A SINGULAR SEMILINEAR EQUATION IN $L^1(\mathbb{R})$

BY

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Abstract. Let $\beta$ be a positive and nondecreasing function on $\mathbb{R}$. The boundary-value problem $\beta(u) - u'' = f$, $u'(\pm\infty) = 0$ is considered for $f \in L^1(\mathbb{R})$. It is shown that this problem can have a solution only if $\beta$ is integrable near $-\infty$, and that if this is the case, then the problem has a solution exactly when $\int_{-\infty}^{\infty} f(x) \, dx > 0$.

In [5, Lemma 5.6] T. Kurtz proves that the problem $e^u - u'' = f$ has a solution $u \in C^2(\mathbb{R})$ satisfying $u'(\pm\infty) = \lim_{x \to \pm\infty} u'(x) = 0$, whenever $f$ is nonnegative, continuous, compactly supported, and not identically equal to zero. Herein we study more general problems of the form

$$
\beta(u) - u'' \ni f, \quad u'(\pm\infty) = 0,
$$

where $\beta$ is a maximal monotone graph in $\mathbb{R}$ (see, for example, Brezis [2, §1.8]). In particular, $\beta$ can be any continuous, nondecreasing function on $\mathbb{R}$. If $0 \in \text{int} \, \beta(\mathbb{R})$, this problem is well understood; see Benilan, Brezis and Crandall [1] and Proposition 1 below. When $\beta(\mathbb{R}) \subseteq (0, \infty)$, as for the case $\beta(u) = e^u$, Kurtz's result is the only one known to the authors; and his methods depend very strongly on the explicit form of $\beta(u) = e^u$. We characterize those maximal monotone graphs $\beta$ with $\beta(\mathbb{R}) \subseteq (0, \infty)$ for which (P) has a solution for some $f \in L^1(\mathbb{R})$, and then show that for such $\beta$ (P) has a solution if and only if $\int_{-\infty}^{\infty} f(x) \, dx > 0$. Thus our conclusions are sharp as regards possible $\beta$ and $f$ in (P).

Let us be more precise. If $\beta$ is any maximal monotone graph and $f \in L^1_{\text{loc}}(\mathbb{R})$, by a solution of (P) we understand a function $u$ such that $u$ and $u'$ are locally absolutely continuous on $\mathbb{R}$, $f(x) + u''(x) \in \beta(u(x))$ a.e., and $u'(\pm\infty) = 0$. We denote by $D(\beta)$ the domain of $\beta$ and by $\beta^0$ the minimal section of $\beta$; the function $\beta^0$ assigns to $r \in D(\beta)$ the element in $\beta(r)$ of least modulus (so $\beta = \beta^0$ if $\beta$ is single-valued).

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The main result is

**Theorem 1.** Suppose $\beta$ is a maximal monotone graph in $\mathbb{R}$ with $\beta(\mathbb{R}) \subseteq (0, \infty)$. Then the following are equivalent:

(i) If $f \in L^1(\mathbb{R})$, then $f$ has a solution exactly when $\int_{-\infty}^{\infty} f(x) \, dx > 0$.

(ii) There exists some $f \in L^1(\mathbb{R})$ for which $(P_f)$ has a solution.

(iii) There is an $a \in \mathbb{R}$ for which $(-\infty, a) \subseteq D(\beta)$ and $\int_{-\infty}^{a} \beta^0(x) \, dx < \infty$.

This result is of interest because if (i), (ii), or (iii) holds, then the (possibly multivalued) mapping $f + u'' \mapsto -u''$, $u$ the solution to $(P_f)$, defines an accretive operator in $L^1(\mathbb{R})$: see Lemma 4(c). This operator generates a semigroup of contractions on a subset of $L^1(\mathbb{R})$ associated with the nonlinear partial differential equation $u_t - \beta(\phi)_{xx} = 0$, for $\phi = \beta^{-1}$. (See, for example, [4, §3].)

To obtain Kurtz’s result from Theorem 1 we need only note that $\int_{-\infty}^{0} e^x \, dx < \infty$, and so (iii) is valid for $\beta(x) = e^x$. And conversely if, for example, $\beta(x) = -1/x$ for large negative $x$, the equivalence of (ii) and (iii) implies that $(P_f)$ does not have a solution for any $f \in L^1(\mathbb{R})$.

