TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 225, 1977

ON THE SEQUENCE SPACES $l_{(p_n)}$ AND $\lambda_{(p_n)}$, $0 < p_n \leqslant 1$

BY

S. A. SCHONEFELD AND W. J. STILES

ABSTRACT. Let (p_n) and (q_n) be sequences in the interval $(0, 1]$, let $l_{(p_n)}$ be the set of all real sequences (x_n) such that $\sum |x_n|^{p_n} < \infty$, and let $\lambda_{(q_n)}$ be the set of all real sequences (y_n) such that $\sup_\pi \sum |y_n|^{q_{\pi(n)}} < \infty$ where the sup is taken over all permutations π of the positive integers. The purpose of this paper is to investigate some of the properties of these spaces. Our results are primarily concerned with (1) conditions which are necessary and/or sufficient for $l_{(p_n)}$ (resp., $\lambda_{(p_n)}$) to equal $l_{(q_n)}$ (resp., $\lambda_{(q_n)}$), and (2) isomorphic and topological properties of the subspaces of these spaces.

In connection with (1), we show that the following four conditions are equivalent for any sequence (ε_n) which decreases to zero and has $\varepsilon_1 < 1$. (a) There exists a number $K > 1$ such that the series $\sum 1/K^{1/\varepsilon_n}$ converges; (b) the elements ε_n of the sequence satisfy the condition $\varepsilon_n = O(1/\ln n)$; (c) the sequence $((\ln n)((1/n) \sum_1^n \varepsilon_j))$ is bounded; and (d) $l_{(1-\varepsilon_n)}$ equals l_1. In connection with (2), we show that the following are true when (p_n) increases to one. (a) $\lambda_{(p_n)}$ contains an infinite-dimensional closed subspace where the $l_{(p_n)}$-topology and the $\lambda_{(p_n)}$-topology agree; (b) $l_{(p_n)}$ and $\lambda_{(p_n)}$ contain closed subspaces isomorphic to l_1; and (c) $\lambda_{(p_n)}$ contains no infinite-dimensional subspace where the $\lambda_{(p_n)}$-topology agrees with the l_1-topology if and only if

$$\lim((1/n)^{p_1} + (1/n)^{p_2} + \cdots + (1/n)^{p_n}) = \infty.$$

1. Introduction and summary. If (p_n) is a sequence of numbers in the interval $(0, 1]$, the space $l_{(p_n)}$ is the set of all real sequences $x = (x_n)$ such that $\|x\|_{(p_n)} = \sum |x_n|^{p_n}$ is finite, and the space $\lambda_{(p_n)}$ is the set of all real sequences $y = (y_n)$ such that $|y|_{(p_n)} = \sup_\pi \sum |y_{\pi(n)}|^{p_n}$ is finite where the supremum is taken over the set of all permutations π of the natural numbers. The $l_{(p_n)}$-spaces have been used or studied in many places, e.g., in [2], [7], [8], [10] and [11]; and the $\lambda_{(p_n)}$-spaces, which are nonlocally convex analogues of the symmetric sequence spaces studied in [1], [5], and [6], have been used in [8] and [9].

The purpose of this paper is to investigate a few of the properties of $l_{(p_n)}$ and $\lambda_{(p_n)}$, and we summarize now some of our results. Assume for the

Received by the editors March 29, 1974 and, in revised form, January 7, 1975 and September 9, 1975.

AMS (MOS) subject classifications (1970). Primary 46A45.

Key words and phrases. Sequence spaces, nonlocally convex spaces, symmetric spaces, Schauder bases.

remainder of this section that $0 < p_n < q_n \leqslant 1$. We first generalize some of the results obtained in [10]. In particular, we show that $l_{(p_n)}$ equals $l_{(q_n)}$ if and only if there exists a positive number K such that the series $\sum 1/K^{p_n/(q_n-p_n)} < \infty$. We then use this result to help us show that $l_{(p_n)}$ equals $l_{(q_n)}$ when the sequence $(n(1/n)^{p_n/q_n})$ is bounded, and $l_{(p_n)}$ does not equal $l_{(q_n)}$ when $\lim n(1/n)^{p_n/q_n} = \infty$. If (p_n) increases to p, and $j(n) < k(n) < j(n+1)$, then $l_{(p_{j(n)})}$ equals $l_{(p_{k(n)})}$ (while it is not necessarily true that $l_{(p_n)}$ equals $l_{(p_{2n})}$), and $l_{(p_n)}$ equals l_p if and only if the sequence $(n(1/n)^{a_n/p})$ is bounded where a_n is the arithmetic mean given by $a_n = (1/n)(p_1 + \cdots + p_n)$.

When (p_n) increases to p, we write $l_{(p_n)} \neq^s l_p$ if and only if

$$\lim((1/n)^{p_1/p} + (1/n)^{p_2/p} + \cdots + (1/n)^{p_n/p}) = \infty.$$

Using this definition, we show that when $p = 1$, $l_{(p_n)} \neq^s l_1$ if and only if there does not exist any infinite-dimensional subspace of $\lambda_{(p_n)}$ on which the $\lambda_{(p_n)}$-topology and the l_1-topology agree. We show that every closed infinite-dimensional subspace of $l_{(p_n)}$ (resp., $\lambda_{(p_n)}$) contains an isomorphic copy of l_p when (p_n) increases to p, and $\lambda_{(p_n)}$ always contains an infinite-dimensional subspace where the $\lambda_{(p_n)}$ and the $l_{(p_n)}$ topologies agree when (p_n) increases to p. We also show that when $\lambda_{(p_n)}$ is not equal to $\varnothing$ or to l_1, then $\lambda_{(p_n)}$ is not locally convex, and $l_{(p_n)}$ contains no locally bounded infinite-dimensional subspaces if and only if $\lim p_n = 0$. Finally, we show that any time $\lambda_{(p_n)}$ is isomorphic to $\lambda_{(q_n)}$, where (p_n) and (q_n) increase to one, and $l_{(q_n)} \neq^s l_1$, then $\lambda_{(p_n)}$ must equal $\lambda_{(q_n)}$.

2. **Notation.** In addition to the terminology used in §1, we find it convenient to use the following notation and conventions. The set R_p, $0 < p \leqslant 1$, is the collection of all sequences (x_n) of real numbers in $(0,p)$ which increase to p. The letter R denotes R_1. Unless otherwise stated, the letters p and q will always represent numbers in the interval $(0, 1]$. The symbol $\varnothing$ will represent the space of all finitely nonzero sequences of real numbers equipped with the strongest vector topology. The vector e_n is the vector $(0,\ldots,0,1,0,\ldots)$ where the nonzero entry is in the nth position. A block basic sequence $\{z_n\}$ is a sequence of nonzero vectors of the form $z_n = \sum_{i=k_{n-1}}^{k_n} a_i e_i$ where (n_k) is a strictly increasing sequence of nonnegative integers. The notation $[x_n]_\tau$ denotes the τ-closed linear span of the sequence $\{x_n\}$. The letter E represents the set of all real numbers, and the space $E \oplus l_{(p_n)}$ is the space of all sequences $(x_0, x_1, \ldots)$ such that $x_0 \in E$ and $(x_1, x_2, \ldots) \in l_{(p_n)}$. The equality $(x_n) = O(y_n)$ means that $|x_n| \leqslant M|y_n|$ for some M. For $0 < p \leqslant 1$, l_p has the usual meaning and $\|x\|_p$ is $\sum_1^\infty |x_n|^p$ when $x = (x_1, x_2, \ldots)$. Finally, $\| \|_\infty$ is the usual l_∞-norm of a bounded sequence.

