HAUSDORFF MEASURE FUNCTIONS IN THE
SPACE OF
COMPACT SUBSETS OF THE UNIT INTERVAL

BY
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ABSTRACT. The work done in this paper is the result of an attempt to
classify those functions $h$ for which the corresponding Hausdorff measure of
$\mathcal{F}[0,1]$ is zero. A partial characterization is achieved and in doing this some
problems of E. Boardman are solved.

Introduction. If $C$ is a compact nonempty subset of $\mathbb{R}^1$ then $\mathcal{F}[C]$ denotes
the compact metric space consisting of all nonempty compact subsets $C$
edowed with the Hausdorff metric. If $h$ is a continuous, increasing function
defined on the nonnegative real numbers with $h(0) = 0$ we shall denote by
$h - m$ the Hausdorff measure generated by $h$ (see [3]). In [1] it is shown that
for each function $h$ either $h - m(\mathcal{F}[0,1]) = 0$ or $\mathcal{F}[0,1]$ has non-\sigma-finite $h$-
measure. For $\alpha > 0$, the functions $h_{\alpha}, g_{\alpha}$ are defined by

$$h_{\alpha}(t) = 2^{-\alpha t^{-1}} \quad \text{and} \quad g_{\alpha}(t) = 2^{-t^{-\alpha}} \quad \text{for} \quad t > 0$$

and $h_{\alpha}(0) = g_{\alpha}(0) = 0$. Then Boardman shows that $g_{\alpha} - m(\mathcal{F}[0,1]) = \infty$ for
all $0 < \alpha < 1$ and that $h_{1} - m(\mathcal{F}[0,1]) = 0$. The evaluation of

$$h_{\alpha} - m(\mathcal{F}[0,1]) \quad \text{for} \quad 0 < \alpha < 1$$

is left as an open problem.

The work done in this paper is the result of an attempt to classify those
functions $h$ for which $h - m(\mathcal{F}[0,1]) = 0$. The main result is

THEOREM 1. Let $h$ be such that $\lim \inf_{x \to 0} \left(-x \log h(x)\right)^{-1} < \infty$, then
$h - m(\mathcal{F}[0,1]) = 0$.

Thus it follows that $h_{\alpha} - m(\mathcal{F}[0,1]) = 0$ whenever $0 < \alpha < 1$. We observe,
from [1], that if there is an $\alpha < 1$ with $\lim \inf_{x \to 0} \left(-x^\alpha \log h(x)\right)^{-1} > 0$, then
$h - m(\mathcal{F}[0,1]) = \infty$. It is clear, therefore, that this characterization is only

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partial as it gives no information about those functions for which \( \lim_{x \to 0} \left( -\frac{(x \log h(x))^{-1}}{x} \right) = \infty \), and yet \( \lim \inf_{x \to 0} \left( -(x^\alpha \log h(x))^{-1} \right) = 0 \) for all \( \alpha < 1 \). We shall denote by \( K \) the collection of all such functions. Now if \( f_n(x) = 2^{-n x^{-1}} \), then, by Theorem 1, \( f_n - m(\mathbb{F}[0, 1]) = 0 \) for \( n = 1, 2, \ldots \). But by a result of Rogers and Taylor \([4]\) there is a function \( h \) such that \( h - m(\mathbb{F}[0, 1]) = 0 \) and \( \lim_{x \to 0} f_n(x) (h(x))^{-1} = 0 \) for \( n = 1, 2, \ldots \). So there is a function \( h \in K \) for which \( h - m(\mathbb{F}[0, 1]) = 0 \). Conversely, it follows, by a slight generalization of the methods of \([1]\), that

**Theorem 2.** There is a function \( h \in K \) for which \( h - m(\mathbb{F}[0, 1]) = \infty \).

If we combine these results with some work of Dvoretzky \([2]\) we see that if the unit interval has \( \sigma \)-finite \( h \)-measure then \( 2^{-h^{-1}} - m(\mathbb{F}[0, 1]) = 0 \). This is interesting in that it relates some properties of \([0, 1]\) to some of \( \mathbb{F}[0, 1] \), and thus might suggest working in the more general setting of an arbitrary compact set \( K \) in place of \([0, 1]\). Unfortunately, one of our examples, combined with Dvoretzky's result also shows there is a function \( h \) such that the unit interval has non-\( \sigma \)-finite \( h \)-measure and yet \( 2^{-h^{-1}} - m(\mathbb{F}[0, 1]) = 0 \).

