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SUMS OF SOLID n-SPHERES(1)

BY

LOIS M. BROUSSARD

ABSTRACT. We prove that the sum of two solid Antoine n-spheres ($n > 3$) by the identity on the boundary is homeomorphic to the n-sphere S^n.

1. Background.

1.1. *Introduction*. In 1952, R. H. Bing [3] showed that S^3 is obtained from sewing a solid Alexander horned sphere to itself with the identity homeomorphism on the horned sphere. He obtained this surprising result by showing that the decomposition space of an u.s.c. decomposition of E^3 into points and tame arcs associated with the sewing is (topologically) E^3.

In 1920–1921, M. L. Antoine [1], [2] described a wild Cantor set in E^3 that is now known as "Antoine's necklace." In 1951, W. A. Blankenship [6] constructed a wild Cantor set in E^n ($n > 3$) which is a generalization of Antoine's necklace. We call these sets "Antoine-Blankenship necklaces." There are wild n-cells ($n > 3$) in E^n containing Antoine (-Blankenship) Cantor sets [5], [6]. The technique for obtaining these n-cells is called *tubing out* and is illustrated in [5]. Hence there are wild $(n - 1)$-spheres, "Antoine (-Blankenship) spheres", containing these same Cantor sets. Each of these wild $(n - 1)$-spheres has a nonsimply connected complementary domain in S^n. The union of such a wild $(n - 1)$-sphere with its *bad* complementary domain will be called a *solid Antoine (-Blankenship) n-sphere*.

In this paper we show that the identity sewing of a solid Antoine-Blankenship n-sphere with itself yields S^n. The author thanks the referee for comments and suggestions.

1.2. *Definitions and notation*. Euclidean n-space is denoted by E^n. We use ρ for the Euclidean metric, Diam for diameter, Bd, Ext and Int for boundary, exterior and interior (point set or combinatorial), and $N(X, \varepsilon)$ for the (open)

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ε-neighborhood of X in E^n, where $\varepsilon > 0$. An *ε-set* is a set of diameter less than ε and an *ε-map* or *ε-homeomorphism* moves points less than ε. The symbol Cl denotes topological closure.

We assume as familiar the notions of n-cells (B^0, B^1, $B^2, \ldots$ and their homeomorphic images), n-spheres (S^0, S^1, $S^2, \ldots$ and their homeomorphic iamges), complexes, manifolds, disks (2-cells), arcs (1-cells), simple closed curves (1-spheres), tamely (and flatly) embedded complexes, knotted arcs and simple closed curves, general position, and cut and paste techniques. We suggest [7] for a basic reference.

A *torus* is a 2-manifold homeomorphic with $S^1 \times S^1$; an *n-torus* is homeomorphic to $S_1^1 \times S_2^1 \times \cdots \times S_n^1$ where S_i^1 is a 1-sphere; a *solid torus* is homeomorphic to $B^2 \times S^1$; and a *solid n-torus* is homeomorphic to $B^2 \times S_1^1 \times \cdots \times S_{n-2}^1$.

A *Cantor set* is any set homeomorphic with the standard middle-third Cantor set. A Cantor set C is *tame* in a manifold M if C lies on a tame arc in M. If C is not tame, then it is *wild*. A sequence of compact n-manifolds with boundary $M_1 \supset \operatorname{Int} M_1 \supset M_2 \supset \operatorname{Int} M_2 \supset \ldots$ is called a *defining sequence* for a Cantor set C if $\bigcap M_i = C$. If the components of M_i for each i are n-cells (solid n-tori, etc.), then we say that C is *definable by n-cells* (*solid n-tori*, etc.). We have the following useful characterization of tame Cantor sets [4], [11].

THEOREM 1.1. *A Cantor set in E^n is tame iff it is definable by n-cells.*

We define a very simple kind of linking that will be sufficient for our purposes. We say that two unknotted simple closed curves J and K are *linked* in E^3 if there is a homeomorphism h on E^3 taking J and K onto the canonically linked pair $h(J) = \{(x, y, z) | x^2 + y^2 = 1, z = 0\}$ and $h(K) = \{(x, y, z) | (y - 1)^2 + z^2 = 1, x = 0\}$. We say that two disjoint compact sets X and Y link in E^3 if there exist unknotted simple closed curves $J \subset X$ and $K \subset Y$ such that J and K link.

A *crumpled n-cell* or *solid n-sphere* K is a space homeomorphic to the union of an $(n - 1)$-sphere (topologically embedded in S^n) and one of its complementary domains, U. We call U the *interior* of K (Int K) and S the *boundary* of K (Bd K). If K_1 and K_2 are crumpled n-cells and h: Bd $K_1 \to$ Bd K_2 is a homeomorphism, then the space $K_1 \cup_h K_2$, called the *sum* of K_1 and K_2 by h, is the space obtained from the disjoint union of K_1 and K_2 by identifying each point $p \in \operatorname{Bd} K_1$ with $h(p) \in \operatorname{Bd} K_2$. The homeomorphism h is called the *sewing* of K_1 and K_2. If each of K_1 and K_2 is an n-cell, then any sewing of K_1 to K_2 yields S^n. This need not be the case if K_1 and K_2 are not n-cells.

1.3. *The Antoine* (*-Blankenship*) *sphere*. The Antoine-Blankenship necklaces are definable by solid n-tori ($n \geqslant 3$). The notation we use is that of [9]. First we define Antoine's necklace. Let M_1 be a solid 3-torus in E^3, represented by the large solid 3-torus in Figure 1.1. Let $A_1, \ldots, A_k$ be k solid 3-tori embedded in Int M_1, *linked around the factor* S^1 of $M_1 \approx B^2 \times S^1$ as indicated in Figure 1.1. The integer k may be arbitrarily large but at least 2. (For nice pictures, we use $k = 3$.) The embedding of the solid 3-tori $\{A_j\}_{j=1}^k$ in the solid 3-torus M_1 is called an *Antoine embedding*. Let $M_2 = A_1 \cup \cdots \cup A_k$. We obtained M_2 by using an Antoine embedding of k solid 3-tori in M_1. This process iterates. If A is a component of M_i, then the k components (k need not be the same at every stage) of M_{i+1} in A are Antoine embedded in A, as indicated in Figure 1.1.

