LINEAR OPERATORS FOR WHICH $T^*T$ AND $T + T^*$ COMMUTE. II

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ABSTRACT. Let $\theta$ denote the set of bounded linear operators $T$, acting on a separable Hilbert space $\mathcal{H}$, such that $T^*T$ and $T + T^*$ commute. It is shown that such operators are $G_1$. A complete structure theory is developed for the case when $\sigma(T)$ does not intersect the real axis. Using this structure theory, several nonhyponormal operators in $\theta$ with special properties are constructed.

1. Let $\theta$ denote the set of bounded linear operators $T$, acting on a separable Hilbert space $\mathcal{H}$, such that $T^*T$ and $T + T^*$ commute. It is shown that such operators are $G_1$. A complete structure theory is developed for the case when $\sigma(T)$ does not intersect the real axis. Using this structure theory, nonhyponormal operators in $\theta$ are constructed. Some results on the structure of $\sigma(T)$ are also obtained.

2. Introduction. The class $\theta$ has been studied in [3], [4], [5], and considered in [8], [9]. Our notation and terminology will be that of [5]. We shall review it briefly. If $T \in \theta$, then $4T^*T - (T^* + T)^2 \geq 0$ [5]. Define

$$C = \frac{(T^* + T) + i\sqrt{4T^*T - (T^* + T)^2}}{2}.$$  

Then $C$ is normal, $\sigma(C)$ is contained in the closed upper half-plane, $C^*C = T^*T$, and $T + T^* = C + C^*$ [5]. In particular,

$$(\lambda - T^*)(\lambda - T) = (\lambda - C^*)(\lambda - C)$$

for all $\lambda$. If $T \in \theta$ and $T$ is completely nonnormal, then $\sigma(T) = \sigma(T^*)$, $\sigma(C) \subseteq \sigma(T)$, $\partial \sigma(T) \subseteq \sigma(C) \cup \sigma(C^*)$, and $\sigma_p(T) = 0$ [4], [5]. The spectral measure for $C$ is denoted by $\mu(\cdot)$. Any operator $E$ such that $E^2 = E$ will be called a projection. The real numbers are denoted by $\mathbb{R}$. UHP (LHP) is the open
upper (lower) half-plane, $\overline{\text{UHP}}$ ($\text{LHP}$) are their closures. The restriction of an operator $B$ to subspace $\mathfrak{N}$ is denoted $B|\mathfrak{N}$.

3. $T \in \Theta$ with $\sigma(T) \cap \mathfrak{R} = \emptyset$. Our first result will be fundamental in the sequel.

**Theorem 1.** Suppose that $C$ is a normal operator on $\mathfrak{H}$ and $E$ is a projection such that

(i) $C^*(I - E) = (I - E)C^*(I - E)$, $\{EC^*(I - E) = 0\}$,

(ii) $CE = ECE$, $\{(I - E)CE = 0\}$,

(iii) $E^*(C - C^*)(I - E) = 0$.

Let

$$T = CE + C^*(I - E).$$

Then $T \in \Theta$.

**Proof.** Suppose that $C$, $E$ satisfy conditions (i), (ii), (iii). Note that by (iii) and (i):


Let $T = CE + C^*(I - E)$. Then

$$T + T^* = CE + C^*(I - E) + E^*C^* + (I - E^*)C = C^* + C + [CE - C^*E + E^*C^* - E^*C].$$

But,

$$CE - C^*E + E^*C^* - E^*C = (C - C^*)E + E^*(C^* - C) = (C - C^*)E + E^*(C^* - C)E = (I - E^*)(C - C^*)E = 0.$$

Thus $T + T^* = C + C^*$. Hence $T^* = C + C^* - T$, or

$$(3) \quad T^* = C^*E + C(I - E).$$

Using (2), (3) we get

$$T^*T = [(C^*E + C(I - E))[ECE + (I - E)C^*(I - E)] = C^*C.$$

Thus $T \in \Theta$. □

Our next result shows that if $\sigma(T) \cap \mathfrak{R} = \emptyset$, then $T$ is in the form of Theorem 1.

**Theorem 2.** Suppose that $T \in \Theta$ and $\sigma(T) \cap \mathfrak{R} = \emptyset$. Let $E$ be the projection
obtained by integrating \((\lambda - T)^{-1}\) around that portion of \(\sigma(T)\) in the upper half-plane. Let \(C\) be as in (1). Then \(C, E\) satisfy (i), (ii), (iii) and \(T = CE + C^*(I - E)\).