**Proof of Theorem 1.** We prove Theorem 1 by establishing (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i), in that order; the implications are arranged in increasing levels of difficulty. We begin with some simple remarks.

Let $\beta$ satisfy the assumption of Theorem 1. Define

$$L^1(\mathbb{R})_+ = \left\{ f \in L^1(\mathbb{R}) \mid \int_{-\infty}^{\infty} f(x) \, dx > 0 \right\}.$$

We note first of all that $f \in L^1(\mathbb{R})_+$ is a necessary condition for the solvability of $(P_f)$. If $u$ solves $(P_f)$, then $f + u'' \in \beta(u)$ implies $f + u'' > 0$ a.e. and so

$$0 < \int_{-\infty}^{\infty} f(x) + u''(x) \, dx = \lim_{R \to \infty} \int_{-R}^{R} f(x) + u''(x) \, dx = \int_{-\infty}^{\infty} f(x) \, dx,$$

since $\lim_{R \to \infty} u'(\pm R) = 0$. Moreover this same calculation and Fatou’s Lemma imply $f + u'' \in L^1(\mathbb{R})$ and $\|f + u''\|_1 \leq \|f\|_1$ (denoting the norm in $L^p(\mathbb{R})$, $1 \leq p \leq \infty$). Thus if $u$ is a solution of $(P_f)$, $u'' \in L^1(\mathbb{R})$ and $\|u''\|_1 \leq 2\|f\|_1$.

**Definition.** $E$ is the linear subspace of functions $u$ defined on $\mathbb{R}$ such that $u$ and $u'$ are locally absolutely continuous, $u'' \in L^1(\mathbb{R})$, and $u'(\pm \infty) = 0$.

We have proved that if $u$ solves $(P_f)$ for some $f \in L^1(\mathbb{R})$, then $f \in L^1(\mathbb{R})_+$, $u \in E$, and $\|u''\|_1 \leq 2\|f\|_1$, $\|f + u''\|_1 \leq \|f\|_1$. Also $u \in E$ clearly implies $\|u''\|_\infty \leq \|u''\|_1$.

**Proof of (i) $\Rightarrow$ (iii).** If $f \in L^1(\mathbb{R})_+$ and $u$ solves $(P_f)$, then $u' \in L^\infty(\mathbb{R})$ by the preceding; and so there is a positive constant $c$ such that $u(x) \geq cx$ for $x \leq -1$. Furthermore,
A SINGULAR SEMILINEAR EQUATION IN $L^1(\mathbb{R})$  

$$f(x) + u''(x) \geq \beta^0(u(x)) \geq \beta^0(cx) > 0$$

a.e. for $x < -1$, since $\beta^0$ is positive and nondecreasing. Therefore

$$\|f\|_1 \geq \|f + u''\|_1 > \int_{-\infty}^{-1} \beta^0(cx) \, dx = \frac{1}{c} \int_{-\infty}^{-c} \beta^0(y) \, dy;$$

and (iii) follows.

Proof of (iii) $\Rightarrow$ (ii). This is a bit more subtle. Suppose $\int_{-\infty}^{a} \beta^0(x) \, dx < \infty$.

Let us for the moment assume that such a $g$ exists. Define $v: (-\infty, -1] \to \mathbb{R}$ by

$$v(x) = \frac{1}{g(v(x))}, \quad x < -1,$$

$$v(-1) = a - 1.$$ 

Since $g$ is positive, nonincreasing, and continuously differentiable, $v$ is increasing, convex, and twice continuously differentiable. In addition, it is clear that $v(x) \to -\infty$ as $x \to -\infty$, because $g$ is bounded above on compact sets. Since $g(x) \to \infty$ when $x \to -\infty$, $v'(-\infty) = 0$. Moreover

$$v'' \in L^1(-\infty, -1)$$

and

$$\int_{-\infty}^{-1} \beta^0(v(x)) \, dx = \int_{-\infty}^{a-1} \beta^0(y) \frac{1}{v'(v^{-1}(y))} \, dy = \int_{-\infty}^{a-1} \beta^0(y)g(y) \, dy < \infty.$$

Let $u$ be any even, twice continuously differentiable function on $\mathbb{R}$ which satisfies $u(x) = v(x)$ for $x < -1$ and $u < a$ everywhere. Then, by the construction, $f(x) = u''(x) + \beta^0(u(x)) \in L^1(\mathbb{R})$ and $u'(\pm \infty) = 0$, $u$ is a solution of (P).