3. **Main results.** Before proving our first theorem, we make some observations. For a fixed p, $0 < p \leqslant 1$, there is an uncountable number of distinct

spaces $l_{(p_n)}$ such that (p_n) is in R_p. Hence the $l_{(p_n)}$ spaces occur in great abundance. If $\inf p_n > 0$, then $l_{(p_n)}$ is locally bounded and a set is bounded in $l_{(p_n)}$ if and only if it is metrically bounded (cf. [10]). Also if (p_n) is an enumeration of the rational numbers in (0, 1), then $l_{(p_n)}$ contains a complemented isomorphic copy of each $l_{(q_n)}$. Hence $l_{(p_n)}$ is universal, in the terminology of [3], for the class $\{l_{(q_n)}: 0 < q_n \leqslant 1\}$. Our first theorem is a standard type result which is useful in the following.

PROPOSITION 1. *Let X be an infinite-dimensional closed subspace of $l_{(p_n)}$ (resp., $\lambda_{(p_n)}$) where $0 < p_n \leqslant 1$ and $\inf p_n > 0$. Then X contains an infinite-dimensional subspace Y which is $l_{(p_n)}$ (resp., $\lambda_{(p_n)}$) isomorphic to a subspace Z of $l_{(p_n)}$ (resp., $\lambda_{(p_n)}$) where the subspace Z is the closed linear span of a block basic sequence.*

PROOF. Let $\{x_j\}$ be a sequence of linearly independent elements of X such that $\|x_j\|_{(p_n)} = 1$. By taking linear combinations and normalizing if necessary, we can assume that $x_n = (0, \ldots, 0, x^n_{k_{n-1}+1}, x^n_{k_{n-1}+2}, \ldots)$ where (k_n) is a strictly increasing sequence of nonnegative integers such that

$$\|(0, \ldots, 0, x^n_{k_n+1}, x^n_{k_n+2}, \ldots)\|_{(p_n)} < \varepsilon_n.$$

Since $l_{(p_n)}$ is locally bounded when $\inf p_n > 0$, we can apply Theorem 1′ of [4]. This theorem implies that $[x_n]_{(p_n)}$ is isomorphic to $[y_n]_{(p_n)}$ if (ε_n) is chosen sufficiently small and $y_n = (0, \ldots, 0, x^n_{k_{n-1}}, \ldots, x^n_{k_n}, 0, \ldots)$; and the isomorphism is the natural mapping taking x_n to y_n for each n.

Since $\lambda_{(p_n)}$ is also locally bounded when $\inf p_n > 0$, the proof just given for $l_{(p_n)}$ also applies to $\lambda_{(p_n)}$. □

The following theorem is a generalization of a result in [10]. Since the proof is similar, it will be omitted.

THEOREM 2. *Suppose that $0 < p_n < q_n \leqslant 1$. Then $l_{(p_n)}$ is equal to $l_{(q_n)}$ if and only if there exists a number $K > 1$ such that $\sum (1/K^{p_n/(q_n - p_n)}) < \infty$ (equivalently $\sum (1/K^{q_n/(q_n - p_n)}) < \infty$).*

COROLLARY 3. *Suppose that $0 < p_n < q_n \leqslant 1$ and $\inf p_n > 0$. Then $l_{(p_n)}$ is equal to $l_{(q_n)}$ if and only if there exists a number $K > 1$ such that $\sum (1/K^{1/(q_n - p_n)}) < \infty$.*

COROLLARY 4. *Suppose (p_n) is in R_p, and suppose $(j(n))$ and $(k(n))$ are increasing sequences of positive integers such that $j(n) < k(n) \leqslant j(n+1)$. Then $l_{(p_{j(n)})} = l_{(p_{k(n)})}$ (and hence $\lambda_{(p_{j(n)})} = \lambda_{(p_{k(n)})}$).*

PROOF. Let b_n be defined by $p_{k(n)} = p_{j(n)} + 1/b_n$. Since $(p_n) \in R_p$, it follows that $\sum_{n=1}^{\infty} (1/b_n) < \infty$. This implies that

$$\sum_{n=1}^{\infty} \frac{1}{2^{1/(p_{k(n)} - p_{j(n)})}} = \sum \frac{1}{2^{b_n}} < \infty.$$

Hence Corollary 3 implies that $l_{(p_{j(n)})} = l_{(p_{k(n)})}$.

It is easy to see that $\lambda_{(r_n)}$ equals $\lambda_{(s_n)}$ when $l_{(r_n)}$ equals $l_{(s_n)}$ because $|x|_{(r_n)} = \infty$ implies $\sum |x_{\pi(n)}|^{r_n} = \infty$ for some π. □

If (p_n) is not required to be in R_p, the conclusion of Corollary 4 does not necessarily follow. For example, consider the following: Let $p_n = 1/2^n, j(n) = n$, and $k(n) = n + 1$. Then $l_{(p_n)} \neq l_{(p_{n+1})}$ follows from Theorem 2. Note also that for this choice of the sequence (p_n), $l_{(p_n)}$ is not equal to $E \oplus l_{(p_n)}$. (However, $l_{(p_n)}$ must be equal to $E \oplus l_{(p_n)}$ when $(p_n) \in R_p$, and $\lambda_{(p_n)}$ must be equal to $E \oplus \lambda_{(p_n)}$ always.)

We will show next that there are $l_{(p_n)}$ spaces where (p_n) is in R_p such that $l_{(p_n)} \neq l_{(p_{2n})}$. This is perhaps somewhat surprising in view of Corollary 4.

THEOREM 5. *There exists a sequence (p_n) in R such that $l_{(p_n)}$ is not equal to $l_{(p_{2n})}$.*

PROOF. Choose a sequence (b_n) of positive numbers such that $\sum_{n=0}^{\infty} (1/b_n) = B < 1$ and $\sum_{n=0}^{\infty} (2^n/K^{b_n})$ diverges for $K = 1, 2, 3, \ldots$. Let $p_1 = 1 - B$, let $p_{2^{k+1}} = p_{2^k} + 1/b_k$ for $k = 0, 1, 2, \ldots$, and let $p_n = p_{2^k}$ for $2^k \leqslant n < 2^{k+1}$, $k = 0, 1, 2, \ldots$. Then for $K > 1$, the series $\sum_{n=1}^{\infty} (1/K^{1/(p_{2n}-p_n)})$ equals the series $\sum_{n=0}^{\infty} (2^n/K^{b_n})$, and hence diverges. Corollary 3 implies that $l_{(p_n)}$ is not equal to $l_{(p_{2n})}$. □

THEOREM 6. *Suppose $0 < p_n < q_n \leqslant 1$ for $n = 1, 2, \ldots$. If $(n(1/n)^{p_n/q_n})$ is a bounded sequence, then $l_{(p_n)}$ is equal to $l_{(q_n)}$. If $\lim n(1/n)^{p_n/q_n} = \infty$, then $l_{(p_n)}$ is not equal to $l_{(q_n)}$.*

PROOF. Let $M_n = n(1/n)^{p_n/q_n}$. Then $p_n/q_n = 1 - (\ln M_n)/(\ln n)$. Hence

$$q_n/(q_n - p_n) = (\ln n)/(\ln M_n).$$

Thus $\sum 1/(K^{q_n/(q_n-p_n)}) = \sum 1/K^{(\ln n)/(\ln M_n)})$. If r is chosen such that $K = e^r$, this last series becomes $\sum 1/(n^{r/\ln M_n})$. The result now follows from Theorem 2.□

It is easy to see and will prove useful to note now that one can actually construct a sequence (p_n) in R_p such that $\lim n(1/n)^{p_n/p} = \infty$.