**Proof.** Since \((\lambda - T^*)(\lambda - T) = (\lambda - C^*)(\lambda - C)\) for all \(\lambda\), we have for all \(\lambda \in \sigma(C) \cup \sigma(C^*)\)

\[
(C - C^*)(\lambda - T)^{-1} = [(\lambda - C)^{-1} - (\lambda - C^*)^{-1}](\lambda - T^*).
\]

Integrating this first around the upper portion of \(\sigma(T)\) and then the lower portion of \(\sigma(T)\) gives

\[
(C - C^*)E = C - T^* \quad \text{or} \quad E = (C - C^*)^{-1}(C - T^*),
\]

and

\[
(C - C^*)(I - E) = -(C^* - T^*) \quad \text{or} \quad I - E = (C - C^*)^{-1}(T^* - C^*).
\]

By definition of \(E\), we have \(TE = ET\). Now

\[
CE = C(C - C^*)^{-1}(T - C^*) = (C - C^*)^{-1}(CT - C^*C)
\]

\[
= (C - C^*)^{-1}(C - T^*) = ET = TE.
\]

Thus (ii) holds. Similarly, \(C^*(I - E) = (I - E)T = T(I - E)\). Thus \(T = CE + C^*(I - E)\). There remains only to check (iii);

\[
E^*(C - C^*)(I - E) = (C^* - T)(C^* - C)^{-1}(C - C^*)(I - E)
\]

\[
= -C^*(I - E) + T(I - E) = 0. \quad \square
\]

One might suppose that the existence of the \(C, E\) in Theorem 1 is restrictive. The next theorem shows it is not.

**Theorem 3.** Let \(C\) be any normal operator such that \(\sigma(C) \subseteq \text{UHP}\). Let \(M_1\) be any invariant subspace for \(C\). Let \(M_2 = (C - C^*)^{-1}M_1^\perp\). Let \(E\) be the projection onto \(M_1\) along \(M_2\). Then \(T = CE + C^*(I - E) \in \theta\) and \(C, E\) satisfy (i), (ii), (iii).

**Proof.** Let \(C, M_1, M_2\) be as in the statement of the theorem. Clearly \(M_1\) is \(C\) invariant. Thus \(M_2\) is \(C^*\) invariant since \(M_1^\perp\) is \(C^*\) invariant. Let \((C - C^*)^{1/2}\) denote an analytic square root of \(C - C^*\). Now \((C - C^*)^{1/2}M_1 \oplus (C - C^*)^{-1/2}M_1^\perp = K\). Multiplying by \((C - C^*)^{-1/2}\) we see that \(M_1 + M_2 = K, +\) denoting a direct sum. Thus \(E\) is bounded. Conditions (i), (ii) are now immediate. Condition (iii) is equivalent to \((C - C^*)M_2 \subseteq M_1^\perp\). But this follows from the definition of \(M_2\). \(\square\)
Corollary 1. If $T \in \theta$, and $\sigma(T) \cap R = \emptyset$, then $T$ is similar to the orthogonal sum of two subnormal operators, $T_1$, $T_2$ and $\sigma(T) = \sigma(T_1) \cup \sigma(T_2)$, $\sigma(T_1) \cap \sigma(T_2) = \emptyset$.

Thus if $T \in \theta$, $\sigma(T) \cap R = \emptyset$, and $T$ is completely nonnormal, any results about the spectra of subnormal operators may be applied to $T$.

For a compact set $X$, let $C(X)$ denote the continuous functions on $X$ and $\mathcal{R}(X)$ the functions on $X$ which are uniformly approximable by rational functions with poles off $X$. Then from [6] and the results of this section we have:

**Proposition 1.** A compact set $\Sigma$ such that $\Sigma \cap R = \emptyset$ is the spectrum of a completely nonnormai $T \in \theta$ if and only if $\Sigma$ is symmetric with respect to the real axis and $\mathcal{R}(\Sigma \cap D) \neq C(\Sigma \cap D)$ for every open disc $D$ such that $\Sigma \cap D \neq \emptyset$.

The only part that needs to be proved is that if $C$ is normal with an invariant subspace $\mathcal{M}$, $C$ is the minimal normal extension of $C|\mathcal{M}$, and $C|\mathcal{M}$ is completely nonnormnal, then the $T$ generated by $C$, $\mathcal{M}$ is completely nonnormal. We now examine the relationship between the complete nonnormality of $T$ and the complete nonnormality of $C|\mathcal{M}$.

