THE SPECTRAL GEOMETRY OF SYMMETRIC SPACES\(^{(1)}\)

BY

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Abstract. Let \( M \) be a compact Riemannian manifold without boundary. Let \( D \) be a differential operator on \( M \). Let \( \text{spec} (D, M) \) denote the eigenvalues of \( D \) repeated according to multiplicity. Several authors have studied the extent to which the geometry of \( M \) is reflected by \( \text{spec} (D, M) \) for certain natural operators \( D \). We consider operators \( D \) which are convex combinations of the ordinary Laplacian and the Bochner or reduced Laplacian acting on the space of smooth functions and the space of smooth one forms. We prove that it is possible to determine if \( M \) is a local symmetric space from its spectrum. If the Ricci tensor is parallel transported, the eigenvalues of the Ricci tensor are spectral invariants of \( M \).

Introduction. Let \( M \) be a compact connected Riemannian manifold without boundary and let \( D_0 = d^* d \) be the Laplacian acting on the space of smooth functions. Let \( \text{spec} (D_0, M) \) denote the set of eigenvalues \( 0 \leq \lambda_1 \leq \lambda_2 \cdots \); each eigenvalue is repeated according to the multiplicity. The basic question we will be considering is to what extent the geometry of the manifold \( M \) is reflected by \( \text{spec} (D_0, M) \) and by the spectra of certain other natural differential operators acting on \( M \). Sakai [5] has proved

**Theorem (Sakai).** Let \( M \) and \( M' \) be Einstein manifolds of dimension 6. Suppose that \( M \) and \( M' \) have the same Euler characteristic and that \( \text{spec} (D_0, M) = \text{spec} (D_0, M') \). Then if \( M \) is a local symmetric space, so is \( M' \).

If we enlarge the class of differential operators which we are willing to consider, other geometrical properties of the manifold \( M \) are reflected by the spectrum. Let \( D_p = d^* d + dd^* \) be the Laplacian acting on the space of smooth \( p \)-forms; let \( \text{spec} (D_p, M) \) denote the eigenvalues of \( D_p \) repeated according to multiplicity. Patodi [4] has proved

**Theorem (Patodi).** Suppose that \( \text{spec} (D_p, M) = \text{spec} (D_p, M') \) for \( p = 0, 1, 2 \). Then:
(a) if \( M \) has constant scalar curvature \( c \), so does \( M' \);
(b) if \( M \) is Einstein, so is \( M' \);

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(c) if \( M \) has constant sectional curvature \( c \), so does \( M' \).

Donnelly [1] has applied this result of Patodi's to generalize Sakai's theorem as follows:

**Theorem (Donnelly).** Let \( \text{spec} (D_p, M) = \text{spec} (D_p, M') \) for \( p = 0, 1, 2 \). If \( M \) is an Einstein local symmetric space, so is \( M' \).

In this paper we will remove the hypothesis that \( M \) is Einstein by considering a still more general class of natural differential operators on \( M \). If \( M \) is a local symmetric space, the Ricci tensor is parallel transported. Consequently, the eigenvalues of the Ricci tensor do not depend upon the point of evaluation. We will show that if the Ricci tensor is parallel transported, then the eigenvalues of the Ricci tensor are also spectral invariants of the manifold.

1. The theorems discussed in the Introduction are all proved by computing certain asymptotic invariants of the spectrum. We will use the following notation: \( M \) is a compact connected Riemannian manifold of dimension \( m \); let \( G \) denote the Riemannian metric on \( M \) and let \( ds^2 = g_{ij} dx_i \circ dx_j \) in some system of coordinates. Indices \( i, j, \ldots \) run from 1 through \( m \) and index a frame for the tangent bundle of \( M \); we sum over repeated indices. Let \( g^{ij} \) denote the metric on \( T^* M \) and let \( |d \text{vol}| \) be the Riemannian measure.

Let \( V \) be a smooth vector bundle over \( M \) and let \( D: C^\infty (V) \to C^\infty (V) \) be a second order differential operator with leading symbol given by the metric tensor. If we choose a system of local coordinates for \( M \) and a local frame for \( V \), we can express \( D \) in the form

\[
D = -(g^{ij} \partial^2/\partial x_i \partial x_j + M_k \partial/\partial x_k + N).
\]

The \( M_k \) and \( N \) are square matrices which are not invariantly defined but depend upon the choice of frame and local coordinates.