COROLLARY 7. *If $(p_n) \in R_p$, $0 < p \leqslant 1$, and if the sequence $\{x_n\}$ is $l_{(p_n)}$-bounded where*

$$x_n = (\underbrace{(1/n)^{1/p}, \ldots, (1/n)^{1/p}}_{n \text{ terms}}, 0, \ldots)$$

then $l_{(p_n)}$ is equal to l_p.

PROOF. The conditions imply that $\sup n(1/n)^{p_n/p} < \infty$, and hence the corollary follows from Theorem 6. □

The following example shows that the boundedness condition of Theorem 6 is not necessary.

EXAMPLE. There are sequences (p_n) and (q_n) such that $0 < p_n < q_n \leqslant 1$ for which the sequence $(n(1/n)^{p_n/q_n})$ is not bounded but $l_{(p_n)}$ is equal to $l_{(q_n)}$.

PROOF. One can select the sequences (p_n) and (q_n) and a strictly increasing sequence (n_k) of positive integers such that, in the notation of the proof of Theorem 6, $M_{n_k} \geqslant k$ while $\sum (1/n^{r/\ln M_n})$ converges for some $r > 0$. □

It is not possible to construct an example like the one above when $q_n = q$ for all n and (p_n) is in R_q. This is true because the inequality $n(1/n)^{p_n/q} \leqslant (1/n)^{p_1/q} + (1/n)^{p_2/q} + \cdots + (1/n)^{p_n/q}$ implies that the sequence $\{((1/n)^{1/q}, (1/n)^{1/q}, \ldots, (1/n)^{1/q}, 0, \ldots)\}$ with n nonzero terms would not be $l_{(p_n)}$-bounded while the members of this sequence are l_q-normalized.

The preceding discussion suggests the following question: Suppose (p_n) and (q_n) are in R_p and $p_n \leqslant q_n$ for all n. If there exists a number M such that $\|x\|_{(q_n)} = 1$ implies $\|x\|_{(p_n)} \leqslant M$ for all x of the form $x = (r, r, \ldots, r, 0, \ldots)$, must $\lambda_{(p_n)}$ equal $\lambda_{(q_n)}$?

THEOREM 8. *Suppose that* $0 < p_n < q_n \leqslant 1$. *Then* $l_{(p_n)}$ *equals* $l_{(q_n)}$ *if and only if there exists a permutation* π *of the positive integers such that* $(q_{\pi(n)} - p_{\pi(n)})/p_{\pi(n)} = O(1/\ln n)$.

PROOF. Suppose there exists a permutation π having the above property. We can assume that π is the identity. Then there exists an M such that $(q_n - p_n)/p_n \leqslant M/\ln n$. Thus

$$p_n/(q_n - p_n) \geqslant (\ln n)/M$$

and

$$\sum 1/(K^{p_n/(q_n - p_n)}) \leqslant \sum 1/(K^{(\ln n)/M}) \leqslant \sum 1/(n^{(\ln n)/M}).$$

Hence the series $\sum 1/(K^{p_n/(q_n-p_n)})$ converges for all large K. This implies, by Theorem 2, that $l_{(p_n)} = l_{(q_n)}$.

Conversely, if $l_{(p_n)}$ equals $l_{(q_n)}$, there exists a $K > 1$ such that the series $\sum 1/(K^{p_n/(q_n-p_n)})$ converges by Theorem 2. Choose a permutation π such that the terms of the series $\sum 1/(K^{p_{\pi(n)}/(q_{\pi(n)}-p_{\pi(n)})})$ are in decreasing order. Then

$$\lim n(1/(K^{p_{\pi(n)}/(q_{\pi(n)}-p_{\pi(n)})})) = 0.$$

Thus $\ln n - (p_{\pi(n)}/(q_{\pi(n)} - p_{\pi(n)}))(\ln K) < 0$ for all large values of n. This implies that $(q_{\pi(n)} - p_{\pi(n)})/p_{\pi(n)} \leqslant (\ln K)/(\ln n)$ for all large values of n. □

We omit the proof by Corollary 9 and Corollary 10 because these proofs are implicitly contained in [10].

COROLLARY 9. *If* $(p_n) \in R_p$, *then* $l_{(p_n)}$ *equals* l_p *if and only if* $(p - p_n) = O(1/\ln n)$.

COROLLARY 10. *If* $0 < p_n \leqslant q_n \leqslant 1$ *and if* $\sum (q_n/p_n - 1) < \infty$, *then* $l_{(p_n)}$ *equals* $l_{(q_n)}$.

PROPOSITION 11. *Suppose* $(p_n) \in R_p$ *and* $a_n = (1/n)(p_1 + p_2 + \cdots + p_n)$; *then* $l_{(p_n)} = l_p$ *if and only if the sequence* $(n(1/n)^{a_n/p})_{n=1}^{\infty}$ *is bounded.*

PROOF. It is a well-known fact that the geometric mean is less than or equal to the arithmetic mean, i.e., $(c_1 c_2 \cdots c_n)^{1/n} \leqslant (1/n)(c_1 + c_2 + \cdots c_n)$. If we let $c_k = (1/n)^{p_k/p}$, $k = 1, 2, \ldots, n$, in this last expression, we obtain the inequality

$$[(1/n)^{p_1/p}(1/n)^{p_2/p} \cdots (1/n)^{p_n/p}]^{1/n}$$
$$\leqslant (1/n)[(1/n)^{p_1/p} + (1/n)^{p_2/p} + \cdots + (1/n)^{p_n/p}].$$

Hence $n(1/n)^{a_n/p} \leqslant (1/n)^{p_1/p} + \cdots + (1/n)^{p_n/p}$. The sequence $\{x_n\}$ where $x_n = ((1/n)^{1/p}, (1/n)^{1/p}, \ldots, (1/n)^{1/p}, 0, \ldots)$ is (topologically) bounded in l_p. Thus if $l_{(p_n)} = l_p$, $\{x_n\}$ is bounded in $l_{(p_n)}$, and this implies that $(n(1/n)^{a_n/p})$ is bounded by the last inequality. Conversely, suppose $(n(1/n)^{a_n/p})$ is bounded. Since $a_n \leqslant p_n$, $n(1/n)^{a_n/p} \geqslant n(1/n)^{p_n/p}$. Hence Theorem 6 implies that $l_{(p_n)} = l_p$. □

Combining Corollary 3, Corollary 9, and Proposition 11, we see the following four conditions are equivalent for any sequence (ε_n) converging monotonically to zero such that $\varepsilon_1 < 1$: (1) There exists a number $K > 1$ such that $\sum 1/K^{1/\varepsilon_n} < \infty$; (2) $\varepsilon_n = O(1/(\ln n))$; (3) $((\ln n)((1/n)(\sum_{j=1}^{n} \varepsilon_j)))$ is a bounded sequence; and (4) $l_{(1-\varepsilon_n)} = l_1$.

COROLLARY 12. *Suppose* $(p_n) \in R_p$ *and* $l_{(p_n)} = l_p$; *then* $l_{(a_n)} = l_p$ *where* $a_n = (1/n)(p_1 + p_2 + \cdots + p_n)$.

PROOF. If $l_{(p_n)}$ equals l_p, then Proposition 11 implies that $n(1/n)^{a_n/p} \leqslant M$, for some M. Hence, the proof of Theorem 6 implies that $l_{(a_n)}$ equals l_p. □

Unfortunately we do not know the answer to the following question: If (p_n) is in R_p and if $a_n = (1/n)(p_1 + \cdots + p_n)$, is $l_{(a_n)}$ equal to $l_{(p_n)}$?

THEOREM 13. *If* $(p_n) \in R_p$, *then* $\lambda_{(p_n)} = l_{(p_n)}$ *if and only if* $l_{(p_n)} = l_p$.