First we need the following well-known result whose proof we omit.

**Proposition 2.** Suppose $T$ is hyponormal. If the subspace $\mathcal{L}$ is invariant under $T$ and $T|\mathcal{M}$ is normal, then $\mathcal{M}$ reduces $T$.

**Proposition 3.** Let $\mathcal{M}_1 \subseteq N(T - C)$, $(\mathcal{M}_2 \subseteq N(T - C^*))$ be $C$, $(C^*)$ invariant subspaces. If $C|\mathcal{M}_1$ $(C^*|\mathcal{M}_2)$ has a normal summand, then $T$ has a normal summand.

The proof follows from Proposition 2 and the fact that $T\phi = C\phi$, $T^*\phi = C^*\phi$ for $\phi \in \mathcal{M}_1$ $(T\phi = C^*\phi$, $T^*\phi = C\phi$ for $\phi \in \mathcal{M}_2$).

**Theorem 4.** Suppose that $T \in \theta$, $\sigma(T) \cap R = \emptyset$, and $C$, $E$ are as in Theorem 1. Let $\mathcal{M}_1 = E\mathcal{M}$ and $\mathcal{M}_2 = (I - E)\mathcal{M}$. Then $T$ is completely nonnormal if and only if both $C|\mathcal{M}_1$, and $C^*|\mathcal{M}_2$ are completely nonnormal.

**Proof.** Proposition 3 takes care of the only if part. Suppose now that $T$ has a normal summand so that $T = T_1 \oplus T_2$ where $T_2$ is normal. Since $(\lambda - T)^{-1} = (\lambda - T_1)^{-1} \oplus (\lambda - T_2)^{-1}$, one of $E$ or $(I - E)$ has a normal summand and $C$ has a corresponding normal summand. Hence either $C|\mathcal{M}_1$ or $C^*|\mathcal{M}_2$ has a normal summand. □

Theorem 4 has the following interesting consequence.

**Theorem 5.** Let $T$, $C$, $E$, $\mathcal{M}_1$ be as in Theorem 4. Then $T$ is completely nonnormal if and only if $C|\mathcal{M}_1$ is completely nonnormal and $C$ is the minimal normal extension of $C|\mathcal{M}_1$.
Proof. $C$ is not the minimal normal extension of $C|\mathfrak{M}$, if and only if there is a subspace $\mathfrak{M} \subseteq \mathfrak{M}^+$ which reduces $C$. But from Theorem 3, $\mathfrak{M}_2 = (C - C^*)^{-1}\mathfrak{M}^+$. Clearly $(C - C^*)^{-1}\mathfrak{M} = \mathfrak{M}$. Thus $C$ is not the minimal normal extension of $C|\mathfrak{M}$, if and only if $C^*|\mathfrak{M}_2$ has a normal summand. Theorem 5 now follows from Theorem 4. □

Theorems 1, 2, and 3 completely characterize $T \in \theta$ with $\sigma(T) \cap \mathfrak{R} = \emptyset$. When considering some specific examples in §5 we will need the following results.

**Theorem 5.** Suppose that $T \in \theta$, there exists $C, E$ satisfying (i), (ii), (iii), and $C - C^*$ is one-to-one. If $T$ is also hyponormal, then $T$ is normal.

**Proof.** Suppose that $T \in \theta$, $C$ and $E$ satisfy (i)-(iii), $C - C^*$ is one-to-one, and $T$ is hyponormal. Then

\[
T^*T - TT^* = C^*C - [CE + C^*(I - E)][E^*C^* + (I - E)^*C]
\]

\[
= C^*C - CEE^*C^* - E(I - E)^*C
\]

\[
- C^*(I - E)E^*C^* - C^*(I - E)(I - E)^*C
\]

\[= CEE^*(C - C^*) + C^*EE^*(C^* - C)
\]

\[+ (C^* - C)EC + C^*E^*(C - C^*)
\]

\[= (C^* - C)EE^*(C^* - C)
\]

\[+ (C^* - C)EC + C^*E^*(C - C^*).
\]