I should like to thank E. Boardman for providing me with a preprint of her paper \([1]\) and for the interesting correspondence we had concerning this work. I am also indebted to the referee for his constructive criticism of an earlier form of this paper.

**Preliminaries.** Let \( C \) be a compact nonempty subset of \( \mathbb{R}^1 \) and let \( x_i \in C \) for \( i = 1, 2, \ldots, n \); then \( \{x_1, x_2, \ldots, x_n\} \in \mathbb{F}[C] \). The sphere in \( \mathbb{F}[C] \), centre \( \{x_1, x_2, \ldots, x_n\} \) and radius \( r \) is precisely the set of compact subsets \( K \) of \( C \) such that \( K \cap [x_i - r, x_i + r] \neq \emptyset \) for \( i = 1, 2, \ldots, n \). Also if \( C \subset \bigcup_{i=1}^l I_i \) where the \( I_i \) are closed intervals each of length \( l \), then for \( K \in \mathbb{F}[C], K \in \mathbb{F}[\bigcup_{i \in A} I_i] \) and \( K \cap I_i \neq \emptyset \) for each \( i \in A \) where \( A \) is some nonempty subset of \( \{1, 2, \ldots, l\} \). In fact \( K \) belongs to the sphere in \( \mathbb{F}[\bigcup_{i=1}^l I_i] \), centre \( \{x_i\}_{i \in A} \) and radius \( \frac{l}{2} \) where \( x_i \) is the midpoint of \( I_i \). Thus \( \mathbb{F}[C] \) is contained in a union of \( 2^l - 1 \) spheres each of diameter \( l \).

**Proof of Theorem 1.**

**Lemma 1.** Let \( \alpha \in (0, 1) \), \( S = \{x_i\} \) be a sequence of positive real numbers with \( \lim_{i \to \infty} x_i = 0 \) and \( I_j \) (\( j = 1, 2, \ldots, l \)) be closed intervals such that \( \sum_{j=1}^l d(I_j) < \alpha \). Then, given any \( e > 0 \), there is a sequence \( \{\delta_n\}_{n=1}^N \) of sets such that \( \mathbb{F}[\bigcup_{i=1}^l I_j] \subset \bigcup_{n=1}^N \delta_n, \sum_{n=1}^N h_{\alpha}(d(\delta_n)) < e \) and \( d(\delta_n) \in S \) for \( n = 1, 2, \ldots, N \).

**Proof.** Let \( \eta > 0 \) be such that \( \sum_{i=0}^{i_0} d(I_j) < \alpha - \eta \). Choose \( i_0 \) such that, for \( i > i_0 \), \( 2^{-n x_i^{-1}} < e \). Now \( I_j \) can be covered by \( [d(I_j)x_i^{-1}] + 1 \) closed intervals each length \( x_i \), where the square brackets denote the integer part
function. Thus \( \bigcup_{j=1}^{t} I_j \) can be covered by \( N(i) = \sum_{j=1}^{t} [(d(I_j) x_i^{-1}) + 1] \)
closed intervals each of length \( x_i \). Now

\[
N(i) \leq x_i^{-1} \sum_{j=1}^{t} d(I_j) + t < (\alpha - \eta) x_i^{-1} + t
\]

and so

\[
(2^{N(i)} - 1) h_a(x_i) < 2(\alpha - \eta) x_i^{-1} + t - x_i < \varepsilon.
\]

Denote by \( \{\theta_n\} \) any enumeration of the sets of compact subsets of the
nonempty subcollections of the covering intervals each of length \( x_i \). This
completes the proof of the lemma.

**Lemma 2.** Let \( \alpha \) and \( S \) be as in Lemma 1 and denote by \( \mathcal{F}(\alpha) \) those compact
sets of Lebesgue measure \( < \alpha \). Then, given any \( \varepsilon > 0 \), there is a sequence \( \{\theta_n\} \) of
sets such that \( \mathcal{F}(\alpha) \subset \bigcup_{n=1}^{\infty} \theta_n \), \( \sum_{n=1}^{\infty} h_a(d(\theta_n)) < \varepsilon \) and \( d(\theta_n) \in S \) for \( n = 1, \ 2, \ldots \).