It remains to prove the existence of $g$ with the properties (1). Select a sequence $(a_n)_{n=1}^{\infty}$ which satisfies $a_n < a_{n-1} < a$ for $n = 1, 2, \ldots$ and $\int_{-\infty}^{a_n} \beta^0(x) \, dx < 1/n^2$. Now take $g$ to be any nonincreasing continuously differentiable function so that $g(a_n) = \sqrt{n}$, $n = 1, 2, \ldots$, and $g = 1$ on $[a_1, a]$. Then
\[
\int_{-\infty}^{a} \beta^0(x)g(x) \, dx = \int_{a_1}^{a} \beta^0(x) \, dx + \sum_{n=1}^{\infty} \int_{a_{n+1}}^{a_n} \beta^0(x)g(x) \, dx \\
\leq \int_{a_1}^{a} \beta^0(x) \, dx + \sum_{n=1}^{\infty} \sqrt{n + 1} \int_{a_{n+1}}^{a_n} \beta^0(x) \, dx \\
\leq \int_{a_1}^{a} \beta^0(x) \, dx + \sum_{n=1}^{\infty} \frac{\sqrt{n + 1}}{n^2} < \infty;
\]

\(g\) has the desired properties.

**Proof of (ii) \(\Rightarrow\) (i).** This implication is the most difficult and its proof requires several steps. The lemmas following outline the program.

**Lemma 1.** Let \(f, g \in L^1(\mathbb{R})_+\) and \(\int_{-\infty}^{\infty} f(x) \, dx > \int_{-\infty}^{\infty} g(x) \, dx\). If \((P_g)\) has a solution, then so does \((P_f)\).

**Lemma 2.** If (ii) holds, then

\[
\left\{ f \in L^1(\mathbb{R})_+ \mid \exists g \in L^1(\mathbb{R})_+, \int_{-\infty}^{\infty} f(x) \, dx \right. \left\{ \int_{-\infty}^{\infty} g(x) \, dx, \text{ and } (P_f) \text{ has a solution} \right\} = L^1(\mathbb{R})_+.
\]

The combined implications of Lemmas 1–2 prove that (ii) \(\Rightarrow\) (i). If (ii) is valid, Lemmas 1 and 2 demonstrate that \((P_f)\) has a solution for all \(f \in L^1(\mathbb{R})_+\). Again we prove these results in order of ascending difficulty.

**Proof of Lemma 2.** Choose \(f \in L^1(\mathbb{R})\) so that \((P_f)\) has a solution \(u\); by (ii) there is at least one such \(f\) (and in fact \(f \in L^1(\mathbb{R})_+\)). Now for fixed \(\epsilon > 0\) we prove that there is some \(g \in L^1(\mathbb{R})_+, \|g\|_1 < \epsilon\), for which \((P_f)\) also has a solution. If \(\delta, M > 0\), define \(u_{\delta,M}(x) = u(\delta x) - M\). Then \(u_{\delta,M}\) solves \((P_{f_{\delta,M}})\), where

\[
f_{\delta,M}(x) = \beta^0(u_{\delta,M}(x)) - (u_{\delta,M})''(x) = \beta^0(u(\delta x) - M) - \delta^2 u''(x).
\]

We have \(\|u_{\delta,M}'\|_1 = \delta \|u''\|_1 < \epsilon/2\) for a fixed \(\delta\) small enough. Moreover \(\lim_{M \to \infty} \beta^0(u(\delta x) - M) = 0\) since \(\beta^0(x) \to 0\) as \(x \to -\infty\) (otherwise (ii) could not hold). By the Dominated Convergence Theorem we can choose \(M\) so large that \(\|\beta^0(u(\delta x) - M)\|_1 < \epsilon/2\). Then \(g = f_{\delta,M}\) satisfies \(\|g\|_1 < \epsilon\).

Therefore \((P_f)\) has a solution for \(g\)'s with arbitrarily small \(L^1\)-norm. Now take any \(f \in L^1(\mathbb{R})_+\) and let \(g\) be as above and satisfy \(\int_{-\infty}^{\infty} f(x) \, dx > \|g\|_1\). Then \(\int_{-\infty}^{\infty} f(x) \, dx > \int_{-\infty}^{\infty} g(x) \, dx\). The proof is complete.