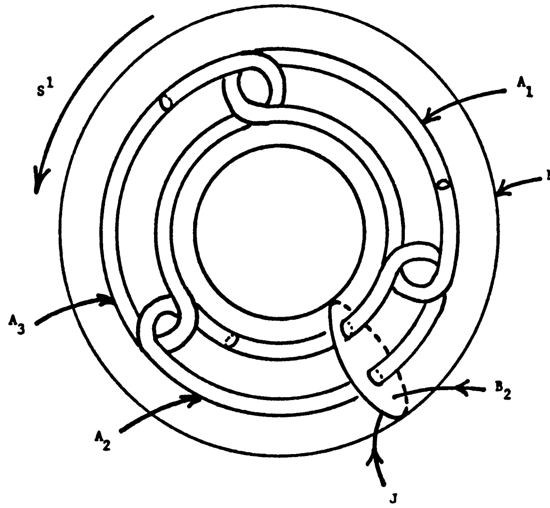

FIGURE 1.1

By choosing k large when constructing M_i, we can make the components of M_i small in relation to the components of M_{i-1} ($i > 2$). Hence the above process gives the defining sequence for a Cantor set $C = \bigcap M_i$. The Cantor set C is called Antoine's necklace.

Now we describe the Antoine-Blankenship necklaces. Let P_i ($i = 1, \ldots, n - 2$) be the natural projection of the solid n-torus $A = B^2 \times S_1^1 \times \cdots \times S_i^1 \times \cdots \times S_{n-2}^1$ onto its factor space $B^2 \times S_i^1$. For $n = 3$, $P_i = 1$ and we get the usual definition of Antoine's necklace. For each integer $i \in \{1, \ldots, n - 2\}$ we will associate an embedding of k solid n-tori $A_1, \ldots, A_k$ in A. Since there exists a homeomorphism of A onto itself which interchanges the S^1 factors, the $n - 2$ embeddings are topologically equivalent. Let i be fixed. The factor space $B^2 \times S_i^1$ is represented by the large solid 3-torus in Figure 1.2. The embeddings of the projections $P_i(A_1), \ldots, P_i(A_k)$ of the solid n-tori $A_1, \ldots, A_k$ are represented by the smaller solid 3-tori linked around the factor S_i^1 in Figure 1.2.

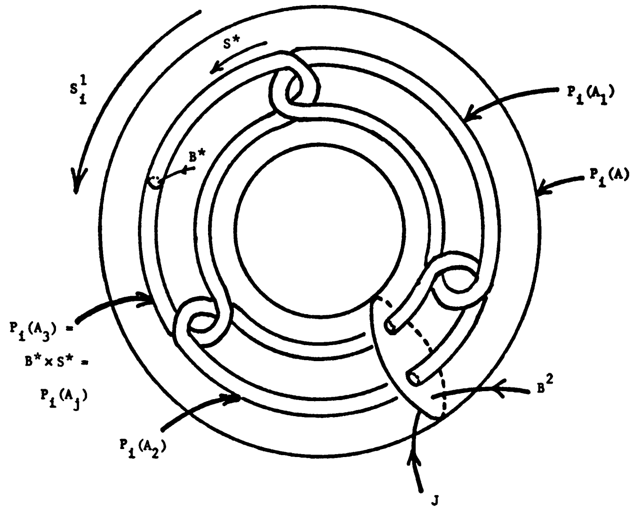

FIGURE 1.2

The solid n-torus A_j is like A in $n-3$ of its factors; in fact, if $B^* \times S^*$ is the solid 3-torus $P_i(A_j)$ in Figure 1.2, then $A_j = B^* \times S_1^1 \times \cdots \times S_{i-1}^1 \times S^* \times S_{i+1}^1 \times \cdots \times S_{n-2}^1$. The integer k of A_j's may be arbitrarily large but always at least 2. (For nice pictures, we use $k = 3$.) The embedding of the solid n-tori $\{A_j\}_{j=1}^k$ in the solid n-torus A is called an *Antoine embedding*. If $A_1, \ldots, A_k$ are Antoine embedded in A using the ith projection P_i, then we say that $A_1, \ldots, A_k$ are *linked around the i-factor of A*.

In order that a sequence $M_1 \supset \operatorname{Int} M_1 \supset M_2 \supset \ldots$ be a defining sequence for a Cantor set, it is necessary that the diameters of the components of M_i get small as i gets large. For $n = 3$, by choosing k large when constructing M_i, we can make the components of M_i small relative to the components of M_{i-1} ($i > 2$). Hence for $n = 3$, we easily obtain a Cantor set, Antoine's necklace. More care is necessary for $n > 3$ to insure that the diameters of the components of M_i get small as i gets large. We take M_1 to be any solid n-torus in E^n. We obtain M_2 by using an Antoine embedding of k solid n-tori in $M_1 = A$. No matter how large the integer k, a component A_j of M_2 can be made small in at most two of its factors. This becomes clear once we note that if $A_1, \ldots, A_k$ are linked around the i-factor of A, then

$$A_j = B^* \times S_1^1 \times \cdots \times S_{i-1}^1 \times S_i^* \times S_{i+1}^1 \times \cdots \times S_{n-2}^1$$

and

$$A = B^2 \times S_1^1 \times \cdots \times S_{i-1}^1 \times S_i^1 \times S_{i+1}^1 \times \cdots \times S_{n-2}^1.$$

That is, only the disk factor and the ith circle factor may be made small which leaves A_j nearly as large as A.

As we noted, if $n = 3$, then we can make all of the components of the second stage M_2 small. However, if $n > 3$, then we need $n - 2$ stages to make components small relative to the diameter of M_i. That is, we may link the solid n-tori of M_2 around the 1-factor of M_1, then in each component of A of M_2 we may link the solid n-tori of M_3 in A around the 2-factor of A, etc. Thus the components of M_{n-1} may be made of arbitrarily small diameter. By repeating this we can insure that $C = \bigcap M_i$ is a Cantor set.

The Cantor set $C = C_n$ definable by n-manifolds in E^n is easily seen to be wild in E^n. We obtain wild $(n-1)$-spheres Σ^{n-1} in E^n such that Σ^{n-1} is locally flat modulo C_n by a simple tubing out technique.