PROOF. Clearly $\lambda_{(p_n)} \subset l_{(p_n)} \subset l_p$. Suppose $l_{(p_n)} = l_p$ and $(x_n) \in l_{(p_n)}$. Then $(x_{\pi(n)}) \in l_{(p_n)}$ for any permutation π. If $\{y_n\} \in l_{(p_n)} \backslash \lambda_{(p_n)}$, then one can find a permuation π such that $\{y_{\pi(n)}\} \notin l_{(p_n)}$. Hence $\lambda_{(p_n)} = l_p$. Conversely, suppose that $\lambda_{(p_n)} = l_{(p_n)}$ and $l_{(p_n)} \neq l_p$. Choose an element $x = (x_1, x_2, \ldots)$ in $l_p \backslash l_{(p_n)}$. Clearly one can find an element of the form $\tilde{x} = (x_1, 0, \ldots, 0, x_2, 0, \ldots, 0, x_3, \ldots)$ which is in $l_{(p_n)}$. Since $l_{(p_n)} = \lambda_{(p_n)}$, $\tilde{x}$ is in $\lambda_{(p_n)}$. However,

this implies that x is in $\lambda_{(p_n)}$ which is clearly impossible. □

We remarked after the proof of Theorem 6 that there are sequences (p_n) in R_p such that $\lim n(1/n)^{p_n/p} = \infty$. One can also construct a sequence (p_n) in R such that $l_{(p_n)} \neq l_1$ while $l_{(p_n)}$ has the following property: There exists a subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ (where $x_n = (1/n, \ldots, 1/n, 0, \ldots)$ has n nonzero terms) such that $(\|x_{n_k}\|_{(p_n)})$ is bounded. These facts lead us to make the following definition.

DEFINITION. *If (p_n) is in R_p, then $l_{(p_n)}$ is strongly not equal to l_p (written $l_{(p_n)} \neq^s l_p$) if and only if $\lim \|x_m\|_{(p_n)} = \infty$ where $x_m = ((1/m)^{1/p}, (1/m)^{1/p}, \ldots, (1/m)^{1/p}, 0, \ldots)$ has m nonzero entries.*

In order to utilize this definition, we need the following lemmas.

LEMMA 14. *Suppose $0 < b < 1$, $(p_n) \in R$, $rb \geqslant 1$, and $p_n < 1$; then the minimum, $\min\{\|x\|_{(p_n)}: x = (x_1, \ldots, x_r, 0, \ldots), \|x\|_\infty = b, \|x\|_1 = 1$, and $x_1 \geqslant x_2 \geqslant \cdots \geqslant 0\}$ is attained at a point of the form $x = (b, b, \ldots, b, c, c, \ldots, c, 0, \ldots)$ where $c \geqslant 0$.*

PROOF. Assume that the minimum does not occur at a point of the form given above. Then the minimum must occur at a point x having at least three distinct entries. We will show that x cannot be of the form

$$x = (b, \ldots, b, \underset{n_1}{y}, \cdots, \underset{n_2}{y}, z, \cdots, \underset{n_3}{z}, 0, \cdots)$$

where $b > y > z > 0$. It will be clear from the proof of this claim that x cannot have more than two distinct entries. We note if $0 < d < 1$, the function g_d given by $g_d(t) = t^p + (d-t)^q$, $0 \leqslant t \leqslant d$, $0 < p \leqslant q < 1$, is strictly increasing in $[0, t_0]$ and strictly decreasing in $[t_0, d]$ where t_0 is the solution to the equation $p/t^{1-p} = q/(d-t)^{1-q}$. We note further that if $0 < r \leqslant p$ and $q \leqslant s < 1$, then the solution to $r/t^{1-r} = s/(d-t)^{1-s}$ is not greater than t_0. Thus if one cannot lower the value of y in the n_2 entry and raise the value of z in the $n_2 + 1$ entry the same amount to decrease the value of $\|x\|_{(p_n)}$, then it must be true that y is greater than t_0. But if y is greater than t_0, then y is greater than the solution to the equation $p_{n_1}/t^{1-p_{n_1}} = p_{n_3}/t^{1-p_{n_3}}$. Hence we can increase y in the n_1 entry and decrease z in the n_3 entry a comparable amount to decrease the value of $\|x\|_{(p_n)}$. This contradicts the fact that x was chosen so that $\|x\|_{(p_n)}$ was a minimum. □

The following lemma generalizes a result contained in [8].

LEMMA 15. *Given any positive number ε, there exists a positive number δ such that $0 < a < b < \delta$ implies that $a^p + b^q < a^q + b^p$ whenever $\varepsilon \leqslant p < q \leqslant 1$.*

PROOF. Suppose that the lemma is not true. Then there exist convergent sequences (a_n), (b_n), (p_n), and (q_n) such that $\varepsilon \leqslant p_n < q_n \leqslant 1$, $0 < a_n < b_n$

$< 1/n$, and $a_n^{p_n} + b_n^{q_n} \geqslant a_n^{q_n} + b_n^{p_n}$. By the intermediate value theorem, there exists a number c_n in $(0, a_n)$ such that $c_n^{p_n} + b_n^{q_n} = c_n^{q_n} + b_n^{p_n}$. Let $f_n(x) = x^{p_n}$, and let $g_n(x) = x^{q_n}$. Applying the generalized mean value theorem to f_n and g_n on $[c_n, b_n]$, we obtain ξ_n in (c_n, b_n) such that

$$1 = \frac{b_n^{p_n} - c_n^{p_n}}{b_n^{q_n} - c_n^{q_n}} = \frac{p_n \xi_n^{(p_n - 1)}}{q_n \xi_n^{(q_n - 1)}}.$$

Thus $\xi_n = (q_n/p_n)^{1/(p_n - q_n)}$. Let $p = \lim p_n$ and $q = \lim q_n$. If $p \neq q$, then $\lim \xi_n = (q/p)^{1/(p-q)}$, and $\lim \xi_n \neq 0$. This contradicts the fact that $0 < \xi_n < 1/n$. If $p = q$, let $s_n = (q_n - p_n)/p_n$. Then $\xi_n = [(1 + s_n)^{1/s_n}]^{-1/p_n}$. Hence $\lim \xi_n = e^{-1/p}$, $p \geqslant \varepsilon > 0$. Again this contradicts the fact that $0 < \xi_n < 1/n$. □

DEFINITION. *Let $x = (x_1, x_2, \dots)$ be any sequence of real numbers such that $\lim x_n = 0$. Then $\hat{x} = (|x_{j_1}|, |x_{j_2}|, \dots)$ where $|x_{j_1}| = \max_j\{|x_j|\}$, and $|x_{j_n}| = \max_{j \neq j_1, \dots, j_{n-1}}\{|x_j|\}$, for $n = 2, 3, \dots$.*

THEOREM 16. *If $(p_n) \in R_p$, there exists a positive number ε such that $|x|_{(p_n)} = \|\hat{x}\|_{(p_n)}$ whenever $\|x\|_\infty < \varepsilon$.*

PROOF. This follows immediately from Lemma 15. □

We have already observed in the proof of Corollary 4 that $\lambda_{(p_n)}$ equals $\lambda_{(q_n)}$ when $l_{(p_n)}$ equals $l_{(q_n)}$. The converse of this statement is trivially false when we allow $\lim p_n$ to be zero, for in this case $\lambda_{(p_n)}$ equals $\varnothing$, but $l_{(p_n)}$ is not equal to $\varnothing$. We are now in a position to show that the converse is false even when the sequence (p_n) is bounded away from zero. This fact is shown in Theorem 17.