**Proof.** Let \( K \in \mathcal{F}(\alpha) \); then there is a finite covering \( J_1, J_2, \ldots, J_t \) of \( K \) by
open intervals with rational endpoints such that \( \sum_{i=1}^{t} d(J_i) < \alpha \). For \( i = 1, \ 2, \ldots, t \) put \( I_i = J_i \). Then the intervals \( I_1, I_2, \ldots, I_t \) satisfy the conditions of
Lemma 1 and \( K \in \mathcal{F}[\bigcup_{j=1}^{t} I_j] \). Now there are only enumerably many closed
intervals with rational endpoints and thus only enumerably many finite
unions of such closed intervals. So we may denote by \( \theta_1, \theta_2, \ldots \) those of the
finite unions of intervals whose total length is less than \( \alpha \). Then by Lemma 1
there is, for each integer \( j \), a sequence \( \{\theta_{nj}\}_{n=1}^{N(j)} \) of sets such that \( \mathcal{F}[\theta_j] \subset \bigcup_{n=1}^{N(j)} \theta_{nj} \), \( \sum_{n=1}^{N(j)} h_a(d(\theta_{nj})) < \varepsilon 2^{-j} \) and \( d(\theta_{nj}) \in S \) for \( n = 1, 2, \ldots, N(j) \).
But we have shown that \( \mathcal{F}(\alpha) \subset \bigcup_{n=1}^{\infty} \mathcal{F}[\theta_j] \) and so

\[
\mathcal{F}(\alpha) \subset \bigcup_{j=1}^{\infty} \bigcup_{n=1}^{N(j)} \theta_{nj}; \quad \sum_{j=1}^{\infty} \sum_{n=1}^{N(j)} h_a(d(\theta_{nj})) < \varepsilon
\]

and \( d(\theta_{nj}) \in S \) for \( n = 1, 2, \ldots, N(j) \) and \( j = 1, 2, \ldots \). This completes the
proof of the lemma.

For \( s = 3, 4, \ldots \) we put

\[
b_s = \left( \frac{2s - 1}{2s} \right)^{(2s - 1)/2} \left( \frac{1}{2s} \right)^{(1/2s)}, \quad \alpha_s = \frac{5}{6}, \frac{7}{8}, \ldots, \frac{2s - 1}{2s}
\]

and observe, using Stirling's Formula, that \( \alpha_s = O(s^{-1/2}) \) as \( s \to \infty \).

**Lemma 3.** Let \( S = \{x_i\} \) be as in Lemma 1 and let \( I_1, I_2, \ldots, I_t \) be closed
intervals such that \( \sum_{j=1}^{t} d(I_j) < 1 \). For each \( s \geq 3 \), given any \( \varepsilon > 0 \), there is a
sequence \( \{\theta_n\} \) of sets such that \( \bigcap_{j=1}^{\infty} I_j \subset \bigcup_{n=1}^{\infty} \theta_n \), \( \sum_{n=1}^{\infty} h_\alpha(d(\theta_n)) < \epsilon \) and \( d(\theta_n) \in S \) for \( n = 1, 2, \ldots \).

**Proof.** The proof of this lemma is in two sections.

(a) Let \( I_1, I_2, \ldots, I_s \) and \( S \) be as in the statement of the lemma. For each \( s \geq 3 \), given any \( \epsilon > 0 \), there is a sequence \( \{\theta_n\} \) of sets such that \( \bigcap_{j=1}^{s} I_j \subset \bigcup_{n=1}^{\infty} \theta_n \), \( \sum_{n=1}^{\infty} h_\alpha(d(\theta_n)) < \epsilon \) and \( d(\theta_n) \in S \) for \( n = 1, 2, \ldots \).

Now for each \( s \), \( \bigcup_{j=1}^{s} I_j \) can be covered by \( N(i) = 2^{j_i} \) \( \bigcup_{x_i} \bigcup_{x_i} I_x \) closed intervals each of length \( x_i \). Let \( I_1', I_2', \ldots, I_N(i) \) be an enumeration of the intervals. Then if \( K \subset \bigcap_{j=1}^{s} I_j \), cardinality \( \{j : K \cap I_j' \neq \emptyset\} \geq (2s-1)/2s \). Thus, for each \( s \), \( \bigcap_{j=1}^{s} I_j \subset \bigcup_{n=1}^{\infty} \theta_n \) is contained in a set of \( M = \sum \binom{N(i)}{k} : (2s-1)/2s x_i < k \leq N(i) \) spheres each of diameter \( x_i \).