Let \( V_x \) denote the fibre of \( V \) over \( x \). For \( t > 0 \), \( \exp(-tD) \) is a well-defined infinitely smoothing operator which is of trace class in \( L^2(V) \). Let \( K(t, D, x, y) \): \( V_x \to V_y \) be the kernel function of \( \exp(-tD) \). Then

\[
\exp(-tD)u(x) = \int_M K(t, D, x, y)(y(y)) |d \text{vol} (y)|.
\]

\( K(t, D, x, y) \) is smooth in \((t, x, y)\). Define

\[
f(t, D, x) = \text{Trace}_{V_x} (K(t, D, x, x)),
\]

\[
f(t, D) = \text{Trace}_{L^2} (\exp(-tD)) = \int_M f(t, D, x) |d \text{vol} (x)|.
\]

It is well known [6] that as \( t \to 0^+ \) that \( f(t, D, x) \) has an asymptotic expansion of the form

\[
f(t, D, x) \sim (4\pi t)^{-m/2} \sum_{n=0}^\infty A_n(D, x)t^n.
\]
The coefficients $A_n(x,D)$ are smooth functions of $x$ which can be computed functorially in terms of the derivatives of the total symbol of the differential operator $D$; $A_n(D,x)$ is a local invariant of $D$. Let

$$A_n(D) = \int_M A_n(D,x) \text{dvol}(x);$$

then

$$f(t,D) \sim (4\pi t)^{-m/2} \sum_{n=0}^{\infty} A_n(D)t^n.$$

If $V$ has a smooth inner product $(\cdot,\cdot)$ on each fibre and if $D$ is selfadjoint with respect to the fibre metric, let $\{\lambda_\nu, \theta_\nu\}_{\nu=1,\infty}$ be a complete spectral decomposition of $D$ into an orthonormal basis of eigensections $\theta_\nu$ and corresponding eigenvalues $\lambda_\nu$. For such a $D$, we can express

$$f(t,D,x) = \sum_\nu \exp(-i\lambda_\nu)(\theta_\nu,\theta_\nu)(x) \sim (4\pi t)^{-m/2} \sum_n A_n(D,x)t^n,$$

$$f(t,D) = \sum_\nu \exp(-i\lambda_\nu) \sim (4\pi t)^{-m/2} \sum_n A_n(D)t^n.$$

The integrated invariants $A_n(D)$ depend only on the asymptotic behavior of the series $\sum_\nu \exp(-i\lambda_\nu)$ and are therefore spectral invariants. Although we will only be computing the invariants $A_n(D,x)$ for selfadjoint operators, it is technically convenient to have them defined for operators which are not necessarily selfadjoint.

In [3] we defined the invariants $B\nu(D,x)$ and derived explicit formulas for $B_0, B_2, B_4, B_6$. In the notation of this paper;

$$B_{2n}(D,x) = (4\pi)^{-m/2} A_{2n}(D,x) \quad \text{and} \quad B_{2n+1}(D,x) = 0.$$

We use the notation $A_n$ in this paper rather than $B_{2n}$ since it is more classical. We describe these formulas using the following notation: let $\nabla_\xi$ be the Levi-Civita connection on $TM$; extend $\nabla_\xi$ to tensors of all types. Let $e = (e_1, \ldots, e_m)$ be an orthonormal frame for $TM$ defined in a neighborhood of some $x_0 \in M$. We identify $TM$ with the dual $T^*M$ using the metric $G$. Let

$$R_{ijkl} = G((\nabla_i \nabla_j - \nabla_j \nabla_i - \nabla_{[e_i,e_j]}e_k, e_l)$$

be the components of the curvature tensor of the Levi-Civita connection. Let $\nabla$ be any connection on the vector bundle $V$. Define the reduced or Bochner Laplacian $D^\nabla$ by the diagram:

$$D^\nabla: C^\infty(V) \xrightarrow{\nabla} C^\infty(T^*M \otimes V) \xrightarrow{\nabla \otimes 1 + 1 \otimes \nabla} C^\infty(T^*M \otimes T^*M \otimes V) \xrightarrow{-G \otimes 1} C^\infty(V).$$

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In normal coordinates, $D^V = -g^{ij} \nabla_i \nabla_j$. We proved earlier [2]

**Theorem 1.1.** Given $D: C^\infty(V) \to C^\infty(V)$ which is a second order differential operator with leading symbol given by the metric tensor, there is a unique connection $\nabla$ on $V$ such that $E = D^V - D$ is a 0th order operator.

Let $D = D^V - E$ where $E$ is an invariantly defined endomorphism of $V$. If $D_0 = d^*d$ is the Laplacian acting on the space of smooth functions, the connection $\nabla$ induced by $D_0$ is the flat connection and the endomorphism $E$ is zero. Let

$$W_{ij} = \nabla_i \nabla_j - \nabla_j \nabla_i - \nabla_{[i,e_j]}: V \to V$$

be the curvature of the connection on $V$. For each $(i,j)$, $W_{ij}$ is a square matrix. If $\theta$ is a tensor field, let $\theta_{ij}$ denote covariant differentiation in the direction $e_i$. We denote multiple covariant differentiation by $\theta_{i_1 \ldots i_r}$.