THEOREM 17. *There are sequences (p_n) and (q_n) in R such that $0 < p_n \leqslant q_n$ and $\lambda_{(p_n)} = \lambda_{(q_n)}$ while $l_{(p_n)} \neq l_{(q_n)}$.*

PROOF. Choose (p_n) in R as in the proof of Theorem 5. Then $l_{(p_n)}$ is not equal to $l_{(p_{2n})}$; but $l_{(p_{2n})}$ is equal to $l_{(p_{2n-1})}$ by Corollary 4. Let $q_n = p_{2n-1}$ for $n = 1, 2, \dots$. Clearly $p_n \leqslant p_{2n-1} = q_n$, and this implies that $\lambda_{(q_n)}$ contains $\lambda_{(p_n)}$. Let $x = (x_1, x_2, \dots)$ be an element of $\lambda_{(q_n)}$. Without loss of generality, we can assume that $x_1 \geqslant x_2 \geqslant x_3 \geqslant \cdots \geqslant 0$ and $x_1 < \varepsilon$ where $\varepsilon > 0$ is the number given in the statement of Theorem 16. Note that

$$\frac{\sum_{i=1}^n x_i^{p_i}}{2\sum_{i=1}^n x^{q_i}} \leqslant \frac{\sum_{i=1}^n x^{p_i}}{(x_1^{p_1} + x_2^{p_3} + \cdots + x_n^{p_{2n-1}}) + (x_1^{p_2} + x_2^{p_4} + \cdots + x_n^{p_{2n}})}$$
$$\leqslant \frac{\sum_{i=1}^n x_i^{p_i}}{\sum_{i=1}^{2n} x_i^{p_i}} \leqslant 1.$$

Hence $|x|_{(p_n)} \leqslant 2|x|_{(q_n)}$, and this implies that $\lambda_{(p_n)} \supset \lambda_{(q_n)}$. □

We now turn our attention to some theorems involving the notion $l_{(p_n)} \neq^s l_p$.

THEOREM 18. *Suppose* $(p_n) \in R$, $x_k = (x_{k1}, x_{k2}, \dots)$, $x_{k1} \geqslant x_{k2} \geqslant \cdots \geqslant 0$, $\|x_k\|_1 = 1$ *and* $\lim \|x_k\|_\infty = 0$; *then* $\lim \|x_k\|_{(p_n)} = \infty$ *if* $l_{(p_n)} \neq^s l_1$.

PROOF. Suppose $l_{(p_n)} \neq^s l_1$ and x_k satisfies the above conditions. By using Lemma 14, we can construct a sequence $\{y_k\}$ such that $\frac{1}{2}\|y_k\|_\infty = \|x_k\|_\infty = b_k$, $\|y_k\|_1 = 1$, $\|y_k\|_{(p_n)} \leqslant 2\|x_k\|_{(p_n)}$, and $y_k = (2b_k, \dots, 2b_k, c_k, \dots, c_k, 0, \dots)$, $0 \leqslant c_k \leqslant 2b_k$. Since $\lim b_k = 0$ and $\|y_k\|_1 = 1$, the fact that $l_{(p_n)} \neq^s l_1$ implies that $\lim \|y_k\|_{(p_n)} = \infty$. This of course implies that $\lim \|x_k\|_{(p_n)} = \infty$. □

We are unable to determine if Theorem 18 generalizes to l_p in the following way: Suppose (p_n) is in R_p, $\|x_k\|_p = 1$, and $\lim \|x_k\|_\infty = 0$ where x_k is as above. Is $\lim \|x_k\|_{(p_n)} = \infty$ if $l_{(p_n)} \neq^s l_p$?

COROLLARY 19. *Suppose* $(p_n) \in R$, $x_k \in \lambda_{(p_n)}$, $\|x_k\|_1 = 1$, $l_{(p_n)} \neq^s l_1$, *and* $\lim_{k\to\infty} \|x_k\|_\infty = 0$; *then* $\lim |x_k|_{(p_n)} = \infty$.

PROOF. If $l_{(p_n)} \neq^s l_1$, then Theorem 18 implies the result because any rearrangement of a sequence in $\lambda_{(p_n)}$ has the same $\lambda_{(p_n)}$ norm. □

THEOREM 20. *Suppose* $(p_n) \in R$; *then* $l_{(p_n)} \neq^s l_1$ *if and only if there is no infinite-dimensional subspace of* $\lambda_{(p_n)}$ *on which the* $\lambda_{(p_n)}$- *and the* l_1-*topologies agree.*

PROOF. Suppose $l_{(p_n)} \neq^s l_1$ and the $\lambda_{(p_n)}$-topology and the l_1-topology agree on an infinite-dimensional subspace X. By checking the proof of Proposition 1, we observe that we may assume that X is closed and contains a block basic sequence $\{x_n\}$. By taking linear combinations of the x_n's, we can obtain a sequence $\{y_n\}$ in X such that $\|y_n\|_1 = 1$ and $\|y_n\|_\infty \to 0$. By Corollary 19, $|y_n|_{(p_n)} \to \infty$. This is a contradiction.

Conversely, suppose that it is not true that $l_{(p_n)} \neq^s l_1$. Then there exists a strictly increasing sequence (m_k) of positive integers such that

$$x_k = (x^k, x^k, \dots, \underset{m_k}{x^k}, 0, \cdots)$$

has the property that $\|x_k\|_1 = 1$ and $\|x_k\|_{(p_n)} \leqslant M$ for some M and all k. Choose a strictly increasing subsequence (n_k) of the sequence (m_k) such that $l_{(p_{n_k})}$ is equivalent to l_1. Then Theorem 13 implies that $\lambda_{(p_{n_k})}$ is equivalent to l_1. Let

$$y_1 = (y^1, \dots, \underset{n_1}{y^1}, 0, \cdots),$$

$$y_2 = (0, \dots, 0, \underset{n_1}{y^2}, \cdots, \underset{n_1+n_2}{y^2}, 0, \cdots),$$

$$y_3 = (0, \dots, \underset{n_1+n_2+1}{y^3}, \cdots, \underset{n_1+n_2+n_3}{y^3}, 0, \cdots),$$

etc., where y^j is chosen such that $\|y_j\|_1 = 1$. The conditions imply that $|y_j|_{(p_n)} \leqslant M$. Hence if $\sum a_j y_j$ is any element in $[y_j]_{\lambda_{(p_n)}}$, Theorem 16 implies that there is a permutation π of the natural numbers such that

$$\begin{aligned}
|\textstyle\sum a_j y_j|_{(p_n)} &= (|a_{\pi(1)}y^{\pi(1)}|^{p_1} + |a_{\pi(1)}y^{\pi(1)}|^{p_2} + \cdots + |a_{\pi(1)}y^{\pi(1)}|^{p_{n_{\pi(1)}}}) \\
&\quad + (|a_{\pi(2)}y^{\pi(2)}|^{p_{n_{\pi(1)}+1}} + |a_{\pi(2)}y^{\pi(2)}|^{p_{(n_{\pi(1)}+2)}} + \cdots + |a_{\pi(2)}y^{\pi(2)}|^{p_{(n_{\pi(1)}+n_{\pi(2)})}}) + \cdots \\
&\leqslant M(|a_{\pi(1)}|^{p_1} + |a_{\pi(2)}|^{p_{n_{\pi(1)}}} + |a_{\pi(3)}|^{p_{(n_{\pi(1)}+n_{\pi(2)})}} + \cdots) \\
&\leqslant M(|a_{\pi(1)}|^{p_1} + |a_{\pi(2)}|^{p_{n_{\pi(1)}}} + |a_{\pi(3)}|^{p_{n_{\pi(2)}}} + \cdots) \\
&\leqslant M|(a_1, a_2, \ldots)|_{(p_{n_k})}
\end{aligned}$$

when the $|a_j|, j = 1, 2, \ldots$, are sufficiently small. Since $|\ |_{(p_{n_k})}$ is equivalent to $\|\ \|_1$, the above implies that the l_1-topology on $[y_j]_{\lambda_{(p_n)}}$ is stronger than the $\lambda_{(p_n)}$-topology on $[y_j]_{\lambda_{(p_n)}}$. This implies that the two topologies agree on $[y_j]_{\lambda_{(p_n)}}$. □

If (q_n) increases "rapidly" to $q = 1$, then the unit vector basis in $l_{(q_n)}$ is equivalent to the unit vector basis in l_1, and the $l_{(q_n)}$-topology agrees with the l_1-topology on $l_{(q_n)}$. Hence we have the following.