Let C be a Cantor set in E^n. Let $M_1 \supset M_2 \supset \ldots$ be the defining sequence for C. Let B_0 be an n-cell in the complement of M_1 (as usual, M_1 connected). Let α be an arc from B_0 to M_1 such that α intersects each in a single point. Thicken α to an n-tube T_1 joining B_0 to M_1 and intersecting each in an $(n-1)$-cell in the boundary. Let $\alpha_1, \ldots, \alpha_k$ be arcs in M_1 joining $T_1 \cap M_1$ to the k components $A_1, \ldots, A_k$ of M_2, respectively. For each i, α_i intersects Bd M_1 in a single point in $T_1 \cap M_1$. Thicken each of these arcs, obtaining a mutually exclusive collection of n-tubes $T_{21}, \ldots, T_{2k_2}$ in Int $M_1 \cup (T_1 \cap M_1)$

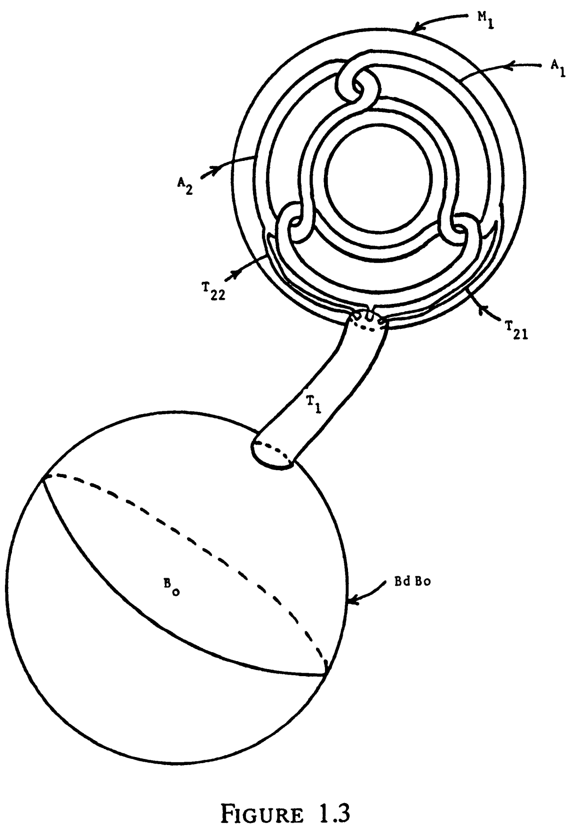

FIGURE 1.3

such that, for each i, T_{2i} joins T_1 to A_i and intersects each in an $(n-1)$-cell in its boundary. Continuing in this fashion, we tube out from $T_{21}, \ldots, T_{2k_2}$ to the components of M_3, and then to the components of M_4, etc. We let $B_1 = B_0 \cup T_1$, $B_2 = B_1 \cup (\bigcup_{i=1}^{k_2} T_{2i})$, $B_3 = B_2 \cup (\bigcup_{i=1}^{k_3} T_{3i})$, etc. For each m, B_m is an n-cell and $S_m = \text{Bd } B_m$ is an $(n-1)$-sphere. In the limit we have $B_m \to B$, an n-cell containing C, and $S_m \to \Sigma^{n-1} = \text{Bd } B$, an $(n-1)$-sphere containing C.

For $n = 3$, it is necessary to tube out carefully as indicated in Figure 1.3. The 3-tubes T_{ij} are joined to the components of M_i. A T_{ij} does not go through the hole of the component of M_i that it intersects and does not wind around the hole of the component of M_{i-1} that contains it. [As each component A of M_α is a solid 3-torus and $\pi_1(A)$ is infinite cyclic, it is natural to consider the "hole" in A.] No T_{ij} winds even halfway around the hole of M_{i-1}. Hence there exists a nonempty family $\mathfrak{F}$ of countable dense subsets of the Cantor set C such that $(E^3 - \Sigma^2) + F$ is 1-ULC if $F \in \mathfrak{F}$. (For $n > 3$, if F is any countable dense subset of the Antoine-Blankenship Cantor set C, then $(E^n - \Sigma^{n-1}) + F$ is 1-ULC.)

2. Identity sewings of Antoine solid n-spheres. We consider the Antoine or Antoine-Blankenship $(n-1)$-sphere S to be embedded in S^n and call $K = S \cup U$ an *Antoine crumpled n-cell* or *Antoine solid n-sphere*, where U is the component of $S^n - S$ that is not simply connected.

THEOREM 2.1. *If K is an Antoine solid n-sphere and h is the identity sewing on K, then $K \cup_h K \approx S^n$.*

Suppose p is an ideal point at infinity such that $E^n + p$ is homeomorphic to S^n. When R. H. Bing sewed together two solid (Alexander) horned spheres [3], he described an upper semicontinuous decomposition of $E^3 + p$ into tame arcs and points. He showed that the continuum obtained from sewing the solid horned spheres by the identity was homeomorphic to the decomposition space of $E^3 + p$. Then he showed the decomposition was homeomorphic to S^3 by shrinking out the nondegenerate elements.

This is the plan of attack here. The bulk of the work is in showing that the decomposition space resulting from the identity sewing of Antoine solid n-spheres is homeomorphic to S^n.

We prove the following version of Bing's sewing theorem [3]:

THEOREM 2.2. *A continuum is homeomorphic with S^n if it is the sum of three mutually exclusive sets S, U^1, U^2 such that* (1) *there is a homeomorphism of $S \cup U^i$* $(i = 1, 2)$ *onto an Antoine solid n-sphere that carries S onto the Antoine $(n-1)$-sphere and* (2) *there is a homeomorphism of $S \cup U^1$ onto $S \cup U^2$ that leaves each point of S fixed.*

PROOF. We first describe a decomposition of $E^n + p = E^n_*$ into tame arcs and points such that the resulting space $X = E^n_*/G$ is homeomorphic with $S \cup U^1 \cup U^2$. For the proof of $E^n_*/G \approx S^n$, which completes the proof of Theorem 2.2, see [3]. (Note that we will establish Bing's shrinking criterion in §2.2.3.)

2.1. *Decomposition of* $E^n + p$. We describe an upper semicontinuous decomposition G' of E^n. With the addition of the ideal point at infinity, p, G' extends to a decomposition G of S^n. Suppose θ is an $(n-1)$-hyperplane in E^n. The plane θ separates E^n into two components H' and H'' which we think of as the "upper half" and the "lower half." Roughly, the arcs are those obtained by locating two Cantor sets on opposite sides of θ and joining the corresponding points with arcs locally PL modulo the end points. The decomposition is definable by n-dimensional dogbones or n-dimensional solid double tori as discussed in [9]. We denote the manifolds used to describe the Cantor set above θ (in H') with primed letters and the manifolds used to describe the Cantor set below θ (in H'') by double primed letters. These Cantor sets are distinct copies of the Antoine Cantor set for $n = 3$ or the Antoine-Blankenship Cantor set for $n \geqslant 4$, the Cantor sets that are used to build the Antoine solid n-spheres.