For the proof of Lemma 1 we require another Lemma 3(a) below. (Parts (b) and (c) are included for interest's sake.)

**Lemma 3.** (a) Let \(v \in \mathcal{C}\), and \(p \in L^\infty(\mathbb{R})\) be locally Lipschitz continuous and nondecreasing. Then \(p'(v)v'^2 \in L^1(\mathbb{R})\) and
A SINGULAR SEMILINEAR EQUATION IN $L^1(R)$

$$\int_{-\infty}^{\infty} p(v(x))v''(x) + p'(v(x))v'(x)^2 dx = 0.$$  

(b) Let

$$\text{Sign } r = \begin{cases} 
1, & r > 0, \\
[-1,1], & r = 0, \\
(-1), & r < 0.
\end{cases}$$

If $a \in L^\infty(R)$, $v \in \mathcal{L}$, and $a(x) \in \text{Sign } v(x)$ a.e., then $\int_{-\infty}^{\infty} v'' a(x) dx \leq 0$.

(c) If $f, \tilde{f} \in L^1(R_+)_u, \tilde{u}$ are solutions of $(P_f)$ and $(P_{\tilde{f}})$, respectively, then $\|f + u'' - \tilde{f} + \tilde{u}''\|_1 \leq \|f - \tilde{f}\|_1$.

Proof of Lemma 3. We adapt arguments used in [1] and [3] to this simple case. If $R > 0$, then

$$\int_{-R}^{R} p(v(x))v''(x) + p'(v(x))v'(x)^2 dx = p(v(R))v'(R) - p(v(-R))v'(-R).$$

Since $p \in L^\infty(R)$ and $v'(-\infty) = 0$, (a) follows from Fatou's Lemma by letting $R \to \infty$ above.

To obtain (b), apply (a) with $p(s) = p_n(s) = p_0(ns)$, where $p_0(s) = s$ for $|s| < 1$ and $p_0(s) = \text{sign } s$ for $|s| \geq 1$. Then by (a) $\int_{-\infty}^{\infty} p_n(v)v'' dx \leq 0$. But $p_n(v) \to \text{sign}_0(v)$, where $\text{sign}_0 s = \text{sign } s$ for $s \neq 0$, $\text{sign}_0 0 = 0$. Therefore we can send $n \to \infty$ to conclude

$$\int_{[v > 0]} v''(x) dx - \int_{[v < 0]} v''(x) dx \leq 0$$

$([v > 0] = \{x|v(x) > 0\}$, etc.). Finally $v'(x) = 0$ a.e. on $[v = 0]$ and so $v''(x) = 0$ a.e. on this set (the derivative of any absolutely continuous function $v$ vanishes a.e. on $[v = c]$ for any $c \in R$). If $a(x) \in \text{Sign } v(x)$ a.e., we therefore have

$$\int_{-\infty}^{\infty} a(x)v''(x) dx = \int_{[v > 0]} v''(x) dx - \int_{[v < 0]} v''(x) dx + \int_{[v = 0]} a(x)v''(x) dx$$

$$= \int_{[v > 0]} v''(x) dx - \int_{[v < 0]} v''(x) dx \leq 0.$$  

(It is not hard to prove that equality actually holds.) To prove (c) let

$$a(x) = \begin{cases} 
1 & \text{on } [f + u'' > \tilde{f} + \tilde{u}''] \cup [u > \tilde{u}], \\
0 & \text{on } [f + u'' = \tilde{f} + \tilde{u}''] \cap [u = \tilde{u}], \\
-1 & \text{on } [f + u'' < \tilde{f} + \tilde{u}''] \cup [u < \tilde{u}].
\end{cases}$$

Then $a$ is well defined since $\beta$ is monotone, $a(x) \in \text{Sign}(u - \tilde{u})(x)$ a.e., and $a(f + u'' - (\tilde{f} + \tilde{u}'')) = |f + u'' - (\tilde{f} + \tilde{u}'')|$ a.e. By (b)
\[ \|f + u'' - (f + \hat{u}'')\|_1 = \int_{-\infty}^{\infty} a(f - \hat{f}) \, dx + \int_{-\infty}^{\infty} a(u - \hat{u})'' \, dx \leq \int_{-\infty}^{\infty} a(f - \hat{f}) \, dx \leq \|f - \hat{f}\|_1, \]

and (c) is proved.