Let $\rho_{ij} = R_{ijk}$ and $\tau = R_{ijij} = \rho_{ii}$. We are using a different sign convention from that used by Sakai in the definition of the curvature tensor; this changes some of the signs in our formulas. Let

$$\rho^n = \rho_{i_1 i_2} \rho_{i_3 i_4} \cdots \rho_{i_{n-1} i_n} \rho_{i_n i_1}$$

be the trace of the $n$th power of the Ricci tensor. Let

$$(\nabla \rho)^2 = R_{ijkl} R_{ijkl}, \quad (\nabla R)^2 = \tau_{ij} \tau_{ij}, \quad R^2 = R_{ijkl} R_{ijkl}.$$ 

We sum over repeated indices. In [3] we derived formulas for $A_n(D, x)$ in terms of the $R_{ijkl}, \ldots, W_{ij}, \ldots, E, \ldots$ tensors for $n = 0, 1, 2, 3$. Since only the integral of these local formulas is a spectral invariant, we simplify the formulas of [3] by integrating by parts.

**Theorem 1.2.** Let $D = D^V - E$; then:

(a) $A_0(D) = \dim(V) \cdot \text{vol}(M)$;

(b) $A_1(D_0) = \int_M -\tau/6 |d \text{vol}|$,

(c) $A_1(D) = \dim(V) \cdot A_1(D_0) + \int_M \text{Tr}(E) |d \text{vol}|$;

(d) $A_2(D_0) = \frac{1}{360} \int_M (5\tau^2 - 2\rho^2 + 2R^2) |d \text{vol}|$;

(e) $A_2(D) = \dim(V) \cdot A_2(D_0) + \frac{1}{360} \int_M \text{Tr}(30W_{ij} W_{ij} - 60\tau E + 180E^2) |d \text{vol}|$;
\[ A_3 D_0 = \frac{1}{9 \cdot 71} \int_M \left\{ -142(\nabla \tau)^2 - 26(\nabla \rho)^2 - 7(\nabla R)^2 - 35 \tau^3 \\
+ 42\rho^2 - 42\tau R^2 + 36\rho^3 - 20\rho \rho_{kl} R_{ijkl} \\
+ 8\rho R_{ikln} R_{jkln} - 24 R_{ijkn} R_{jkw} R_{knu} \right\} |d \text{vol}|. \]

(d) \[ A_3(D) = \dim(V) \cdot A_3(D_0) \]

\[ + \frac{1}{360} \int_M \text{Tr} \left\{ -4W_{ij;k} W_{ji;k} + 2W_{ij;k} W_{jk;i} - 12W_{ij} W_{jk} W_{ki} \\
- 6R_{ijkn} W_{ij} W_{kn} + 4\rho_{jk} W_{jn} W_{kn} - 5\tau W_{kn} W_{kn} \\
+ (5\tau^2 - 2\rho^2 + 2R^2 + 30\rho W_{ij}) E \\
- 4\rho_{jk} E_{jk} + 10\tau_{jk} E_{jk} - 30\tau E^2 \\
- 30E_{jk} E_{jk} + 60E^3 \right\} |d \text{vol}|. \]

2. Let \( D_p = (d^* d + dd^*) \) be the Laplacian acting on the space of smooth \( p \)-forms. We must determine the connection \( \nabla^p \) and the endomorphism \( E^p \) induced by \( D_p \) to apply the formulas of Theorem 1.2. Let \( D_p^V = -g^{ij} \nabla_i \nabla_j \) be the Bochner Laplacian defined by the Levi-Civita connection. The operators \( D_p \) and \( D_p^V \) both have the same second order symbol; \( E_p = D_p^V - D_p \) is at most a first order operator. The first order symbol of \( E^p \) is invariantly defined; it can be computed functorially as a linear combination of the first derivatives of the metric. In normal coordinates, the first derivatives of the metric vanish at the centre. Therefore \( E_p = D_p^V - D_p \) is a 0th order operator. By Theorem 1.1, \( \nabla = \nabla^p \) and \( E^p = E^p \).

