on $Y(n, B)$ by $(L, g_2, g_3) \mapsto (L, \lambda^4 g_2, \lambda^6 g_3)$. There is a natural action of $\text{Aut}(B)$ on $X(n, B)$.

**Theorem 2.** There is a 1-1 correspondence between elliptic surfaces $\pi: S \to B$ which admit a section satisfying $p_g - q + 1 = n$, and the set $X(n, B)$. The set of elliptic surfaces $S$ over $B$ (without specifying a projection) which admit a section satisfying $p_g - q + 1 = n$ is in 1-1 correspondence with $X(n, B)/\text{Aut}(B)$, provid-
ed that we exclude the case of elliptic $K3$ surfaces, i.e., $n = 2, B = P^1$.

Example. Elliptic surfaces over $P^1$. There is a unique line bundle $L_n$ of degree $n$ on $P^1$. We may identify $H^0(P^1, 0, (m))$ with $\mathcal{O}_m = \text{the set of polynomials } P(i) \text{ of degree } \leq m$. Then $Y(n, P^1) = \{(P(i), Q(i)) \in \mathcal{O}_{4n} \times \mathcal{O}_{6n}\}$ satisfying:

(i) $4P(i)^3 - 27Q(i)^2 \neq 0$.

(ii) $\text{min}(3 \text{ ord}_a P, 2 \text{ ord}_a Q) < 12$ for every $a \in \mathbb{C}$.

(iii) $\text{min}(12n - 3 \text{ deg } P, 12n - 2 \text{ deg } Q) < 12$.

$\mathbb{C}^*$ acts on $Y(n, P^1)$ by $(P, Q) \mapsto (\lambda^4 P, \lambda^6 Q)$. $SL(2, \mathbb{C})/\pm 1 = \text{Aut } (P^1)$ acts on $Y(n, P^1)$ by

$$(P, Q) \mapsto \left((ct + d)^4 P\left(\frac{at + b}{ct + d}\right), (ct + d)^6 Q\left(\frac{at + b}{ct + d}\right)\right).$$

References


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