The n-manifolds M_1', M_1'' are solid n-tori, i.e., connected manifolds homeomorphic to $B^2 \times S_1^1 \times \cdots \times S_{n-2}^1$. Join M_1' and M_1'' with an n-tube T_1 so that T_1 intersects each of M_1' and M_1'' in an $(n-1)$-cell in the boundary of each. T_1 is constructed as a thin regular neighborhood of a PL arc joining M_1' to M_1''. Let M_1 denote the union $M_1' \cup T_1 \cup M_1''$. Each component A_i' of M_2' is connected to a unique component A_i'' of M_2'' by an n-tube T_i in Int M_1 so that T_i runs straight through the tube T_1 of M_1 and T_i intersects each of M_2' and M_2'' in an $(n-1)$-cell which lies in A_i' and A_i'', respectively. We denote the union $A_i' \cup T_i \cup A_i''$ by A_i. The T_i's are mutually exclusive and the union of the r n-manifolds $A_1, \ldots, A_r$ is M_2. Inductively, if $A = A' \cup T \cup A''$ is a component of M_k where A' and A'' are solid n-tori and T is an n-tube joining A' and A'', then the r components $\{A_i\}_{i=1}^r$ of M_{k+1} in A are obtained by connecting each of the r solid n-tori A_i' in A' to a unique solid n-torus A_i'' in A'' by an n-tube T_i in Int A that runs straight through T, intersects each of A_i' and A_i'' in an $(n-1)$-cell and misses all other components of A' and A''. The T_i are mutually exclusive, and constructed as thin regular neighborhoods of PL arcs.

The manifolds M_i are located so that they are symmetric with respect to θ. Moreover, each component of M_i intersects θ in an $(n-1)$-cell. It is natural to think of each component of M_i as being made up of an upper handle A', a lower handle A'', and a stem T. We shall consider the image of an upper handle (lower handle, stem) under a homeomorphism on E^n to be the upper handle (lower handle, stem) of the image.

A component of M_i in S^3 is a solid double torus or a cube with two handles. Although it is not easy to "see" a component of M_i in S^4, it is quite easy to visualize the spine of any component of M_i. It is the union of two PL toroidal surfaces and a PL arc joining them. This follows from the fact that a handle is of the form $B^2 \times S_1^1 \times S_2^1$ with spine homeomorphic to $S_1^1 \times S_2^1$ and the stem is a 4-tube with spine an arc. In general, for $n \geqslant 4$ the spine of a component of M_i is the union $W' \cup Y \cup W''$ where W', W'' are handle spines homeomorphic to $S_1^1 \times \cdots \times S_{n-2}^1$ and Y is the stem spine, an arc. Y intersects each of W' and W'' in a point, which we call a *node* of the spine. [For $n \geqslant 3$, each component of M_i is called an n-dogbone.]

The nondegenerate elements of our upper semicontinuous decompositon G' of E^n are the components of $\bigcap M_i$. We let G be the extension of G' to S^n. Let H be the set of nondegenerate elements in G, $H^* = \bigcup H$. As we described the manifolds M_i, we could easily have described homeomorphisms T_1, T_2 between $S \cup U = K$ and each of the two parts of E_*^n/G, i.e., the part that intersects $\overline{H'}$ and the part that intersects $\overline{H''}$. The technique resembles that used in describing an Antoine $(n-1)$-sphere or solid n-sphere. Hence E_*^n/G is homeomorphic with $S \cup U^1 \cup U^2$ where G consists of points of $E^n + p - \bigcap M_i$ and components of $\bigcap M_i$.

2.2. *Shrinking the arcs*. The method of shrinking the arcs resembles those of R. H. Bing [3] and L. O. Cannon [8].

To introduce the reader to the shrinking technique for $n \geqslant 4$, we first obtain a basic lemma for the case $n = 3$. Note that for $n = 3$, Theorem 2.2 also follows from [10].

In the description of the Antoine-Blankenship Cantor sets, the number k of components $\{A_j\}_{j=1}^k$ of M_{i+1} in a component A of M_i may be arbitrarily large and must be at least two. For simplicity, we always take $k = 3$ in the figures.

A set of $(n-1)$-hyperplanes $P_1, \ldots, P_s$ parallel to θ with P_i between P_{i-1} and P_{i+1} $(i = 2, 3, \ldots, s-1)$ is said to be in *good position* relative to $h(M_k)$ for a homeomorphism h on E^n if for each component A of $h(M_k)$,

(1) each handle of A may intersect at most one hyperplane,

(2) if P_i intersects A in the stem of A, then $P_i \cap A$ is an $(n-1)$-cell, and

(3) if A_μ is a component of $A \cap h(M_{k+m})$, $m = 1, 2, \ldots$, then if P_i intersects A in the stem for $i \in \{2, 3, \ldots, s-1\}$, then $P_i \cap A_\mu$ is an $(n-1)$-cell in the stem of A_μ.

We shall consider the images of the planes $P_1, \ldots, P_s$ under a homeomorphism f on E^n to be in good position relative to $fh(M_k)$ if $P_1, \ldots, P_s$ are in good position relative to $h(M_k)$.

In Lemmas 2.3 and 2.4, we use four planes. For notational simplicity, we use P_a, P_b, P_c, and P_d for P_1, P_2, P_3, and P_4, respectively.

2.2.1. *Antoine solid* 3-*spheres*: *A basic lemma*. We do not shrink the arcs of H in one stage of the defining sequence, but we follow a procedure that will eventually shrink the arcs. The shrinking procedure is outlined in the proof of the following lemma.