We compute \( E^p \) as follows: let

\[ D = \bigoplus_p D_p, \quad D^V = \bigoplus_p D_p^V, \quad E = \bigoplus_p E^p. \]

Let \( \{e_1, \ldots, e_m\} \) be an orthonormal frame for \( TM \) defined near \( x_0 \in M \). We have identified \( TM \) with \( T^* M \) using the metric. If \( I = (i_1, \ldots, i_p) \) for \( 1 \leq i_1 < \cdots < i_p \leq m \), let

\[ e_I = e_{i_1} \wedge \cdots \wedge e_{i_p}, \quad |I| = p, \]

\[ \nabla_i(e_I) = \Gamma_{ij}^k e_J \quad (\Gamma_{ij}^k = 0 \text{ if } |I| \neq |J|), \]

\[ G((\nabla_{e_I} \nabla_j - \nabla_j \nabla_i - [e_i, e_j]) e_I, e_J) = G(W_{ij} e_I, e_J) = R_{ij}^k e_k. \]

Let \( \text{Clif}(T^* M) \) denote the Clifford bundle over \( M \). It is defined from the tensor algebra on \( T^* M \) by the relations:
There is a functorial vector bundle isomorphism between $\text{Cliff}(T^*M)$ and the exterior algebra $\Lambda(T^*M)$; this map is not an algebra morphism. This identification defines $\ast$ multiplication on $\Lambda(T^*M)$. Let $\alpha \in \Lambda(T^*M)$ and $\beta \in T^*M$. Let $e(\beta)$ denote exterior and $i(\beta)$ denote interior multiplication. Then $\beta \ast \alpha = (e(\beta) - i(\beta))(\alpha)$. We use Clifford multiplication to construct a first order operator

$$d_0 : C^\infty(\Lambda(T^*M)) \to C^\infty(T^*M \otimes \Lambda(T^*M)) \to C^\infty(\Lambda(T^*M)).$$

The leading symbols of $d_0$ and $(d + d^*)$ agree; the difference is a functorially defined operator of order 0 which can be expressed functorially as a linear combination of the first derivatives of the metric. This difference must vanish so $d_0 = d + d^*$.

We will denote differentiation in the direction $e_i$ by "$/i$".

$$(d + d^*)(f_\ell e_\ell) = f_\ell e_\ell \ast e_\ell + f_\ell \Gamma_{\ell \ell \ell} e_\ell \ast e_\ell + \cdots,$$

$$(d + d^*)^2(f_\ell e_\ell) = f_\ell \Gamma_{\ell \ell \ell} e_\ell \ast e_\ell \ast e_\ell + \cdots,$$

$$D^\ell(f_\ell e_\ell) = f_\ell \Gamma_{\ell \ell \ell} e_\ell \ast e_\ell \ast e_\ell + \cdots.$$
Since $E^1: \Lambda^1(T^*M) \to \Lambda^1(T^*M)$, one of the indices in the pair $(i,j)$ must be $l$. This expression is symmetric in $(i,j)$ so we may assume $i = l$.

$$E^1(e_k) = -R_{ijkl}e_j = \rho_{jk}e_j.$$  

Similarly,

$$E^2(e_k \wedge e_l) = \frac{1}{2}e_j \wedge e_i \cdot W_{ij}(e_k \wedge e_l) = \frac{1}{2}e_j \wedge e_i \cdot (R_{ijkl}e_n \wedge e_l + R_{ijln}e_k \wedge e_n).$$

Since $E^2$ preserves $\Lambda^2(T^*M)$, one of the indices in the pair $(i,j)$ must be either $n$ or $l$ in the first expression, and one of the indices in the pair $(i,j)$ must be either $k$ or $n$ in the second expression.

$$E^2(e_k \wedge e_l) = -e_j \wedge e_i \cdot (R_{ijkl}e_l - R_{ijkn}e_n + R_{kijn}e_n - R_{njln}e_k)$$

$$= \rho_{kn}e_n \wedge e_l + \rho_{ln}e_k \wedge e_n - 2R_{klnp}e_n \wedge e_p$$

$$= \rho_{kn}e_n \wedge e_l + \rho_{ln}e_k \wedge e_n - R_{klnp}e_n \wedge e_p.$$  

If $P(G)$ is an invariant in the derivatives of the metric, let

$$P(M) = \int_M P(G)|d\text{vol}|.$$  

For example, $I(M) = \text{vol}(M)$. If $\epsilon$ is real, let

$$D_p^\epsilon = \epsilon D_p + (1 - \epsilon)D_p^\nabla.$$  

The connection induced by $D_p^\epsilon$ is the Levi-Civita connection and the corresponding endomorphism is $\epsilon D_p^\nabla$. We combine the expressions for $E^1$ and $E^2$ with the formulas of Theorem 1.2 to compute