Thus $(I - E^*)[T^*T - TT^*](I - E) = 0$. But $[T^*T - TT^*] > 0$ so that $[T^*T - TT^*](I - E) = 0$. Thus by (4), we have $(C^* - C)EC(I - E) = 0$. But $C^* - C$ is one-to-one. Hence $EC(I - E) = 0$, or $EC = ECE = CE$. Since $C$ is normal we also have $EC^* = C^*E$ by Fuglede's theorem [10]. Thus (iii) becomes $(C - C^*)E^*(I - E) = 0$ or $E^*(I - E) = 0$. But then $E^* = E^*E$. Hence $E$ is hermitian and reduces $T$. But $\sigma(TE) \subseteq \text{UHP}$, $TE \in \theta$, implies $T$ is normal [5]. □

**Corollary 2.** If $T \in \theta$, $\sigma(T) \cap \mathfrak{R} = \emptyset$, and $T$ is not normal, then $T$ is not seminormal.

**Corollary 3.** If $T \in \theta$ is hyponormal and completely nonnormal, then there does not exist an $E$ satisfying (i), (ii), (iii) where (2) holds.

4. **Operators in $\theta$ are $G_1$.** An operator is called $G_1$ if for all $\lambda \in \sigma(T)$, $\|\lambda - T\|^{-1}$ is the reciprocal of the distance from $\lambda$ to $\sigma(T)$. That is,
\begin{align*}
\|(\lambda - T)^{-1}\| &= 1/\rho(\lambda, \sigma(T)).
\end{align*}

Hyponormal operators are always $G_1$ [16].

**Theorem 7.** If $T \in \theta$, then $T$ is $G_1$.

**Proof.** We may assume that $T \in \theta$ and $T$ is completely nonnormal. Let $C$ be as in (1). Let $D_{\varepsilon}$ be the complement of $\mathbb{R} \times [-\varepsilon, \varepsilon]$. Then $(\lambda - T) \cdot (\lambda - C^*F(D_{\varepsilon}))^{-1}F(D_{\varepsilon})$ is analytic on $\mathbb{UHP}$ and

\begin{align*}
(\lambda - T)(\lambda - C^*F(D_{\varepsilon}))^{-1}F(D_{\varepsilon}) &= (\lambda - T)(\lambda - C^*)^{-1}F(D_{\varepsilon})
\end{align*}

for $\lambda \notin \sigma(C^*) \subseteq \sigma(T)$. But for any vector $\phi \in \mathcal{H}$ and any real $\lambda$,

\begin{align*}
\|(\lambda - T)(\lambda - C^*F(D_{\varepsilon}))^{-1}F(D_{\varepsilon})\| &= \langle (\lambda - T)(\lambda - C^*F(D_{\varepsilon}))^{-1}F(D_{\varepsilon})\phi, (\lambda - T)(\lambda - C^*F(D_{\varepsilon}))^{-1}F(D_{\varepsilon})\phi \rangle \\
&= \langle (\lambda - C^*)(\lambda - C)(\lambda - C^*F(D_{\varepsilon}))^{-1}F(D_{\varepsilon})\phi, (\lambda - C^*F(D_{\varepsilon}))^{-1}F(D_{\varepsilon})\phi \rangle \\
&= \langle F(D_{\varepsilon})\phi, F(D_{\varepsilon})\phi \rangle = \|F(D_{\varepsilon})\phi\|^2 \leq \|\phi\|^2.
\end{align*}

Also $\lim_{|\lambda| \to \infty} \|(\lambda - T)(\lambda - C^*F(D_{\varepsilon}))^{-1}\| = 1$. Thus

\begin{align*}
\|(\lambda - T)(\lambda - C^*F(D_{\varepsilon}))^{-1}F(D_{\varepsilon})\| &\leq 1
\end{align*}

for all $\lambda \in \mathbb{UHP}$. Hence $\|(\lambda - T)(\lambda - C^*)^{-1}\| \leq 1$ for all $\lambda \in \mathbb{UHP}$, $\lambda \notin \sigma(T)$ since $F(D_{\varepsilon})$ converges strongly to $I$ as $\varepsilon \to 0$ [5]. Similarly $\|(\lambda - T) \cdot (\lambda - C)^{-1}\| \leq 1$ for all $\lambda \in \mathbb{LHP}$, $\lambda \notin \sigma(T)$. Now if $\lambda \in \mathbb{UHP}$, $\lambda \notin \sigma(T)$, we have

\begin{align*}
\|(\bar{\lambda} - T)^{-1}\| &= \|(\bar{\lambda} - T^*)^{-1}\| = \|(\lambda - T)(\lambda - C^*)^{-1}(\lambda - C)^{-1}\| \\
&\leq \|(\lambda - T)(\lambda - C^*)^{-1}\| \|(\lambda - C)^{-1}\| \\
&\leq \|(\lambda - C)^{-1}\| = 1/\rho(\lambda, \sigma(C)) = 1/\rho(\bar{\lambda}, \sigma(C^*)) \\
&= 1/\rho(\bar{\lambda}, \sigma(T)).
\end{align*}