**Proof of Lemma 1.** Suppose \( f, g \in L^1(\mathbb{R}) \) and \( \int_{-\infty}^{\infty} f(x) \, dx > \int_{-\infty}^{\infty} g(x) \, dx \). Assume \((P_g)\) has a solution. To prove that then \((P_f)\) has a solution we employ the following result of Benilan, Brezis and Crandall [1, §4]:

**Proposition 1.** Suppose \( \gamma \) is a maximal monotone graph in \( \mathbb{R} \) with 0 \( \in \gamma(0) \) and 0 \( \in \text{int} \gamma(\mathbb{R}) \). Then for every \( f \in L^1(\mathbb{R}) \) there is a function \( v \) such that

- \( v, v' \in L^\infty(\mathbb{R}) \) and \( v'' \in L^1(\mathbb{R}) \),
- \( f(x) + v''(x) \in \gamma(v(x)) \) a.e.,
- \( v'(\pm \infty) = 0, \|v'\|_\infty \leq \|v''\|_1 \leq 2\|f\|_1 \).

**Remark 1.** At this point there is a discontinuity in our presentation: except for Proposition 1 the discussion does not assume the reader to be familiar with [1] or [3]. The interested reader should attempt to prove Proposition 1 for himself, at least for the special case when \( \gamma \) is continuous. (This one-dimensional proposition does not require the machinery of [1].)

Proposition 1 allows us to solve as follows certain problems approximating \((P_f)\).

For \( 0 < \lambda < \sup \beta(\mathbb{R}) \) there is a number \( r_\lambda \in D(\beta) \) with \( \lambda \in \beta(r_\lambda) \). Set \( \beta^\lambda(x) = \beta(x + r_\lambda) - \lambda \); then \( \beta^\lambda \) satisfies the assumptions on \( \gamma \) in Proposition 1. And so there exists a \( w_\lambda \) satisfying (a), (b), (c), with \( \beta^\lambda \) in place of \( \gamma \). Define \( u_\lambda = w_\lambda + r_\lambda \). Then we have

- \( u_\lambda, u_\lambda' \in L^\infty(\mathbb{R}), u_\lambda'' \in L^1(\mathbb{R}) \),
- \( f(x) + u_\lambda''(x) \in \beta^\lambda(w_\lambda(x)) = \beta(u_\lambda(x)) - \lambda \) a.e.,
- \( u_\lambda'(\pm \infty) = 0, \|u_\lambda'\|_\infty \leq \|u_\lambda''\|_1 \leq 2\|f\|_1 \).

The solution \( u \) of \((P_f)\) will be constructed as the limit of the \( u_\lambda \) as \( \lambda \downarrow 0 \). First we show the \( u_\lambda \) decreases as \( \lambda \) decreases. Let \( p \) be a smooth, nondecreasing function defined on \( \mathbb{R} \) such that \( p(x) = 0 \) for \( x \geq 0 \), \( p(x) < 0 \) for \( -1 < x < 0 \), \( p(x) = -1 \) for \( x \leq -1 \). Now \( (u_\lambda - u_\eta)' = \beta(u_\lambda(x)) - \beta(u_\eta(x)) + \eta - \lambda \); and so, by the monotonicity of \( \beta \),

\[ p(u_\lambda - u_\eta)(u_\lambda - u_\eta)'' \geq (\eta - \lambda) p(u_\lambda - u_\eta). \]

Lemma 3(a) implies \( \int_{-\infty}^{\infty} p(u_\lambda - u_\eta)(u_\lambda - u_\eta)'' \, dx \leq 0 \). Letting \( \lambda > \eta \) we conclude that \( u_\lambda \geq u_\eta \) a.e.
To discover a (pointwise) lower bound for the $u_\lambda$ we recall that the problem $(P_\beta)$ has a solution $v$:

$$(P_\beta) \quad g(x) + v(x)^\prime \in \beta(v(x)) \quad \text{a.e.}, \quad v'(\pm \infty) = 0.$$  

As in the preceding we construct approximate functions $u_\lambda$ which satisfy conditions like (2), with $g$ replacing $f$. The $u_\lambda$, like the $u_\lambda$, decrease as $\lambda \searrow 0$. In addition, the $u_\lambda$ are bounded from below by $v$; this is proved by the same method as above.