Now the number of terms in \( M \) is \( O(x_i^{-1}) \) and the binomial coefficients can be estimated by Stirling’s Formula. We find, for any fixed \( \eta > 0 \), an estimate of the form

\[
M h_\alpha(x_i) = o((b_\eta^{-1} 2^{-\alpha_\eta + \eta})^{x_i^{-1}}) \quad \text{as } i \to \infty.
\]

So we need to prove that \( 2^{-\alpha_\eta} < b_\eta \) for all \( s \geq 3 \). This can easily be verified for \( s = 3 \). For \( s \geq 4 \) we use the inequality

\[
(2s)^{1/2} > \epsilon > (1 + 1/(2s-1))^{2s-1}
\]

to show that

\[
\frac{2s-1}{2s} \log \frac{2s-1}{2s} > \frac{1}{4s} \log \frac{1}{2s}.
\]

It is thus sufficient to prove that, for \( s \geq 4 \) we have

\[
\alpha_\eta \log 2 + (3/4s) \log (1/2s) > 0.
\]

We observe that since \( 2s > e^2 > (1 + 1/s)^{2s} \) we have

\[
\frac{3}{2(2s+1)} \log \frac{1}{2s+2} > \frac{3}{4s} \log \frac{1}{2s}.
\]

Using this fact we can easily deduce the required inequalities by induction on \( s \) and so (a) is proved.

(b) Let \( S \) be as in the statement of the lemma and let \( I_1, I_2, \ldots, I_s \) be closed intervals such that \( \sum_{j=1}^{s} d(I_j) < (2s-1)/2s \). For each \( s \geq 3 \), given any \( \epsilon > 0 \), there is a sequence \( \{\theta_n\} \) of sets such that \( \bigcap_{j=1}^{s} I_j \subset \bigcup_{n=1}^{\infty} \theta_n \); \( \sum_{n=1}^{\infty} h_\alpha(d(\theta_n)) < \epsilon \) and \( d(\theta_n) \in S \) for \( n = 1, 2, \ldots \).
This is proved by induction on $s$. For $s = 3$ the statement is trivial since in that case $\mathcal{S}[\bigcup_{j=1}^{t} I_j] \setminus \mathcal{S}(\alpha_s) = \emptyset$. Now assume it is true for some $s \geq 3$ and let $I_1, I_2, \ldots, I_t$ be closed intervals with $\sum_{j=1}^{t} d(I_j) < (2s + 1)/(2s + 2)$. Then

$$\sum_{j=1}^{t} d\left(\frac{2s + 2}{2s + 1}I_j\right) < 1$$

and

$$\mathcal{S}\left[\bigcup_{j=1}^{t} \frac{2s + 2}{2s + 1}I_j\right] \setminus \mathcal{S}(\alpha_s) \subset \left(\mathcal{S}\left[\bigcup_{j=1}^{t} \frac{2s + 2}{2s + 1}I_j\right] \setminus \mathcal{S}\left(\frac{2s - 1}{2s}\right)\right)$$

$$\cup \left(\mathcal{S}\left(\frac{2s - 1}{2s}\right) \setminus \mathcal{S}(\alpha_s)\right).$$

But, by (a), there is a sequence $\{g_n^0\}$ of sets such that

$$\mathcal{S}\left[\bigcup_{j=1}^{t} \frac{2s + 2}{2s + 1}I_j\right] \setminus \mathcal{S}\left(\frac{2s - 1}{2s}\right) \subset \bigcup_{n=1}^{\infty} g_n^0; \quad \sum_{n=1}^{\infty} h_\alpha(d(g_n^0)) < \frac{1}{2} \varepsilon$$

and $d(g_n^0) \in (2s + 2)\mathcal{S}/(2s + 1)$ for $n = 1, 2, \ldots$. Let $g_1, g_2, \ldots$ be an enumeration of all the finite unions of closed intervals with rational endpoints whose total length is less than $(2s - 1)/2s$. Then

$$\mathcal{S}\left(\frac{2s - 1}{2s}\right) \setminus \mathcal{S}(\alpha_s) \subset \bigcup_{i=1}^{\infty} \mathcal{S}[g_i] \setminus \mathcal{S}(\alpha_s).$$