Similarly, if $\lambda \in \mathbb{LHP}$, $\lambda \notin \sigma(T)$,

\begin{align*}
\|(\bar{\lambda} - T)^{-1}\| &\leq 1/\rho(\bar{\lambda}, \sigma(T)).
\end{align*}

Hence $T$ is $G_1$. $\Box$

From [17, Theorem 1] and Theorem 7 we have:
**Proposition 4.** Suppose that $T \in \Theta$ is completely nonnormal. Then for any $z_0 \in \sigma(T)$ and disc $D$ centered at $z_0$, $D \cap \sigma(T)$ cannot lie on a Jordan arc.

While Propositions 1 and 4 are similar, they are not equivalent.

Knowing that $F \in \Theta$ is $G_1$ allows alternative proofs of some of our earlier results. For example, that isolated points of $\sigma(T)$ are reducing eigenvalues for $G_1$ operators is known [14]. It also tells us that the convex hull of $\sigma(T)$ is the closure of the numerical range of $T$, $\text{Cl} \, W(T)$ [12]. That is, $T$ is convexoid. It does not however, provide an alternative proof of the fact that all eigenvalues of $T$ are reducing [4]. Note that there are nonnormal compact $G_1$ operators [16], though there are no nonnormal compact operators in $\Theta$ [4].

If $T \in \Theta$, then $T$ restricted to any reducing subspace is also in $\Theta$. Thus $T \in \Theta$ are not only $G_1$ but also reduction-$G_1$ [1].

5. **Examples and extension of the model.** Our first example is, in a certain sense, canonical for $T \in \Theta$, $T$ completely nonnormal, $\sigma(T) \cap \mathcal{R} = \emptyset$. Theorem 3 will be the basis for most of our constructions.

**Example 1.** Let $H^2$ be the usual Hardy space of the circle. Let $C$ be multiplication by $e^{i\theta} + 2i$ in $L^2$ of the circle. Let $\mathcal{M}_1 = H^2$ and $\mathcal{M}_2 = (2 + \sin \theta)^{-1} H^2$. Let $T$ be the operator generated by $C$, $\mathcal{M}_1$, $\mathcal{M}_2$. Then $T \in \Theta$, $T$ is completely nonnormal and $\sigma(T)$ is the union of two discs centered at $2i$, $-2i$ and of radius one. By Corollary 2, $T$ is not hyponormal.

Example 1 shows that Conjecture (C) of [4] is false and the class of operators in $\Theta$ is nontrivially larger then was conjectured there. It also shows that $\sigma(T)$ need not be connected as was suggested in [5].

The point spectrum of the adjoint of an operator is preserved by similarity. Hence $\sigma_p(T^*) = \{z| |z - 2i| < 1\} \cup \{z| |z + 2i| < 1\}$ for the $T$ in Example 1 since $C|H^2$ is just $2i + S$, $S$ a unilateral shift.

If $\alpha, \beta$ are real scalars and $T \in \Theta$, then $\alpha T + \beta \in \Theta$. By taking direct sums of these operators, $T$ as in Example 1, it is possible to build a completely nonnormal nonhyponormal operator $T \in \Theta$ whose spectrum is any closed set $\Sigma$ whose interior is dense in $\Sigma$, and which is symmetric with respect to the real axis. Let $\Delta$ be a subset of the unit disc, equipped with a measure $\mu$, so that $\mathcal{R}(\Delta)$ is not dense in $L^2(\Delta, d\mu)$. Let $\mathcal{Q}(\Delta)$ be the $L^2$ closure of $\mathcal{R}(\Delta)$. If $\Delta$ has no interior and we repeat the construction of Example 1 using $\mathcal{Q}(\Delta)$ instead of $H^2$, we get a $T \in \Theta$, $T$ completely nonnormal, $T$ not hyponormal, and $\sigma(T)$ with no interior. For example, $\Delta$ could be chosen as a 'Swiss Cheese' space [14].