We claim that there is some $x_0 \in \mathbb{R}$ such that $\{u_\lambda(x_0)\}$ is bounded. If not, then $u_\lambda(x) \to -\infty$ as $\lambda \searrow 0$ for every $x \in \mathbb{R}$. Subtract the equation satisfied by $v_\lambda$ from that satisfied by $u_\lambda$:

$$f(x) - g(x) + (u_\lambda(x) - v_\lambda(x))^\prime \prime \in \beta(u_\lambda(x)) - \beta(v_\lambda(x)).$$

Multiply this by $p(u_\lambda(x) - v_\lambda(x))$ $(p$ as defined above), recall the monotonicity of $\beta$, and integrate:

$$\int_{-\infty}^{\infty} (f(x) - g(x))p(u_\lambda(x) - v_\lambda(x)) + (u_\lambda(x) - v_\lambda(x))^\prime p(u_\lambda(x) - v_\lambda(x)) \, dx \geq 0.$$  

By Lemma 3(a), we have

$$\int_{-\infty}^{\infty} (f(x) - g(x))p(u_\lambda(x) - v_\lambda(x)) \, dx \geq 0.$$

For fixed $x$, $u_\lambda(x) \to -\infty$ and $v_\lambda(x)$ is bounded; therefore $p(u_\lambda(x) - v_\lambda(x)) \to -1$. So the Dominated Convergence Theorem applied to (4) leads to $
int_{-\infty}^{\infty} (g(x) - f(x)) \, dx \geq 0.$ However this contradicts the assumption on $f$ and $g$. Hence there is some $x_0$ for which $\{u_\lambda(x_0)\}$ is bounded; and this implies, since $\|u_\lambda\|_0 \leq 2\|f\|_1$, that the $u_\lambda$ are bounded uniformly on compact sets. They thus converge monotonically and uniformly on compact sets to a limit $u = \lim_{\lambda \searrow 0} u_\lambda$.

Furthermore $u_\lambda(x)^\prime + \lambda + f(x) \in \beta(u_\lambda(x))$ and $u_\eta(x)^\prime + \eta + f(x) \in \beta(u_\eta(x))$ a.e. implies $u_\alpha^\prime + \lambda \leq u_\eta^\prime + \eta$ if $u_\lambda < u_\eta$. Since $u_\alpha^\prime = u_\eta^\prime$ a.e. on $[u_\lambda = u_\eta]$, $u_\lambda^\prime + \lambda \leq u_\eta^\prime + \eta$ a.e. Also $u_\alpha^\prime(x) + \lambda > -f(x)$ a.e. because $0 < \beta^0(u_\lambda(x)) \leq u_\alpha^\prime(x) + \lambda + f(x)$. It follows that the $u_\alpha^\prime$ converge in $L^1_{\text{loc}}(\mathbb{R})$ to $u^\prime$ as $\lambda \searrow 0$, and therefore that $f + u^\prime \in \beta(u)$ a.e.

We must show that $u'(\pm \infty) = 0$. Since $\|u_\lambda\|_0 \leq 2\|f\|_1$ by (2), Fatou's Lemma implies $u^\prime \in L^1(\mathbb{R})$, and therefore $u'(\pm \infty)$ and $u'(-\infty)$ exist. It suffices to prove that $u'(-\infty) = 0$, the same equality for $u'(\pm \infty)$ following by similar arguments. Since $u \leq u_\lambda$ and $u_\lambda(-\infty) = 0$, $u'(-\infty) \geq 0$. We multiply both sides of (3) by $p(u_\lambda - v_\lambda)$ as before and integrate:
\[ \int_{-\infty}^{y} (u_\lambda(x) - u_\Lambda(x))^\prime p(u_\lambda(x) - v_\lambda(x)) \, dx \leq \int_{-\infty}^{y} (f(x) - g(x)) p(u_\lambda(x) - v_\lambda(x)) \, dx \]
\[ \leq \int_{-\infty}^{y} |f(x) - g(x)| \, dx. \]

Integrate by parts on the left and recall that \( u'_\lambda(-\infty) = v'_\lambda(-\infty) = 0 \):

\[ [u'_\lambda(y) - u'_\Lambda(y)] p(u_\lambda(y) - v_\lambda(y)) \leq \int_{-\infty}^{y} |f(x) - g(x)| \, dx. \]