**Theorem 2.2.** Let $\overline{A}_n(D) = A_n(D) - \text{dim}(V)A_n(D_0)$; then

(a) $\overline{A}_1(D^\epsilon) = \epsilon r(M)$,

(b) $\overline{A}_2(D^\epsilon) = \frac{1}{360}(-30R^2 - 60\epsilon r^2 + 180\epsilon^2 r^2)(M),$

$$\overline{A}_2(D^\epsilon) = \frac{1}{360}(-30(m - 2)R^2 - 60(m - 2)\epsilon r^2$$

$$+ 180\epsilon^2(r^2 + (m - 6)\rho^2 + R^2))(M);$$
\[ \overline{A_3}(D^s) = \frac{1}{360}( -2(\nabla_\tau)^2 + 8(\nabla_\rho)^2 + (\nabla R)^2 + 5\tau R^2 - 8\rho^3 ) \]
\[
+ 8\rho_j\rho_{kl} R_{ijkl} + 2\rho_j R_{ikln} R_{jkln} + 3 R_{ijkl} R_{ijuw} R_{knw} \]
\[
+ \epsilon(12(\nabla_\tau)^2 + 5\tau^3 - 2\tau R^2 + 2\tau R^2 - 30\rho_j R_{ikln} R_{jkln}) \]
\[
+ \epsilon^2(-30(\nabla_\rho)^2 - 30\tau R^2 + 60\epsilon^3 \rho^3)(M); \]
\[ \overline{A_3}(D^e) = \frac{1}{360}((m - 2)(-2(\nabla_\tau)^2 + 8(\nabla_\rho)^2 + (\nabla R)^2 + 5\tau R^2 - 8\rho^3 ) \]
\[
+ (m - 2)(8\rho_j\rho_{kl} R_{ijkl} + 2\rho_j R_{ikln} R_{jkln} + 3 R_{ijkl} R_{ijuw} R_{knw}) \]
\[
+ \epsilon((m - 2)(12(\nabla_\tau)^2 + 5\tau^3 - 2\tau R^2 + (2m - 34)\tau R^2) \]
\[
+ \epsilon(30(6 - m)\rho_j R_{ikln} R_{jkln} - 30 R_{ijkl} R_{ijuw} R_{knw}) \]
\[
+ \epsilon^2(-30(\nabla_\tau)^2 - 30(m - 6)(\nabla_\rho)^2 - 30(\nabla R)^2) \]
\[
+ \epsilon^2(-30\tau^3 - 30(m - 6)\tau R^2 - 30\tau R^2) \]
\[
+ \epsilon^3(180\tau R^2 + 60(m - 10)\rho^3 - 360\rho_j\rho_{kn} R_{ikjn}) \]
\[
+ \epsilon^3(360\rho_j R_{ikln} R_{jkln} - 60 R_{ijkl} R_{ijuw} R_{knw})(M). \]

We will need one more result concerning the invariants \( A_n \) for general \( n \). Let \( D: C^\infty(\mathcal{V}) \to C^\infty(\mathcal{V}) \) be a second order differential operator with leading symbol given by the metric tensor. Let \( D^e = \epsilon D + (1 - \epsilon) D^\nabla \).

**Theorem 2.3.** We may decompose \( A_n(D^e, x) = \sum_{k=0}^n \epsilon^k A_{n,k}(D, x) \). Furthermore, \( A_{n,n}(D, x) = (1/n!) \text{Tr}(E^n) \).

The proof is by inspection of the formulas of Theorem 1.2 for \( n = 0, 1, 2, 3 \). For the general case, we use some results of [2]. Consider the collection of endomorphism valued tensors:

\[
R_{i_1 i_2 i_3 i_4 i_5} \ldots I, \quad W_{i_1 i_2 i_3} \ldots, \quad E_{i_1} \ldots
\]

Let \( K(t, D, x, y) \) be the kernel function for \( \exp(-tD) \); there is an asymptotic series for \( K(t, D, x, x) \) as \( t \to 0^+ \) of the form

\[
K(t, D, x, x) \sim (4\pi t)^{-m/2} \sum_{n=0}^\infty E_n(D, x) t^n.
\]

The endomorphisms \( E_n(D, x) \) are local invariants of the operator \( D \); \( \text{Tr}(E_n(D, x)) = A_n(D, x) \).

By applying H. Weyl's theorem [7] we showed [2] that \( E_n \) could be expressed in terms of contractions of various noncommutative polynomial expressions in the \( R_{ijkl} \ldots, W_{ij} \ldots, \) and \( E_{i_1} \ldots \) tensors. For example, \( E_2 \) has the form

\[
E_2(D, x) = A_2(D_0, x) I + A_1(D_0, x) E + E^2/2 + (W_{ij} W_{ij} + 2 E_{i_1 i_2})/12.
\]
Let
\[ \text{ord} \left( R_{i_1 i_2 i_3 \cdots j_s} \right) = \text{ord} \left( W_{i_1 i_2 j_1 \cdots j_s} \right) = \text{ord} \left( E_{j_1 \cdots j_s} \right) = 2 + s. \]