COROLLARY 21. *Suppose $(p_n) \in R$, $l_{(p_n)} \neq^s l_1$, and (n_k) is an increasing sequence of positive integers chosen such that the $l_{(p_n)}$-basic sequence $\{e_{n_k}\}$ is equivalent to the unit vector basis in l_1. Then the $\lambda_{(p_n)}$-closed linear span of $\{e_{n_k}\}$ contains no infinite-dimensional subspace where the $l_{(p_n)}$-topology and the $\lambda_{(p_n)}$-topology agree.*

If $l_{(p_n)} \neq l_1$, then $l_{(p_n)}$ is not locally convex. This is easy to show, and is shown in [10]. Proposition 22 shows that the analogous result holds in $\lambda_{(p_n)}$.

PROPOSITION 22. *If $\lambda_{(p_n)}$ is not equal to $\varnothing$ or to l_1, then $\lambda_{(p_n)}$ is not locally convex.*

PROOF. If $\lambda_{(p_n)} \neq \varnothing$ there exists p, $0 < p \leqslant 1$, such that $p \leqslant p_n$ for all n. The set $\{x \in \lambda_{(p_n)}: |x|_{(p_n)} < \varepsilon\}$ contains the set $\{x \in \lambda_{(p_n)}: \|x\|_p < \varepsilon\}$. Since the convex hull of the last set contains an "l_1 ball", the convex hull of the first set contains an "l_1 ball". This implies that the strongest locally convex topology weaker than the $\lambda_{(p_n)}$-topology is the l_1-topology. Hence, if $\lambda_{(p_n)}$ is locally convex it must be equal to l_1. □

PROPOSITION 23. *Suppose (p_n) is in R_p and $\{x_n\}$ is a block basic sequence such that $|x_n|_{(p_n)} = 1$ and $0 < b \leqslant \|x_n\|_\infty \leqslant B$. Then the $\lambda_{(p_n)}$-closed linear span of $\{x_n\}$ is isomorphic to $\lambda_{(p_n)}$.*

PROOF. Let T be the linear mapping of $[x_n]$ into $\lambda_{(p_n)}$ defined by $Tx_n = e_n$. If $(c_1, c_2, \ldots, c_k, 0, \ldots)$ is any finitely nonzero sequence such that $|c_j| \leqslant 1$,

then there exists a permutation π of the natural numbers such that $|(c_1, c_2, \ldots, c_k, 0, \ldots)|_{(p_n)} = |c_1|^{p_{\pi(1)}} + \cdots + |c_k|^{p_{\pi(k)}}$. Let $\|x_i\|_\infty = a_i$. Then $\varepsilon \leqslant a_i \leqslant 1$ implies that $a_i^{p_{\pi(i)}}/\varepsilon \geqslant 1$. Hence

$$\begin{aligned} |(c_1, c_2, \ldots, c_k, 0, \ldots)|_{(p_n)} &= c_1^{p_{\pi(1)}} + \cdots + c_k^{p_{\pi(k)}} \\ &\leqslant (1/\varepsilon)[(c_1 a_1)^{p_{\pi(1)}} + \cdots + (c_k a_k)^{p_{\pi(k)}}] \\ &\leqslant (1/\varepsilon)[|c_1 x_1 + \cdots + c_k x_k|_{(p_n)}]. \end{aligned}$$

This shows that T is continuous. Also

$$|c_1 x_1 + \cdots + c_k x_k|_{(p_n)} \leqslant |(c_1, \ldots, c_k, \quad 0, \ldots)|_{(p_n)}$$

implies that T^{-1} is continuous. Since T is one-to-one, the theorem follows. □

The $l_{(p_n)}$ part of the next theorem appears in [11], but its proof is also included here for convenience.

THEOREM 24. *If (p_n) is in R_p, then any infinite-dimensional closed subspace X of $l_{(p_n)}$ (resp., $\lambda_{(p_n)}$) contains a subspace isomorphic to l_p.*

PROOF. We first prove the theorem for $l_{(p_n)}$. By Proposition 1, we can assume that X contains an $l_{(p_n)}$-normalized block basic sequence $\{x_n\}$, and by taking linear combinations if necessary, we can assume that

$$x_n = (0, \ldots, 0, x_{k_n}^n, \ldots, x_{k_{n+1}-1}^n, 0, \ldots)$$

where $\{k_n\}$ is an increasing sequence of positive integers such that $l_{(p_{k_n})} = l_p$.

Let T: $[x_n] \to l_p$ be the linear map satisfying $T(x_k) = e_k$. Since

$$\|c_1 x_1 + c_2 x_2 + \cdots\|_{(p_n)} \leqslant |c_1|^{p_{k_1}} + |c_2|^{p_{k_2}} + |c_3|^{p_{k_3}} + \cdots$$

when $|c_j| \leqslant 1$, and since $l_{(p_{k_n})} = l_p$, T must map onto all of l_p. Since T is clearly continuous, the open mapping theorem implies that T is an isomorphism, and this completes the proof for $l_{(p_n)}$.

Let X be a closed infinite-dimensional subspace of $\lambda_{(p_n)}$. Because of Proposition 1, we can assume that X contains a block basic sequence $\{x_n\}$. By taking linear combinations of the x_n's, we can obtain a block basic sequence $\{y_k\}$ such that $\|y_k\|_\infty = a_k$, $|y_k|_{(p_n)} = 1$, and each y_k is of the form

$$y_k = (0, \ldots, 0, *, \ldots, *, a_k, *, \ldots, *, a_k, *, \ldots, *, a_k, *, \ldots, *, 0, \ldots)$$

where y_k contains m_k a_k's and (m_k) is a strictly increasing sequence of positive integers chosen so that $l_{(p_{s_j})} = l_p$ where $s_j = \sum_{i=1}^{j} m_i$. Then Theorem 13 implies that $\lambda_{(p_{s_j})} = l_p$. Let T be the mapping of the $\lambda_{(p_n)}$-closed linear span of $\{y_k\}$ into l_p satisfying $T(y_k) = e_k$. We will show that T is a $\lambda_{(p_n)}$-to-(l_p)

isomorphism. Let (c_k) be an element of l_p. Then by Theorem 16, if $\|(c_k)\|_p$ is sufficiently small, we have

$$\begin{aligned}|\textstyle\sum c_k y_k|_{(p_n)} &= \|(\textstyle\sum c_k y_k)^{\wedge}\|_{(p_n)} \\ &= |c_1 a_1|^{p_{k_1}} + \cdots + |c_1 a_1|^{p_{k_1+m_1-1}} + (\text{other } c_1 y_1 \text{ terms}) \\ &\quad + |c_2 a_2|^{p_{k_2}} + \cdots + |c_2 a_2|^{p_{k_2+m_2-1}} + (\text{other } c_2 y_2 \text{ terms}) + \cdots \\ &\leqslant |c_1|^{p_{s_{\pi(1)}}} + |c_2|^{p_{s_{\pi(2)}}} + |c_3|^{p_{s_{\pi(3)}}} + \cdots\end{aligned}$$

for some permutation π of the natural numbers. This series converges since $\lambda_{(p_{s_i})}$ equals l_p, and thus T is onto. Clearly T is one-to-one, and the graph of T is closed because $\{y_k\}$ is a $\lambda_{(p_n)}$-Schauder basis and $\{e_k\}$ is an l_p-Schauder basis. Hence T is continuous by the Closed Graph Theorem. The Open Mapping Theorem then implies that T is an isomorphism. □

It is interesting to compare Theorem 20 and Theorem 24: Together these theorems show that there are cases where (p_n) is in R and $l_{(p_n)}$ has subspaces isomorphic to l_1 but these subspaces do not have the topology "inherited" from l_1. It is also interesting to note that there are choices of (p_n) in R_p such that the $\lambda_{(p_n)}$- and the l_p-topology do not agree on any infinite-dimensional subspaces (cf. Theorem 20) while Theorem 25 below shows that there is always an infinite-dimensional subspace where the $\lambda_{(p_n)}$- and the $l_{(p_n)}$-topology agree when (p_n) is in R.