LEMMA 2.3. *If h' is a homeomorphism of E^3 and P_a, P_b, P_c, and P_d are 2-hyperplanes in good position relative to $h'(M_i)$, then there exists a homeomorphism h of E^3 such that* (1) *$h = 1$ on $E^3 - h'(M_i)$,* (2) *$h = 1$ between P_b and P_c,* (3) *h takes each component of $h'(M_{i+1})$ onto a set which intersects at most one of P_a and P_d, and* (4) *P_a, P_b, P_c, and P_d are in good position relative to $hh'(M_{i+1})$.*

PROOF. It suffices to consider one component A of $h'(M_i)$, i.e., if A_1, $A_2, \ldots, A_k$ are components of $h'(M_{i+1})$ in A, it suffices to obtain a homeomorphism g of E^3 such that (1) $g = 1$ on $E^3 - A$, (2) $g = 1$ between P_b and P_c, (3) g takes each A_j onto a set which intersects at most one of P_a and P_d, and (4) P_a, P_b, P_c, and P_d are in good position relative to

$$g(A_1 \cup A_2 \cup \cdots \cup A_k).$$

There are two cases to consider; either no plane intersects a handle of A or

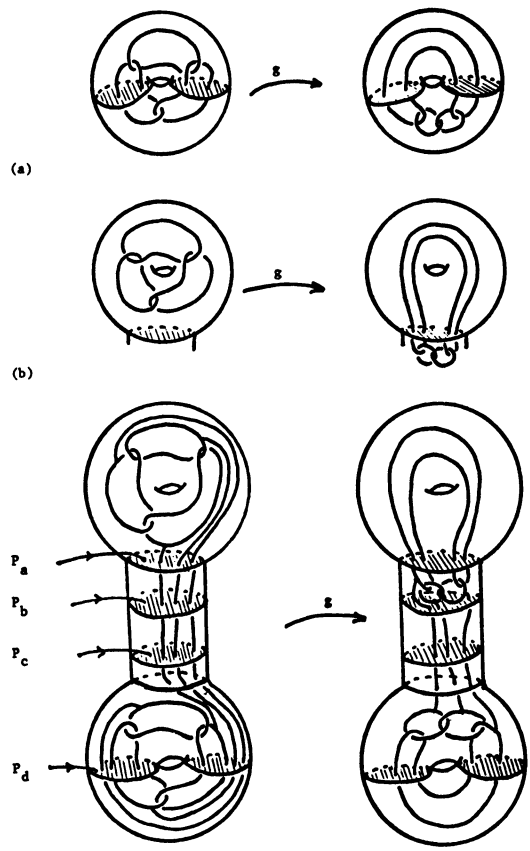

FIGURE 2.1

some plane does. The idea of the proof is contained in the following sequence of pictures in which the A_j's are represented by their 1-spines. The effect of g is that $g(A_2)$, $g(A_3), \ldots, g(A_k)$ do not intersect P_a and $g(A_1)$, $g(A_2), \ldots, g(A_{k-1})$ do not intersect P_d.

We need to consider two sets of handle pictures: (a) A plane intersects a handle. (b) A plane does not intersect a handle. The planes are illustrated by the shaded disks.

Although the figures are sufficient to describe the action of g, we give another description of g which is analogous to the g used later in sewing an Antoine solid n-sphere to itself by the identity. The homeomorphism g can be defined as the composition of two homeomorphisms: g_1, which rearranges the handles of $\{A_j\}_j$ in the handles of A, and g_2, which pulls the handles of the A_j toward the stem of A.

We may assume that the spines of the A_j's are very close to the spine of A. It will be convenient later to consider g_1 and g_2 as moving spines or neighborhoods of spines. The map g_1 acts on the handles of A moving the spines of the upper (lower) handles of $A_2, \ldots, A_k$ $(A_1, \ldots, A_{k-1})$ into a close neighborhood of the upper (lower) node of the spine of A. The map g_2 shortens the spine of the stem of A, pulling the neighborhoods of the nodes of the spine of A beyond the outermost planes intersecting A. Alternatively, one might consider the stem of A as reaching out and engulfing small ball neighborhoods of the nodes of the spine of A, and then pulling them back into the stem.

We now have that (1) $g = 1$ on $E^3 - A$, (2) $g = 1$ between P_b and P_c, and (3) g takes each A_j onto a set which intersects at most one of P_a and P_d. Consider a component A_j of $h'(M_{i+1}) \cap A$. Since $g(A_j) \cap (P_b \cup P_c) = A_j \cap (P_b \cup P_c)$ and the nodes of the spine of A_j are close to the nodes of the spine of A, it follows that P_a and P_d do not intersect the stem of $g(A_j)$. (The map g pulls the nodes of the spine of A beyond P_a and P_d.) We easily obtain g so that conditions (2) and (3) of good position are met. Hence, the planes P_a, P_b, P_c, and P_d are in good position relative to $g(A_j \cap h'(M_{i+1}))$.

2.2.2. *Antoine solid n-spheres*: *A basic lemma*. In this section we prove a lemma that is the higher dimensional analogue of Lemma 2.3.

LEMMA 2.4. *If h' is a homeomorphism of E^n and P_a, P_b, P_c, and P_d are $(n-1)$-hyperplanes in good position relative to $h'(M_i)$, then there exists a homeomorphism h of E^n such that* (1) $h = 1$ *on* $E^n - h'(M_i)$, (2) $h = 1$ *between* P_b *and* P_c, (3) *h takes each component of $h'(M_{i+n-2})$ onto a set which intersects at most one of P_a and P_d, and* (4) *P_a, P_b, P_c, and P_d are in good position relative to $hh'(M_{i+n-2})$.*

A *generalized n-annulus* is an n-manifold homeomorphic with $S_1^1 \times \cdots \times$

$S_{n-1}^1 \times I$, $I = [0, 1]$. A *generalized* (n, k)-*annulus* is an n-manifold homeomorphic with $S_1^1 \times \cdots \times S_{n-k}^1 \times I_1 \times \cdots \times I_k$.

PROOF (LEMMA 2.4). It suffices to consider one component A of $h'(M_i)$. The components of $h'(M_{i+1})$ in A are denoted by $A_1, \ldots, A_s$, the components of $h'(M_{i+2})$ in A by $A_{11}, \ldots, A_{st}$ where $A_{jv} \subset A_j$, and the components of $h'(M_{i+n-2})$ in A by $A_{j_1 j_2 \cdots j_{n-2}}$ where $A_{j_1 \cdots j_{n-2}} \subset A_{j_1 \cdots j_{n-3}}$. If A_α represents a component of $h'(M_{i+u})$, then α is a finite sequence of integers of length u. The components of $h'(M_{i+u+1})$ in A are then denoted by $A_{\alpha 1}, \ldots, A_{\alpha v}$. We assume that A intersects all four planes.

As in the proof of Lemma 2.3, it suffices to find a homeomorphism g of E^n such that (1) $g = 1$ on $E^n - A$, (2) $g = 1$ between P_b and P_c, (3) g takes each component of $h'(M_{i+n-2})$ in A onto a set which intersects at most one of the two hyperplanes P_a and P_d, and (4) P_a, P_b, P_c, and P_d are in good position relative to $g(A \cap h'(M_{i+n-2}))$.