\( E_n \) is homogeneous of order \( 2n \) in these tensors. By replacing \( E \) by \( \epsilon E \), we can express
\[ E_n(D^\epsilon, x) = \sum_{k=0}^{n} \epsilon^k E_{n,k}(D, x). \]
The coefficient of \( \epsilon^n \) is a multiple of \( (E)^n \) by the homogeneity property. Therefore,
\[ A_n(D^\epsilon, x) = \sum_{k=0}^{n} \epsilon^k A_{n,k}(D, x) \quad \text{for} \quad A_{n,k} = \text{Tr} \left( E_{n,k} \right). \]
\[ A_{n,n} = c_n \text{ Tr} \left( (E)^n \right). \]
We complete the proof of Theorem 2.3 by evaluating the constant \( c_n \).

Consider the operator \( D - \epsilon I \). The corresponding endomorphism is \( E + \epsilon I \).
By the functional calculus,
\[ \exp(-t(D - \epsilon I)) = \exp(\epsilon t) \exp(-tD), \]
\[ K(t, D - \epsilon I, x, y) = \exp(\epsilon t) K(t, D, x, y). \]
Therefore,
\[ \sum_{n=0}^{\infty} E_n(D - \epsilon I, x) t^n \sim \exp(\epsilon t) \sum_{n=0}^{\infty} E_n(D, x) t^n \]
\[ \sim \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{n} \epsilon^k E_k(D, x)/k! \right\} t^n. \]

By comparing powers of \( t \) in the two asymptotic expansions we compute
\[ E_n(D - \epsilon I, x) = \sum_{k=0}^{n} \epsilon^k E_k(D, x)/k!. \]
We compute \( E_n(D - \epsilon I) \) by replacing \( E \) by \( E + \epsilon I \). Therefore, the coefficient of \( (E)^n \) in \( E_n \) is \( E_0(D, x)/n! = 1/n! \) by the normalization which we have chosen. This completes the proof of Theorem 2.3.

We apply Theorem 2.3 to the operator \( D^\epsilon_1 \): the endomorphism \( E^1 \) is the Ricci tensor up to a possible sign; \( \text{Tr} \left( (E^1)^n \right) = \rho^n \).

**Theorem 2.4.**
\[ A_n(D^\epsilon_1) = \sum_{k=0}^{n} \epsilon^k A_{n,k}(D_1), \]
\[ A_{n,n}(D_1) = (\rho^n/n!)(M). \]
3. In this section, we will apply the combinatorial theorems which were proved in the first two sections. We will show that certain geometrical properties are reflected by the spectrum. Let $M$ and $M'$ be two Riemannian manifolds.

**Lemma 3.1.** Let $p > 0$ be given and suppose $\text{spec } (D_p^\varepsilon, M) = \text{spec } (D_p^\varepsilon, M')$ for $n + 1$ distinct values of $\varepsilon$. Then $A_{n,k}(D_p^\varepsilon)(M) = A_{n,k}(D_p^\varepsilon)(M')$ for $k = 0, \ldots, n$.

**Proof.** Since the two spectra are the same, $A_n(D_p^\varepsilon)(M) = A_n(D_p^\varepsilon)(M')$ for $n + 1$ distinct values of $\varepsilon$. Therefore,

$$
\sum_{k=0}^n e^k A_{n,k}(D_p^\varepsilon)(M) = \sum_{k=0}^n e^k A_{n,k}(D_p^\varepsilon)(M'),
$$

$$
\sum_{k=0}^n e^k (A_{n,k}(D_p^\varepsilon)(M) - A_{n,k}(D_p^\varepsilon)(M')) = 0.
$$

Since this polynomial vanishes for $n + 1$ distinct values of $\varepsilon$, the coefficients vanish identically.

We apply the formulas we have derived previously to show

**Theorem 3.2.** Let

$$
\text{spec } (D_0^\varepsilon, M) = \text{spec } (D_0^\varepsilon, M') \quad \text{and} \quad \text{spec } (D_1^\varepsilon, M) = \text{spec } (D_1^\varepsilon, M')
$$

for $3$ distinct values of $\varepsilon$. Then:

(a) $1(M) = 1(M')$, $\tau(M) = \tau(M')$, $\rho^2(M) = \rho^2(M')$, $\tau^2(M) = \tau^2(M')$, $R^2(M) = R^2(M')$;

(b) if $M$ has constant scalar curvature $c$, so does $M'$;

(c) if $M$ is Einstein, so is $M'$;

(d) if $M$ has constant sectional curvature $c$, so does $M'$.