By induction hypothesis, for each $i$, there is a sequence $\{g_n^i\}$ of sets such that

$$\mathcal{S}[g_i] \setminus \mathcal{S}(\alpha_s) \subset \bigcup_{n=1}^{\infty} g_n^i; \quad \sum_{n=1}^{\infty} h_\alpha(d(g_n^i)) < \varepsilon 2^{-i-1}$$

and $d(g_n^i) \in (2s + 2)\mathcal{S}/(2s + 1)$ for $n = 1, 2, \ldots$. Hence

$$\mathcal{S}\left[\bigcup_{j=1}^{t} I_j\right] \setminus \mathcal{S}(\alpha_{s+1}) \subset \bigcup_{i=0}^{\infty} \bigcup_{n=1}^{\infty} \frac{2s + 1}{2s + 2} g_n^i;$$

$$d((2s + 1)g_n^i/(2s + 2)) = (2s + 1)d(g_n^i)/(2s + 2) \in \mathcal{S} \text{ for } n = 1, 2, \ldots \text{ and } i = 0, 1, 2, \ldots; \text{ and}$$

$$\sum_{i=0}^{\infty} \sum_{n=1}^{\infty} h_\alpha_{s+1}\left((d(2s + 1)g_n^i)/(2s + 2)\right) = \sum_{i=0}^{\infty} \sum_{n=1}^{\infty} h_\alpha_{s+1}(d(g_n^i)) < \varepsilon.$$

This completes the induction and so (b) is proved.

We saw, in the proof, how (b) implies that given $\varepsilon > 0$ there is a sequence $\{g_n\}$ of sets such that
and \( d(\theta_n) \in S \) for \( n = 1, 2, \ldots \). Combining this fact with (a) and Lemma 2 we easily deduce Lemma 3.

**Lemma 4.** Let \( \alpha \) and \( S \) be as in Lemma 1 and let \( \epsilon > 0 \) be given. Then there are sets \( \{\theta_n\} \) such that \( \mathcal{H}[0,1] \subset \bigcup_{n=1}^{\infty} \theta_n; \sum_{n=1}^{\infty} h_\alpha(d(\theta_n)) < \epsilon \) and \( d(\theta_n) \in S \) for \( n = 1, 2, \ldots \).

**Proof.** Choose \( s \geq 3 \) so that \( 0 < \alpha_s < \alpha \). By Lemma 3 there are sets \( \{\theta_n\} \) such that \( \mathcal{H}[0,1] \subset \bigcup_{n=1}^{\infty} \theta_n; \sum_{n=1}^{\infty} h_\alpha(d(\theta_n)) < \epsilon \) and \( d(\theta_n) \in S \) for \( n = 1, 2, \ldots \). Thus

\[
\sum_{n=1}^{\infty} h_\alpha(d(\theta_n)) < \sum_{n=1}^{\infty} h_\alpha(d(\theta_n)) < \epsilon
\]
as required.

**Proof of Theorem 1.** Let \( A = \liminf_{x \to 0} \left( -x \log h(x) \right)^{-1} \), then \( 0 < A < \infty \). Choose \( \alpha \in (0,1) \) so that \( A \log 2 < \alpha^{-1} \). Then there is a sequence \( S = \{x_i\} \) of positive real numbers with \( \lim_{i \to \infty} x_i = 0 \) and \( -(x_i \log h(x_i))^{-1} < (\alpha \log 2)^{-1} \) for \( i = 1, 2, \ldots \). Hence \( h(x_i) < h_\alpha(x_i) \) for \( i = 1, 2, \ldots \). Thus, by Lemma 4, given any \( \epsilon > 0 \) there are sets \( \{\theta_n\} \) such that \( \mathcal{H}[0,1] \subset \bigcup_{n=1}^{\infty} \theta_n; \sum_{n=1}^{\infty} h_\alpha(d(\theta_n)) < \epsilon \) and \( d(\theta_n) \in S \) for \( n = 1, 2, \ldots \). Hence

\[
\sum_{n=1}^{\infty} h(d(\theta_n)) < \sum_{n=1}^{\infty} h_\alpha(d(\theta_n)) < \epsilon
\]
and so \( h - m(\mathcal{H}[0,1]) = 0 \) as required.

**References**


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