The idea of the proof is contained in a sequence of pictures. In the figures, $n = 4$ and the manifolds A_α are represented by their spines. In general, a partition of the $(n-2)$-dimensional portion of the spine of A_α, a component of $h'(M_{i+n-3})$, is used to indicate the approximate location of the spines of $A_{\alpha 1}, A_{\alpha 2}, \ldots, A_{\alpha k}$.

In the figure, a handle spine of A represented by $S_1^1 \times S_2^1$ (we drop the superscripts for the remainder of this proof) and the handles of A_1, A_2, A_3 in that handle of A are shown linked around the S_1-factor. In general if the handles of $h'(M_{i+1})$ link around the S_{β_1}-factor of $h'(M_i)$ for some β_1, then the handles of $h'(M_{i+2})$ link around the S_{β_2}-factor of $h'(M_{i+1})$, where $\beta_2 = \beta_1 + 1$. To save notation, we shall write i for β_i. With this in mind, we have that the handles of $h'(M_{i+j})$ link around the S_j-factor of the handles of $h'(M_{i+j-1})$ for $j = 1, 2, \ldots$.

The map g is obtained as the composition of two homeomorphisms, g_1 and g_2. For each A_α in $h'(M_{i+n-3}) \cap A$, g_1 rearranges the handles of $A_{\alpha 1}, \ldots, A_{\alpha k}$ in the handles of A_α. For each component A_α of $h'(M_{i+n-3}) \cap A$, g_2 pulls the handles of $A_{\alpha 1}, \ldots, A_{\alpha k}$ into or toward the stem of A_α, achieving the desired relationship with the $(n-1)$-hyperplanes P_a, P_b, P_c, and P_d. Moreover, g_1 is fixed outside $h'(M_{i+n-3}) \cap A$ and g_2 is fixed outside A.

We now set up the machinery necessary to describe g_1 and g_2. Consider a component A_α of $h'(M_{i+n-3})$ in A containing $A_{\alpha 1}, \ldots, A_{\alpha k}$ that are in turn components of $h'(M_{i+n-2})$. Let the spine of A_α be the union of Z_α, Z'_α, and X_α where Z_α, Z'_α are the spines of the upper and lower handles, respectively. Each of Z_α and Z'_α is homeomorphic to $S_1 \times \cdots \times S_{n-2}$. The set X_α is an arc meeting Z_α and Z'_α in points p_α and p'_α, respectively. Similarly, let the spine of A be $Z \cup Z' \cup X$ with nodes $\{p, p'\}$.

Let R_α, R'_α be PL generalized $(n-2)$-annular neighborhoods of p_α, p'_α in Z_α, Z'_α, respectively. If Z_α is represented by $S_1^* \times \cdots \times S_{n-3}^* \times S_{n-2}$ [the

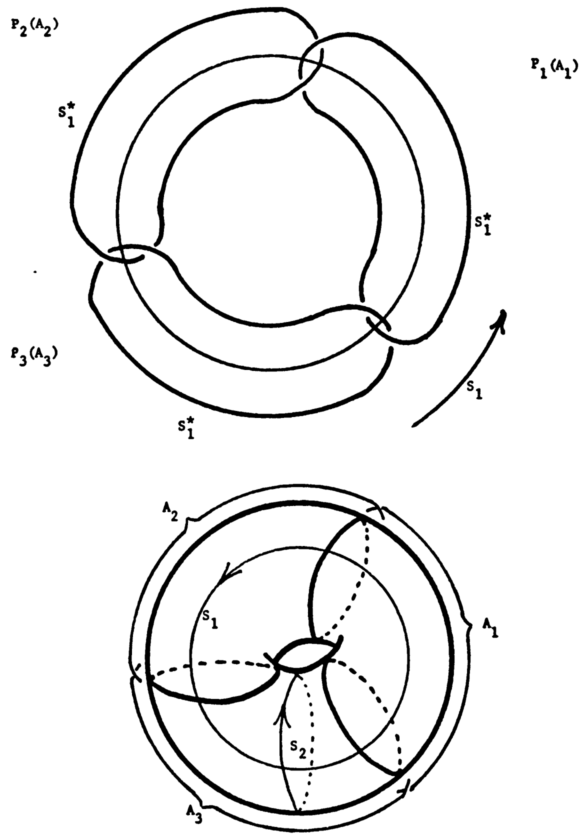

FIGURE 2.2

asterisk notation is used to indicate that the handles of the components of $h'(M_{i+n-3})$ are small in the first $(n-3)$ factors and still large in the last factor. Also, if Z is represented by $S_1 \times \cdots \times S_{n-2}$, then each S_i^* is represented by an interval I_i in S_i, $i = 1, \ldots, n-3$; see Figure 2.2], then R_α is represented by $S_1^* \times \cdots \times S_{n-3}^* \times I$ where I is an appropriate interval in S_{n-2}. Since we may assume that the spine of $h'(M_{j+1})$ is in a very small neighborhood of the spine of $h'(M_j)$ for all j, we may assume that R_α is close to a set $D_\alpha \subset Z$ represented by $I_1 \times \cdots \times I_{n-3} \times I$, where I_k is the interval in S_k corresponding to S_k^*. The neighborhood R'_α has a similar representation; see Figure 2.3. Clearly the set $D_\alpha \subset Z$ $(D'_\alpha \subset Z')$ is an $(n-2)$-cell.