We shall now briefly consider two possible ways of extending the structure theory of Theorems 2 and 3 to operators with $\sigma(T) \cap \mathcal{R} \neq \emptyset$. Note from the proof of Theorem 2, that if $\sigma(T) \cap \mathcal{R} = \emptyset$, then $\mathcal{M}_2 = N(C - T^*)$ while $\mathcal{M}_1 = N(C - T)$. Conversely;
Proposition 5. Suppose that $T \in \theta$ and $C$ is (1). Let $M_1 = N(C - T)$, $M_2 = N(C^* - T)$. Then $M_1$, $M_2$ are $T$ invariant, $T|M_1 = C|M_1$, and $T|M_2 = C^*|M_2$. Furthermore, if $C - C^*$ is one-to-one, then $M_1 \cap M_2 = \{0\}$.

Proof. Note that $C^* - T^* = T - C$, $T^* - C = C^* - T$, and $C^* C = T^* T$. Thus $C^*(T - C) = (T - C)T$ and $C(T - C^*) = (T - C^*)T$. □

There need not, however, exist a nontrivial null space for either $C - T$ or $C^* - T$.

Proposition 6. Let $S$ be a unilateral shift. Let $C$ be as in (1). Then $N(C - S) = \{0\}$ and $N(C - S^*) = \{0\}$.

Proof. Since $S^* S = I$, $C$ is a unitary operator with spectrum on the upper half of the unit circle. Thus $C|M$ is normal for any invariant subspace $M$ of $C$. By Proposition 2, $N(C - S)$ and $N(C - S^*)$ reduce $C$. But $C - S = S^* - C^*$ and $C - S^* = S - C^*$. Thus $N(C - S)$, $N(C - S^*)$ reduce $S$. Since $S$ is completely nonnormal, we have $N(C - S) = \{0\}$ and $N(C - S^*) = \{0\}$. □

Since operators in $\theta$ are $G_1$, another possible extension is to use the results of Stampfli [18] to generalize Theorem 2. In [18] a method is developed to integrate a scalar multiple of the resolvent around pieces of $\sigma(T)$. For example, if $\sigma(T) \subseteq D_1 \cup D_2$ where $D_1$ are two discs, tangent say at 0, then [18] gives hyperinvariant subspaces $M_1$, $M_2$ for $T$ such that $\sigma(T|M_1) \subseteq D_1$, $\sigma(T|M_2) \subseteq D_2$. If $\sigma(T) \cap \mathbb{R} = \emptyset$, then this $M_1$, $M_2$ are complementary. In general, however, they need not be complementary. This difficulty is implicit in [18].

Example 2. Let $C_\varepsilon = e^{i\theta} + (1 + \varepsilon)i$ for $\varepsilon > 0$ on $L^2$ of the circle. Let $M_1(\varepsilon) = H^2$, $M_2(\varepsilon) = (\sin \theta + 1 + \varepsilon)^{-1}H^2$, and $E_\varepsilon$ be the projection onto $M_1(\varepsilon)$ along $M_2(\varepsilon)$. Assume for the moment that $\|E_\varepsilon\| \to \infty$ as $\varepsilon \to 0$. Define $T_\varepsilon$ using $C_\varepsilon$, $M_1(\varepsilon)$, $M_2(\varepsilon)$. If $T_\varepsilon$, $C_\varepsilon$ are multiplied by the same real scalar, then $T_\varepsilon = C_\varepsilon E_\varepsilon + C_\varepsilon^*(I - E_\varepsilon)$ still holds. Define

$$T = \sum_{\varepsilon \to 0} T_\varepsilon / \|E_\varepsilon\|$$

where $\varepsilon \to 0$.

If $\varepsilon \to 0$ not too fast, we have $T \in \theta$, $\sigma(T)$ is connected, and $\sigma(T) \cap \mathbb{R} = \{0\}$. Let $M_1$, $M_2$ be the subspaces generated by Stampfli's theorem. Using $f_1$, $f_2$ nonzero except at zero, we have $M_1$, $M_2$ are hyperinvariant for $T$, $\sigma(T|M_1) \subseteq \sigma(T) \cap \overline{\text{UHP}}$, $\sigma(T|M_2) \subseteq \sigma(T) \cap \overline{\text{LHP}}$, $M_1 + M_2$ is dense, and $M_1 \cap M_2 = \{0\}$. The integrals used to define $M_1$, $M_2$ are the orthogonal sum of the corresponding integral on each $L^2$ space. Since $f_1$, $f_2$ were assumed nonzero away from zero, we have