THEOREM 25. *If (p_n) is in R_p, then $\lambda_{(p_n)}$ contains an infinite-dimensional subspace where the $\lambda_{(p_n)}$-topology and the $l_{(p_n)}$-topology agree.*

PROOF. Let

$$\begin{aligned}x_1 &= (x^1, \ldots, \underbrace{x^1}_{m_1}, 0, \cdots), \\ x_2 &= (0, \ldots, 0, \underbrace{x^2, \cdots, x^2}_{m_1+1}, \underbrace{0, \cdots}_{m_1+m_2}), \\ x_3 &= (0, \ldots, 0, \underbrace{x^3, \cdots, x^3}_{m_1+m_2+1}, \underbrace{0, \cdots}_{m_1+m_2+m_3}),\end{aligned}$$

etc; where (m_k) is a strictly increasing sequence of positive integers chosen so that $m_0 = 1$, $\|x_j\|_{(p_n)} = 1$, $|x_j|_{(p_n)} \leqslant 2$, and $l_{(p_{m_k})}$ equals l_p. Note that Theorem 13 implies that $\lambda_{(p_{m_k})}$ also equals l_p. Let $x = \sum a_j x_j$ where $a = (a_1, a_2, \ldots)$ is a finitely nonzero sequence. If $|a_j|, j = 1, 2, \ldots,$ is sufficiently small, Theorem 16 implies that there exists a permutation π of the positive integers such that

$$\begin{aligned}|x|_{(p_n)} &= |a_{\pi(1)} x^{\pi(1)}|^{p_1} + \cdots + |a_{\pi(1)} x^{\pi(1)}|^{p_{m_{\pi(1)}}} \\ &\quad + |a_{\pi(2)} x^{\pi(2)}|^{p_{(m_{\pi(1)}+1)}} + \cdots + |a_{\pi(2)} x^{\pi(2)}|^{p_{(m_{\pi(1)}+m_{\pi(2)})}} + \cdots \\ &\leqslant 2(|a_{\pi(1)}|^{p_1} + |a_{\pi(2)}|^{p_{m_{\pi(1)}}} + |a_{\pi(3)}|^{p_{m_{\pi(2)}}} + \cdots).\end{aligned}$$

The above implies that $|x|_{(p_n)} \leqslant 2|a|_{(p_{m_k})}$. Also when $|a_j| \leqslant 1$, we have

$$\begin{aligned}
&|a_1|^{p_{m_1}} + |a_2|^{p_{(m_1+m_2)}} + |a_3|^{p(m_1+m_2+m_3)} + \cdots \\
&\quad = |a_1|^{p_{m_1}}\|x_1\|_{(p_n)} + |a_2|^{p_{(m_1+m_2)}}\|x_2\|_{(p_n)} + |a_3|^{p_{(m_1+m_2+m_3)}}\|x_3\|_{(p_n)} + \cdots \\
&\quad \leqslant (|a_1 x^1|^{p_1} + \cdots + |a_1 x^1|^{p_{m_1}}) + (|a_2 x^2|^{p_{(m_1+1)}} + \cdots + |a_2 x^2|^{p_{(m_1+m_2)}}) + \cdots \\
&\quad = \|x\|_{(p_n)}.
\end{aligned}$$

If we let $s_j = \sum_1^j m_i$, then $\| \ \|_{(p_{s_j})}$ is equivalent to $\| \ \|_p$. Since $| \ |_{(p_{m_k})}$ is also equivalent to $\| \ \|_p$, the above inequalities imply that the $l_{(p_n)}$-topology is stronger than the $\lambda_{(p_n)}$-topology on $sp(x_n)$. This means that the two topologies agree on $sp(x_n)$ and hence on the closure of $sp(x_n)$. □

THEOREM 26. *Suppose (p_n) and (q_n) are in R, $l_{(q_n)} \neq^s l_1$, and $\lambda_{(p_n)}$ is isomorphic to $\lambda_{(q_n)}$; then $\lambda_{(p_n)}$ equals $\lambda_{(q_n)}$.*

PROOF. Let T: $\lambda_{(p_n)} \to \lambda_{(q_n)}$ be the given isomorphism, and let $f_k = T(e_k)$, $k = 1, 2, \ldots$. Since $\{f_n\}$ is a $\lambda_{(q_n)}$-bounded sequence, the set of the kth coordinates of the sequence $\{f_n\}$ forms a bounded set. Thus we can apply the proof of Proposition 1 to construct two strictly increasing sequences (m_j) and (n_j) of positive integers such that $m_j < n_j < m_{j+1}$ and the sequence $\{f_{m_j} - f_{n_j}\}$ is a $\lambda_{(q_n)}$-basic sequence equivalent to the "truncated" block basic sequence $\{\tilde{f}_{m_j} - \tilde{f}_{n_j}\}$. Since T can be extended to an l_1-to-l_1 isomorphism, the sequence $\{\tilde{f}_{m_j} - \tilde{f}_{n_j}\}$ is l_1-bounded away from zero. Thus Corollary 19 implies that the sequence $(\|\tilde{f}_{m_j} - \tilde{f}_{n_j}\|_\infty)$ is bounded away from zero. Hence Proposition 23 implies that $\lambda_{(q_n)}$ is isomorphic to $[\tilde{f}_{m_j} - \tilde{f}_{n_j}]_{\lambda_{(q_n)}}$ and hence to $[f_{m_j} - f_{n_j}]_{\lambda_{(q_n)}}$. Since $\lambda_{(p_n)}$ is isomorphic to $[e_{m_j} - e_{n_j}]_{\lambda_{(p_n)}}$ and all of these isomorphisms are given by the natural mappings, $\lambda_{(p_n)}$ equals $\lambda_{(q_n)}$. □

We are unable to answer the following question: If (p_n) and (q_n) are in R_p and if $\lambda_{(p_n)}$ (resp., $l_{(p_n)}$) is isomorphic to $\lambda_{(q_n)}$ (resp., $l_{(q_n)}$), must $\lambda_{(p_n)}$ (resp., $l_{(p_n)}$) equal $\lambda_{(q_n)}$ (resp., $l_{(q_n)}$)?

We mentioned at the beginning of this section that if $p_n \geqslant p > 0$, then $l_{(p_n)}$ is a locally bounded space and a subset of this space is bounded if and only if the subset is metrically bounded. (It is easy to see that the same is also true for $\lambda_{(p_n)}$.) The following is an extension of this idea.

THEOREM 27. *Suppose $0 < p_n < 1$ for all n. Then $l_{(p_n)}$ contains no infinite-dimensional locally bounded subspace if and only if* $\lim p_n = 0$.