Since the spine of $h'(M_{j+1})$ is in a very small neighborhood of the spine of $h'(M_j), j = 1, 2,, \ldots$, we can assume that each generalized annulus R_α (R'_α) is in a PL n-cell neighborhood, B_α (B'_α), of the $(n-2)$-cell, D_α (D'_α), in Z (Z'). (Consider a regular neighborhood of D_α (D'_α).) Moreover, B_α and B'_α are in Int A. Since the handles of A do not intersect P_b and P_c and p_α (p'_α) is in a handle of A_α, B_α (B'_α) does not intersect P_b (P_c). With a minor adjustment (a slight rotation of the R_α (R'_α) or a shortening of the interval factor of R_α (R'_α)),

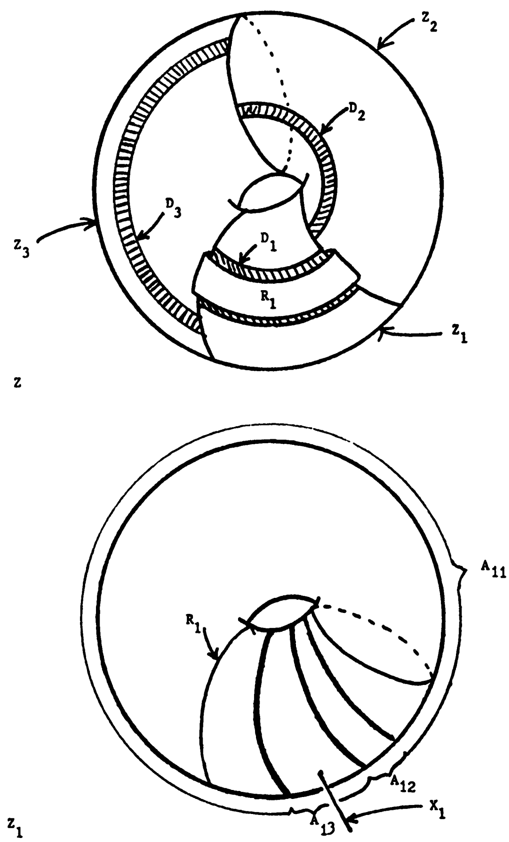

FIGURE 2.3

we can assume that the D_α (D'_α) and hence the B_α (B'_α), for all A_α components of $h'(M_{i+n-3}) \cap A$, form a pairwise disjoint collection. See Figure 2.3.

We can adjust B_α and X_α so that B_α does not intersect the stem of A_β for $\beta \neq \alpha$ and so that X_α intersects Bd B_α in a single point. We make the same adjustments for X_α and B'_α. Then there exists a collection of pairwise disjoint open n-cells $\{U_\alpha\}$ such that (stem A_α) $\cup B_\alpha \cup B'_\alpha \subset U_\alpha \subset \operatorname{Int} A$.

Now we define g_1. Let g_1 be a homeomorphism on E^n, that is the identity outside $h'(M_{i+n-3}) \cap A$ and between P_b and P_c, that moves neighborhoods of the spines of the upper (lower) handles of $A_{\alpha 2}, \ldots, A_{\alpha k}$ ($A_{\alpha 1}, \ldots, A_{\alpha k-1}$) into a small neighborhood of R_α (R'_α). This is possible since the handles of $A_{\alpha 1}, \ldots, A_{\alpha k}$ are linked around the S_{n-2}-factor of A_α. We can assume that the parts of the stems of $g_1(A_{\alpha 2}), \ldots, g_1(A_{\alpha k})$ above P_b ($g_1(A_{\alpha 1}), \ldots, g_1(A_{\alpha k-1})$ below P_c) are very close to X_α. (The small neighborhoods of R_α (R'_α) are in B_α (B'_α).) Note here that although

$$g_1(A_{\alpha 2} \cup \cdots \cup A_{\alpha k-1}) \subset U_\alpha,$$

$A_{\alpha 1}$ or $A_{\alpha k}$ may have a handle that intersects $E^n - U_\alpha$.

We define g_2 in pieces. If P_a intersects A_α, then we obtain an $(n-1)$-cell P_a^α as a slice of the stem of A_α between P_a and P_b. We choose P_a^α sufficiently close to P_b so that P_a^α separates $P_a \cap A$ from P_b. Certainly the cell P_a^α separates the upper handle of A_α from the lower handle. Let $\hat{A}_\alpha$ be the component of $A_\alpha - P_a^\alpha$ containing the upper handle of A_α and let $\hat{A}_\alpha$ be the component of $A_\alpha - P_b$ containing the lower handle of A_α. We define a homeomorphism f_α on E^n such that $f_\alpha = 1$ on $\hat{\hat{A}}_\alpha$, $f_\alpha = 1$ outside U_α, and $f_\alpha(A_\alpha - \hat{A}_\alpha) \supset N(X_\alpha) \cup B_\alpha$ for $N(X_\alpha)$ a neighborhood of X_α in A_α, by letting the stem of A_α below P_a^α "engulf" B_α. Then f_α^{-1} pulls B_α into the stem of A_α, below P_a^α, and hence below P_a. If P_d intersects A_α, we define f_α' to have a similar effect on the lower handle of A_α. If P_a (P_d) does not intersect A_α, then we let $f_\alpha = 1$ $(f_\alpha' = 1)$.

By combining f_α^{-1} and $f_\alpha'^{-1}$, we obtain g_α for each A_α in A. We obtain g_2 by piecing together the g_α.

Now we consider $g = g_2 g_1$. It is clear that $g = 1$ outside A, and between P_b and P_c. The map g_1 acts on each component A_α of $h'(M_{i+n-3}) \cap A$, moving the spines of the upper (lower) handles of $A_{\alpha 2}, \ldots, A_{\alpha k}$ $(A_{\alpha 1}, \ldots, A_{\alpha k-1})$ into a neighborhood of the upper (lower) node of the spine of A_α. The map g_2 shortens the spine of the stem of each A_α, if necessary, pulling the neighborhoods of the nodes of the spine of each A_α beyond the outermost planes intersecting each A_α. Hence $g = g_2 g_1$ carries each component of $h'(M_{i+n-2})$ in A off one of the planes P_a or P_d.

We now obtain that P_a, P_b, P_c, and P_d be in good position relative to $g(A \cap h'(M_{i+n-2}))$. Let $A_{\alpha j}$ be a component of $A \cap h'(M_{i+n-2})$. Since $g_2 g_1 = 1$ between P_b and P_c, we have that each of $P_b \cap g_2 g_1(A_{\alpha j})$ and $P_c \cap g_2 g_1(A_{\alpha j})$ is an $(n-1)$-cell in the stem of $g_2 g_1(A_{\alpha j})$. If P_a (P_d) does not intersect A, then P_a (P_d) does not intersect $g_2 g_1(A_{\alpha j})$. If P_a (P_d) does intersect A_α, then since the nodes of the spine of $g_1(A_{\alpha j})$ are close to the nodes of the spine of A_α, it follows that P_a (P_d) does not intersect the stem of $g(A_{\alpha j}) = g_2 g_1(A_{\alpha j})$. (The map g moves the upper node of the spine of $A_{\alpha j}$ "below" P_a.) Then, since P_b and P_c do not intersect the handles of $g(A_{\alpha j})$, and the handles of $g(A_{\alpha j}) \cap g(M_{i+n-2+m})$, $m = 1, 2, \ldots$, are in the handles of $g(A_{\alpha j})$, the set of planes P_a, P_b, P_c, and P_d is in good position relative to $g(A_{\alpha j})$.