$$\bigcup_n \left[ \sum_{i=1}^n \oplus M_1(\varepsilon_i) \right] \subseteq M_1, \quad \bigcup_n \left[ \sum_{i=1}^n \oplus M_2(\varepsilon_i) \right] \subseteq M_2.$$
Thus to show that \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) are not complementary it suffices to show \( \|E_\varepsilon\| \to \infty \). To see that \( \|E_\varepsilon\| \to \infty \) as \( \varepsilon \to 0 \), let \( \alpha_\varepsilon, \beta_\varepsilon \) be the two roots of \( z^2 + 2(1 + \varepsilon)iz - 1 \). One root has modulus greater than one, the other has modulus less then one. Assume \( |\alpha_\varepsilon| < 1, 1 < |\beta_\varepsilon| \). Note that \( \alpha_\varepsilon, \beta_\varepsilon \to -i \) as \( \varepsilon \to 0 \). Let \( f_\varepsilon = -(\beta_\varepsilon(z - \beta_\varepsilon))^{-1}, g_\varepsilon = (1 - \alpha_\varepsilon z)z \). Note that \( f_\varepsilon \in H^2 \) and \( g_\varepsilon \in H^2_\perp \). Now let

\[
g_\varepsilon = f_\varepsilon + (z - z + 2i(e + 1))^{-1}f_\varepsilon,
\]

and observe that \( f_\varepsilon \in \mathcal{M}_1(\varepsilon), (z - z + 2i(e + 1))^{-1}f_\varepsilon \in \mathcal{M}_2(\varepsilon) \). Thus \( E_\varepsilon g_\varepsilon = f_\varepsilon \) and \( \|f_\varepsilon\| \to \infty \) as \( \varepsilon \to 0 \) since \( \beta_\varepsilon \to -i \). But for \( |z| = 1 \),

\[
g_\varepsilon = -\beta_\varepsilon(z - \beta_\varepsilon)^{-1} + z(z^2 + 2i(e + 1)z - 1)^{-1}(1 - \alpha_\varepsilon z)
= -\beta_\varepsilon(z - \beta_\varepsilon)^{-1} + (z - \alpha_\varepsilon)^{-1}(z - \beta_\varepsilon)^{-1}(1 - \alpha_\varepsilon z)
= (z - \beta_\varepsilon)^{-1}(-\beta_\varepsilon + z).
\]

Thus \( \|g_\varepsilon\| = 1, \|E_\varepsilon g_\varepsilon\| \to \infty \), and hence \( \|E_\varepsilon\| \to \infty \) as desired.

It would be of interest to know if for every completely nonnormal \( T \in \Theta \) such that \( \sigma(T) \cap \mathcal{K} \) is a single point, one has \( \mathcal{M}_1, \mathcal{M}_2 \) as in Example 2. Provided \( \sigma(T) \cap \text{LHP} \) and \( \sigma(T) \cap \text{UHP} \) are separated by the appropriate curves, Stampfli’s result gives an \( \mathcal{M}_1, \mathcal{M}_2 \) hyperinvariant for \( T \) such that \( \mathcal{M}_1 \cap \mathcal{M}_2 = \{0\} \). The difficulty is in showing \( \mathcal{M}_1 + \mathcal{M}_2 \) is dense.

If one considers the special case in [18, Theorem 1] where \( f_i(\lambda) = \lambda^m, i = 1, 2, m \) an integer \( \geq 1 \), one can show that \( \mathcal{M}_1 + \mathcal{M}_2 \) is dense if \( 0 \in \sigma_p(T^{*m}) \), since

\[
T^m = \int_{\partial D_1} \lambda^m(\lambda - T)_1 d\lambda + \int_{\partial D_2} \lambda^m(\lambda - T)_1 d\lambda.
\]

Putnam has shown that if \( 0 \in \sigma_p(T^{*}), 0 \in \partial \sigma(T) \), and there exists \( \lambda_n \to 0 \) such that \( |\lambda_n| \|((T^* - \lambda_n)^{-1} \| \to 1 \) as \( n \to \infty \), then \( 0 \) is a reducing eigenvalue [13]. Putnam’s result is thus one way of getting \( \mathcal{M}_1 + \mathcal{M}_2 = \mathcal{K} \) for completely nonnormal \( T \). However, this result and its subsequent generalizations, force \( \partial \sigma(T) \) to approach \( 0 \) almost vertically in order to apply them. Our next result does much better for operators in \( \Theta \).