**Proof.** We use Theorem 2.2 to prove (a):

$$
1(M) = A_0(D_0^\varepsilon)(M) = A_0(D_0^\varepsilon)(M') = 1(M'),
$$

$$
\tau(M) = -6A_1(D_0^\varepsilon)(M) = -6A_1(D_0^\varepsilon)(M') = \tau(M').
$$

By Lemma 3.1, $A_{2,k}(D_1^\varepsilon)(M) = A_{2,k}(D_1^\varepsilon)(M')$ for $k = 0, 1, 2$. We apply Theorem 2.2 to compute $A_{2,1}(D_1^\varepsilon)$ and $A_{2,2}(D_1^\varepsilon)$:

$$
\rho^2(M) = 2A_{2,2}(D_1^\varepsilon)(M) = 2A_{2,2}(D_1^\varepsilon)(M') = \rho^2(M'),
$$

$$
\tau^2(M) = -6A_{2,1}(D_1^\varepsilon)(M) = -6A_{2,1}(D_1^\varepsilon)(M') = \tau^2(M').
$$

Finally, since $A_2(D_0^\varepsilon)(M) = A_2(D_0^\varepsilon)(M')$,

$$
(5\tau^2 - 2\rho^2 + 2R^2)(M) = (5\tau^2 - 2\rho^2 + 2R^2)(M').
$$

This implies $R^2(M) = R^2(M')$ and completes the proof of (a).
We complete the proof of Theorem 3.2 by expressing the geometrical assumptions about $M$ as integral conditions. $M$ has constant scalar curvature $c$ iff $\tau = -2c$. This is equivalent to assuming that $(\tau + 2c)^2(M) = 0$. Since $(\tau + 2c)^2 = \tau^2 + 4\tau c + 4c^2$, we apply (a) to show that $(\tau + 2c)^2(M') = 0$ which implies $M'$ has constant scalar curvature $c$. If $m = 2$, $M$ and $M'$ are automatically Einstein. We suppose $m > 2$; $M$ is Einstein iff $\rho_{jk} = c\delta_{jk}$ or equivalently $(\rho_{jk} - c\delta_{jk})^2(M) = 0$. Since $(\rho_{jk} - c\delta_{jk})^2 = \rho^2 - 2\tau c + mc^2$,

$$(\rho_{jk} - c\delta_{jk})^2(M') = 0.$$  

This implies $M'$ is Einstein. Finally, $M$ has constant sectional curvature iff $R_{ijkl} = c\delta_{ik}\delta_{jl} - c\delta_{il}\delta_{jk}$, or equivalently,

$$0 = (R_{ijkl} - c\delta_{ik}\delta_{jl} + c\delta_{il}\delta_{jk})^2(M) = (R^2 - 4\tau c + 2c^2(m^2 - m))(M) = 0.$$  

By (a) this invariant also vanishes for $M'$ so $M'$ has constant sectional curvature $c$.

The Ricci tensor $\rho_{ij}$ will be invariant under parallel translation iff $(\nabla \rho)^2(M) = 0$; for such a Riemannian manifold, the eigenvalues of $\rho_{ij}$ are independent of the point of the manifold.

**Theorem 3.3.** Let $\text{spec}(D_0, M) = \text{spec}(D_0, M')$ and $\text{spec}(D_1, M) = \text{spec}(D_1, M')$ for $m + 1$ distinct values of $\epsilon$. If $(\nabla \rho)^2(M) = 0$, then $(\nabla \rho)^2(M') = 0$ and the eigenvalues of the Ricci tensors on $M$ and on $M'$ are the same.

**Proof.** By Lemma 3.1, $A_{n,k}(D_1)(M) = A_{n,k}(D_1)(M')$ for $0 < k < n \leq m$. By Theorem 2.2,

$$((\nabla \rho)^2 + \tau \rho^2)(M) = -12A_{3,2}(D_1)(M) = -12A_{3,2}(D_1)(M') = ((\nabla \rho)^2 + \tau \rho^2)(M').$$  

Since $\rho = 0$ on $M$, $M$ has constant scalar curvature $c$. $M'$ has constant scalar curvature $c$ by Theorem 3.2. Therefore,

$$\tau \rho^2(M) = -2c\rho^2(M) = -2c\rho^2(M') = \tau \rho^2(M).$$  

This implies $0 = (\nabla \rho)^2(M) = (\nabla \rho)^2(M')$.