PROOF. If $\lim p_n \neq 0$, then (p_n) contains a subsequence (q_n) which is bounded away from 0. Since $l_{(q_n)}$ is locally bounded, then $l_{(p_n)}$ contains a locally bounded subspace. Conversely, let $\lim p_n = 0$ and let X be an infinite-dimensional subspace of $l_{(p_n)}$. Select a sequence $\{x_n\}$ in X such that x_n is of

the form $x_n = (0, \ldots, 0, x^n_{k_n}, x^n_{k_n+1}, \ldots)$ where (k_n) is a strictly increasing sequence of positive integers. Suppose that X contains a bounded neighborhood N of 0 and ε is a positive number such that $\|x\|_{(p_n)} \leqslant \varepsilon$ implies that x in is N. Without loss of generality, assume that $\|x_n\|_{(p_n)} = \varepsilon$. Since $N_\varepsilon = \{x\colon \|x_n\|_{(p_n)} \leqslant \varepsilon\}$ is bounded, there exists a positive number α such that $N_{\varepsilon/2} \supset \alpha N_\varepsilon$. However, $\lim \|\alpha x_n\|_{(p_n)} = \lim \|x_n\|_{(p_n)} = \varepsilon$. $\square$

Suppose $0 < p_n \leqslant 1$. Clearly $\lambda_{(p_n)}$ is the intersection $\bigcap_{\pi \in \Pi} l_{p_{\pi(n)}}$ where Π is the set of all permutations of the natural numbers. Let $\mathfrak{T}_\pi$ denote the "sup topology" obtained from the F-seminorms $\| \ \|_{(p_{\pi(n)})}$. With this notation, we have the following.

THEOREM 28. *If $(p_n) \in R_p$, then $\mathfrak{T}_\pi$ lies between the l_p-topology and the $\lambda_{(p_n)}$-topology. Furthermore, $\mathfrak{T}_\pi$ is metrizable if and only if $\lambda_{(p_n)} = l_p$.*

PROOF. It is clear that $\mathfrak{T}_\pi$ is stronger than the l_p-topology and weaker than the $\lambda_{(p_n)}$-topology. Since $\lambda_{(p_n)} = l_p$ implies that the $\lambda_{(p_n)}$-topology is equal to the l_p-topology, $\mathfrak{T}_\pi$ is metrizable when $\lambda_{(p_n)} = l_p$.

Conversely, suppose $\mathfrak{T}_\pi$ is metrizable and $\lambda_{(p_n)} \neq l_p$. Then there exists permutations $\{\pi_k\}_{k=1}^\infty$ and a decreasing sequence of positive numbers $\{\varepsilon_k\}_{k=1}^\infty$ such that the sets $U_k = \{x\colon \|x\|_{(p_{\pi_k(n)})} < \varepsilon_k\}$, $k = 1, 2, \ldots$, form a neighborhood base at zero for $\mathfrak{T}_\pi$. By Theorem 6, there exists a positive integer n_1 such that

$$\left\| \frac{\varepsilon_1}{2} \Big(\underbrace{\left(\frac{1}{n_1}\right)^{1/p}, \left(\frac{1}{n_1}\right)^{1/p}, \ldots, \left(\frac{1}{n_1}\right)^{1/p}}_{n_1 \text{ times}}, 0, \ldots \Big) \right\|_{(p_n)} > 1.$$

Since $\lim_{n\to\infty} p_{\pi_1(n)} = p$, there exists a point z_1 of the form

$$z_1 = \frac{\varepsilon_1}{2}\Big(0, \ldots, 0, \underbrace{\left(\frac{1}{n_1}\right)^{1/p}}_{m_1}, \ldots, \underbrace{\left(\frac{1}{n_1}\right)^{1/p}}_{m_2+n_2-1}, 0, \ \cdots \Big)$$

such that z_1 is in U_1. We will construct a permutation (q_n) of (p_n) in stages. The first $m_1 + n_1 - 1$ terms of (q_n) are

$$(p_{n_1+1}, \ldots, p_{k_1}, \underbrace{p_1}_{m_1}, p_2, \ldots, \underbrace{p_{n_1}}_{m_1+n_1-1})$$

where $k_1 = m_1 + n_1 - 1$. Since $l_{(p_n)} \neq l_p$, $l_{(p_{k_1+n})} \neq l_p$. Again, by Theorem 6, there exists a positive integer n_2 such that

$$\left\| \frac{\varepsilon_2}{2} \Big(\underbrace{\left(\frac{1}{n_2}\right)^{1/p}, \ldots, \left(\frac{1}{n_2}\right)^{1/p}}_{n_2 \text{ times}}, 0, \ldots \Big) \right\|_{(p_{k_1+n})} > 1.$$

Since $\lim_{n\to\infty} p_{\pi_1(n)} = p$ and $\lim_{n\to\infty} p_{\pi_2(n)} = p$, there exists a point z_2 of the form

$$z_2 = \frac{\varepsilon_2}{2}\Bigg(0,\ldots,0,\underbrace{\left(\frac{1}{n_2}\right)^{1/p},\cdots,\left(\frac{1}{n_2}\right)^{1/p}}_{m_2+n_2-1},0,\ \cdots\Bigg)$$

such that z_2 is in U_1 and U_2 and $m_2 > m_1 + n_1$. The first $m_2 + n_2 - 1$ terms of (q_n) are

$$(p_{n_1+1},\ldots,p_{k_1},p_1,p_2,\ldots,p_{n_1},p_{k_1+n_2+1},\ldots,p_{k_2},\underbrace{p_{k_1+1},\ldots,p_{k_1+n_2}}_{m_2+n_2-1=k_2}\qquad)$$

where $k_2 = m_2 + n_2 - 1$. Continue this process inductively to obtain (q_n) and $\{z_n\}$ such that z_n is in $\bigcap_{j=1}^{n} U_j$ and $\|z_k\|_{(q_n)} > 1$. Let U be defined by $U = \{x\colon \|x\|_{(q_n)} < 1\}$. Then clearly U is a neighborhood of 0 in $\mathfrak{T}_\pi$, but there does not exist any integer, n, such that $\bigcap_{k=1}^{n} U_k$ is contained in U. This is a contradiction. □

References

1. Z. Altshuler, P. Casazza and B. Lin, *On symmetric basic sequences in Lorentz sequence spaces*, Israel J. Math. (to appear).
2. B. A. Barnes and A. K. Roy, *Boundedness in certain topological linear spaces*, Studia Math. **33** (1969), 147–156. MR **40** #670.
3. C. Bessaga, *A note on universal Banach spaces of a finite dimension*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **6** (1958), 97–101. MR **22** #4935.
4. C. Bessaga and A. Pełczyński, *On bases and unconditional convergence of series in Banach spaces*, Studia Math. 17 (1958), 151–164. MR **22** #5872.
5. D. J. H. Garling, *On symmetric sequence spaces*, Proc. London Math. Soc. (3) **16** (1966), 85–106. MR **33** #537.
6. J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces*, Springer-Verlag, New York, 1973.
7. S. Rolewicz, *On the characterization of Schwartz spaces by properties of the norm*, Studia Math. **20** (1961), 87–92. MR **25** #1434.
8. S. Schonefeld and W. Stiles, *On some linear topologies on* $\cup l_p$ (submitted).
9. ———, *On the inductive limit of* $\cup l_p$, $0 < p \leqslant 1$, Studia Math. (to appear).
10. S. Simons, *The sequence spaces* $l(p_\nu)$ *and* $m(p_\nu)$, Proc. London Math. Soc. (3) **15** (1965), 422–436. MR **31** #600.
11. W. Stiles, *On non locally p-convex spaces*, Colloq. Math. **23** (1971), 261–262. MR **46** #9683.

Department of Mathematics, Florida State University, Tallahassee, Florida 32306