2.2.3. *Shrinking criterion.* We now show that we have Bing's shrinking criterion [3].

LEMMA 2.5. *If U is an open set in E^n containing $H^* = \bigcup H$ and ε is a positive number, then there is a homeomorphism h of E^n such that* (1) $h = 1$ *on* $E^n - U$ *and* (2) *for any* $g \in H$, $\operatorname{Diam}(h(g)) < \varepsilon$.

PROOF. Since $H^* \subset U$, by a compactness argument, there exists an integer j such that $M_j \subset U$. As we observed in §1, the handles must eventually get small, hence there is an integer $k \geqslant j$ such that the diameter of each handle in

M_{k-1} is less than $\varepsilon/4$. We will show that there is a finite set of $(n-1)$-hyperplanes $P_1, \ldots, P_s$ that are in good position relative to M_m for $m \geqslant k$ and such that each component of $M_k - \bigcup_{i=1}^{s} P_i$ has a diameter less than $\varepsilon/4$.

We obtain the planes $P_1, \ldots, P_s$. The stem of each component A of M_k is of the form $\Phi_A(B^{n-1} \times I)$ where B^{n-1} is the standard $(n-1)$-cell, I is the interval [0, 1], and Φ_A is a homeomorphism. Moreover, there exists an integer $t_A > 0$ such that each component of $A - \bigcup_{i=0}^{t_A} \Phi(B^{n-1} \times i/t_A)$ has diameter less than $\varepsilon/4$. As M_k has only finitely many components, there exists an integer $s = \max\{t_A | A$ is a component of $M_k\}$. Let $P_1', \ldots, P_s'$ be s $(n-1)$-hyperplanes parallel to θ and intersecting the stem of M_1. We can assume that each plane P_i' intersects the stem of each component of M_k in exactly one $(n-1)$-cell. (Consider $P_1', \ldots, P_s'$ to be located close to θ. Let f be a homeomorphism on E^n so that for each component A of M_k, $f(A)$ is in a close regular neighborhood $N(A)$ of the spine of A. We can obtain f so that f pulls A into $N(A)$ radially. Since the stem spine of each A intersects θ in a single point and the P_i' are near θ, we may assume that $N(A) \cap P_i'$ $(i = 1, \ldots, s)$ is an $(n-1)$-cell. We can replace M_k by $\{f^{-1}(A) | A$ is a component of $M_k\}$.) We index the planes $P_1', \ldots, P_s'$ so that they are in good position relative to M_1, and hence relative to M_k, $k > 1$.

There exists a homeomorphism Ψ on E^n such that $\Psi = 1$ outside M_{k-1} and $\Psi(P_i') = P_i$ $(i = 1, \ldots, s)$ is the desired collection of hyperplanes. (For each component A of M_k, Ψ takes a subcollection of the $(n-1)$-cells $\{P_i' \cap A | i = 1, \ldots, s\}$ onto the collection $\{\Phi_A(B^{n-1} \times i/t_A) | i = 0, \ldots, t_A\}$.)

From Lemmas 2.3, 2.4 there is a homeomorphism h_1 on E^n such that $h_1 = 1$ outside M_k and between P_2 and P_{s-1} and h_1 takes each component of M_{k+n-2} onto a set that intersects at most one of P_1 and P_s. Since h_1 preserves good position, we can apply Lemmas 2.3, 2.4 again to each component of $h_1(M_{k+n-2})$ using $P_1, P_2, P_{s-2}, P_{s-1}$ or P_2, P_3, P_{s-1}, P_s for the planes P_a, P_b, P_c, P_d to obtain a homeomorphism h_2 on E^n such that $h_2 = 1$ outside $h_1(M_{k+n-2})$ and $h_2 h_1$ takes each component of M_{k+2n-4} onto a set which intersects at most $s-2$ of the planes $P_1, \ldots, P_s$. In $s-3$ steps we observe that each component of $h_{s-3} \cdots h_2 h_1(M_{k+(s-3)(n-2)})$ hits at most three of the hyperplanes $P_1, \ldots, P_s$ and hence has diameter less than ε. Since every nondegenerate element of G is a subset of $M_{k+(s-3)(n-2)}$, (2) is satisfied. Property (1) follows since h is fixed outside M_j. The proof of Lemma 2.5 is completed.

Bibliography

1. M. L. Antoine, *Sur la possibilité d'etendre l'homeomorphie de deux figures à leurs voisinages*, C. R. Acad. Sci. Paris **171** (1920), 661–663.

2. ———, *Sur l'homeomorphie de deux figures et de leurs voisinages*, J. Math. Pures Appl. **86** (1921), 221–325.

3. R. H. Bing, *A homeomorphism between the 3-sphere and the sum of two solid horned spheres*, Ann. of Math. (2) **56** (1952), 354–362. MR **14**, 192.

4. ———, *Tame Cantor sets in E^3*, Pacific J. Math. **11** (1961), 435–446. MR **24** #A539.

5. ———, *Spheres in E^3*, Amer. Math. Monthly **71** (1964), 353–364. MR **29** #2789.

6. W. A. Blankenship, *Generalization of a construction of Antoine*, Ann. of Math. (2) **53** (1951), 276–297. MR **12**, 730.

7. C. E. Burgess and J. W. Cannon, *Embeddings of surfaces in E^3*, Rocky Mountain J. Math. **1** (1971), 259–344. MR **43** #4008.

8. L. O. Cannon, *On a point-segment decomposition of E^3 defined by McAuley*, Proc. Amer. Math. Soc. **19** (1968), 624–630. MR **37** #897.

9. W. T. Eaton, *A generalization of the dogbone space to E^n*, Proc. Amer. Math. Soc. **39** (1973), 379–387. MR **48** #1238.

10. *Applications of a mismatch theorem to decomposition spaces*, Fund. Math. **89** (1975), 199–224.

11. D. R. McMillan, Jr., *Taming Cantor sets in E^n*, Bull. Amer. Math. Soc. **70** (1964), 706–708. MR **29** #1628.

Department of Mathematics, University of Texas, Austin, Texas 78712

Department of Mathematics, Northern Illinois University, DeKalb, Illinois 60115