Since $1(M) = 1(M')$, we may make a change of scale to assume without loss of generality that $1(M) = 1(M') = 1$. Let

$$P(M)(\lambda) = \det(\rho_{jk} - \lambda_{jk}) = \sum_{i=0}^{m} \lambda c_i(\rho_{jk})(M);$$  

c_i(\rho_{jk}) are the characteristic classes of the matrix $\rho$. Let $\rho^n = \text{Tr}((\rho)^n)$; by Theorem 2.4,

$$\rho^n(M) = n! A_{n,n}(D_1)(M) = n! A_{n,n}(D_1)(M') = \rho^n(M').$$
for $0 \leq n \leq m$. It is well known that we can express the invariants $c_i(p)$ in terms of the invariants $p^n$ for $0 \leq n \leq m$. Since $1(M) = 1(M') = 1$,

$$c_i(1, p, \ldots, p^m)(M) = c_i(1(M), p(M), p^2(M), \ldots, p^m(M)) = c_i(1(M'), p(M'), p^2(M'), \ldots, p^m(M')) = c_i(1, \ldots, p^m)(M').$$

This implies that $P(M)(\lambda) = P(M')(\lambda)$. $\lambda$ is an eigenvalue of the Ricci tensor iff $P(M)(\lambda) = 0$ so the two sets of eigenvalues are the same.

We generalize Donnelly's theorem [1] as follows:

**Theorem 3.4.** Let

$$\text{spec } (D_0, M) = \text{spec } (D_0, M'), \quad \text{spec } (D_2, M) = \text{spec } (D_2, M'),$$

and

$$\text{spec } (D_1^\varepsilon, M) = \text{spec } (D_1^\varepsilon, M')$$

for 4 distinct values of $\varepsilon$. If $M$ is a local symmetric space, so is $M'$.

We use Patodi's theorem [4] and Theorem 3.4 to derive Donnelly's result: let $M$ be Einstein symmetric and let $\text{spec } (D_p, M) = \text{spec } (D_p, M')$ for $p = 0, 1, 2$. If $m = 2$, $M$ is symmetric implies $\tau$ is constant on $M$. This shows $\tau$ is constant on $M'$ so $M'$ is symmetric. We may therefore suppose that $m > 2$. $M$ is Einstein implies $M'$ is Einstein. $E^1(M) = E^1(M') = cI$, $D_1^\varepsilon(M) = D_1^\varepsilon(M') = D_1^\varepsilon(M') + (1 - \varepsilon)cI$. Since $\text{spec } (D_1, M) = \text{spec } (D_1, M')$, $\text{spec } (D_1^\varepsilon, M) = \text{spec } (D_1^\varepsilon, M')$ for all $\varepsilon$. We apply Theorem 3.4 to show $M'$ must be symmetric as well.

We prove Theorem 3.4 as follows: $M$ is symmetric implies $\tau$ is constant on $M$. By Theorem 3.2, $\tau$ is constant on $M'$. Therefore by Theorem 3.2,

$$\nabla^2(M) = \nabla^2(M') = 0, \quad \nabla^3(M) = \nabla^3(M'),$$

$$\tau^2(M) = \tau^2(M') = \tau^2(M'),$$

$$\tau R^2(M) = \tau R^2(M') = \tau R^2(M').$$

By Lemma 3.1, $A_{3,k}(D_1, M) = A_{3,k}(D_1, M')$ for $k = 0, 1, 2, 3$. This shows $A_3(D_1, M) = A_3(D_1, M')$. We use the identities derived above together with the identities $A_{3,k}(D_1, M) = A_{3,k}(D_1, M')$ for $k = 1, 2, 3$ to show

$$\rho_{jk} R_{jnm} R_{kmp} (M) = \cdots (M'), \quad (\nabla \rho)^2(M) = (\nabla \rho)^2(M') = 0,$$

$$\rho^2(M) = \rho^2(M').$$

We use the notation "\cdots" to indicate that the same invariant is to be applied to both $M$ and $M'$ in the equation.

We use the 7 identities derived above together with the identities $A_3(D_p, M) = A_3(D_p, M')$ for $p = 0, 1, 2$ to show
\[-7(\nabla R)^2 - 20\rho_{jk}\rho_{kl} R_{ijkl} - 24 R_{ijkm} R_{kuvw}(M) = \cdots (M'),\]

\[(\nabla R)^2 + 8\rho_{jk}\rho_{kl} R_{ijkl} + 3 R_{ijkm} R_{kuvw}(M) = \cdots (M'),\]

\[(m - 32)(\nabla R)^2 + (8m - 376)\rho_{jk}\rho_{kl} R_{ijkl} + (3m - 96) R_{ijkm} R_{kuvw}(M) = \cdots (M').\]

The matrix of coefficients in these 3 equations is nonsingular. We can solve this system of equations to show

\[(\nabla R)^2(M) = (\nabla R)^2(M').\]

Since \(M\) is symmetric, \((\nabla R)^2(M) = 0\). This shows \((\nabla R)^2(M') = 0\) so \(M'\) is symmetric.

**References**


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