Since \( u'_\lambda \to u'' \) in \( L^1_{\text{loc}}(\mathbb{R}) \), \( u'_\lambda \to u' \) in \( C(\mathbb{R}) \); and similarly for the \( v_\lambda \). So for every \( y \) we can pass to the limit as \( \lambda \searrow 0 \) in (5) to deduce

\[ [u'(y) - u'(y)] p(u(y) - v(y)) \leq \int_{-\infty}^{y} |f(x) - g(x)| \, dx. \]

Suppose that \( u'(-\infty) > 0 \). Then for all \( y \) less than some number, \( u(y) < v(y) - 1 \) and so \( p(u(y) - v(y)) = -1 \). Thus sending \( y \to -\infty \) in (6) implies \( u'(-\infty) \leq v'(-\infty) = 0 \), a contradiction. Therefore \( u'(-\infty) = 0 \), and the proof is complete.

**Remark 2.** We record some additional facts about solutions \( u \) of \( (P) \) and the map \( f \in L^1(\mathbb{R})_+ \mapsto Tf = f + u' \). First, \( T \) is a contraction by Lemma 3(c). Next, if \( u \) is a solution of \( (P) \), then \( u(\pm \infty) = -\infty \). Indeed, if there is a sequence \( x_n, |x_n| \to \infty \) and \( u(x_n) \geq -A \) for some \( A \), then \( u(x) \geq -A - \|u'\|_\infty \) on \( |x - x_n| \leq 1 \) and \( \text{measure}(|u(x) \geq -A - \|u'\|_\infty|) = \infty \). But \( \beta^0(u(x)) \geq \beta^0(-A - \|u'\|_\infty) > 0 \) on this set, contradicting \( \beta^0(u(x)) \in L^1(\mathbb{R}) \).

Second, if \( u \) and \( \hat{u} \) are solutions of \( (P) \), then \( Tf = f + u' = f + \hat{u}' \) implies \( u' - \hat{u}' \) is a constant. Since \( u'(\pm \infty) = \hat{u}'(\pm \infty), \hat{u}' = \hat{u}' \). Thus \( u = \hat{u} + c \) for some \( c \in \mathbb{R}, c \geq 0 \) without loss of generality. Now \( Tf(x) \in \beta(\hat{u}(x)) \cap \beta(\hat{u}(x) + c) \) a.e. Since \( \hat{u}(x) \to -\infty \) as \( |x| \to \infty \), we can choose \( x \) so that \( u(x) \) is a point of strict increase of \( \beta^0, \beta^0(\hat{u}(x)) < \beta^0(\hat{u}(x) + r) \) for \( r > 0 \). For this \( x \) we conclude that \( c = 0 \). Finally, if \( f, \hat{f} \in L^1(\mathbb{R})_+ \), then

\[ \int_{-\infty}^{\infty} (Tf - T\hat{f})^+ \, dx \leq \int_{-\infty}^{\infty} (f - \hat{f})^+ \, dx, \]

\[ m < f < M \text{ a.e. implies } m < Tf < M \text{ a.e.,} \]

and

\[ f \in L^1(\mathbb{R})_+ \implies \int_{-\infty}^{\infty} j(Tf) \, dx \leq \int_{-\infty}^{\infty} j(f) \, dx \]

for every convex lower-semicontinuous function \( j: \mathbb{R} \to [0, \infty] \) satisfying \( j(0) = 0 \). The estimates (7) (which imply that \( T \) is order preserving) and (8)
may be proved directly in a fashion similar to Lemma 3. Alternatively, according to [1], (7), (8) and (9) hold for the mappings $T_\lambda: f \to f + u_\lambda$, where $u_\lambda$ is as in (2), and one just lets $\lambda$ tend to zero. Also, (7) and (8) imply (9) by results of [3].

Added in proof. In a paper to appear in the Israel Journal of Mathematics, S. Fisher shows (among other things) that Theorem 1 remains correct if $\beta \in C(\mathbb{R})$; $\beta(-\infty) = 0$, $\beta > 0$ and $\beta \notin L^1(\mathbb{R})$. We also thank Professor Fisher for a useful remark.

REFERENCES


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