**Theorem 8.** Suppose that there exist lines \( y^2 = ax^2, a > 0 \) and fixed, such that all points in \( \sigma(T) \) except zero lie either above both lines or below both lines. Suppose that \( T \in \Theta \) and \( T \) is completely nonnormal. Then \( 0 \in \sigma_p(T^*) \).

**Proof.** Suppose that \( T^* \phi = 0, \|\phi\| = 1 \). Note that for real \( \varepsilon, \varepsilon \neq 0, (\varepsilon - C^*)^{-1}(\varepsilon - T^n) \) is unitary. Thus \( 1 = \|\phi\| = \|(\varepsilon - C^*)^{-1}(\varepsilon - T^*)\phi\| = \|\varepsilon(\varepsilon - C^*)^{-1}\phi\|. \) Now
\[ \varepsilon (\varepsilon - C^*)^{-1} \phi = \int_{\sigma(C)} \frac{\varepsilon}{(\varepsilon - \lambda)} F(d\lambda) \phi. \]

But \(|\varepsilon (\varepsilon - \lambda)^{-1}| \leq |\varepsilon| |\varepsilon - \lambda_0|^{-1}\) where \(\lambda_0\) is on the two lines. Since the ratio between \(\varepsilon\) and the distance from \(\varepsilon\) to the nearest point on a line is a constant \(K\), we have \(|\varepsilon (\varepsilon - \lambda)^{-1}| \leq K\) all \(\lambda \in \sigma(C)\) and \(K\) is independent of \(\varepsilon\). From \([5]\) we have \(0\) is not a point mass of \(F(\cdot)\). Hence there exists \(\varepsilon_1 > 0\) such that \(\|F(|z| < \varepsilon_1)\phi\| < (2K)^{-1}\). Also there is an \(\varepsilon_0 > 0\) such that \(|\lambda: |\lambda_0 (\varepsilon_0 - \lambda)| > 1/2\) \(\subseteq \{z: |z| < \varepsilon_1\}\). Now

\[ \int_{\sigma(C)} \frac{\varepsilon_0}{(\varepsilon_0 - \lambda)} F(d\lambda) \phi = \int_{|\lambda| < \varepsilon_1} \frac{\varepsilon_0}{\varepsilon_0 - \lambda} F(d\lambda) \phi + \int_{|\lambda| > \varepsilon_1} \frac{\varepsilon_0}{\varepsilon_0 - \lambda} F(d\lambda) \phi. \]

But

\[ \left\| \int_{|\lambda| < \varepsilon_1} \frac{\varepsilon_0}{\varepsilon_0 - \lambda} F(d\lambda) \phi \right\| < K(2K)^{-1} = 1/2 \]

and

\[ \left\| \int_{|\lambda| > \varepsilon_1} \varepsilon_0 (\varepsilon_0 - \lambda)^{-1} F(d\lambda) \phi \right\| < \|\phi\|/2 = 1/2. \]

Thus \(\|\varepsilon_0 (\varepsilon_0 - C^*)^{-1} \phi\| < \|\phi\|\) which is a contradiction. \(\square\)

One can weaken the assumptions of Theorem 9 to only \(T \in \theta\), \(T\) completely nonnormal and there exists real \(\varepsilon_n, \varepsilon_n \not\in \sigma(T), \varepsilon_n \to 0\), such that \(\varepsilon_n \rho(\varepsilon_n, \sigma(T))^{-1}\) is bounded independently of \(n\).

The example on pp. 280–281 of \([13]\) shows that Theorem 8 is not true for \(T\) which are not in \(\theta\) but are \(G_1\).

Regardless of whether or not the subspaces generated by Stampfli’s theorem have dense sum, their existence gives much information about \(\sigma(T)\).

THEOREM 9. Suppose that \(T \in \theta\), \(\sigma(T) \cap \mathbb{R} = \{0\}\), and \(T\) is completely nonnormal. Suppose further that there exist functions \(f_1, f_2\) and domains \(D_1, D_2\) satisfying the assumptions of \([18, \text{Theorem 1 and Theorem 1'}]\). Let \(\mathcal{M}_1, \mathcal{M}_2\) be the closure of the ranges of

\[ A = \int_{\partial D_1}(f_1(\lambda)(\lambda - T)^{-1} d\lambda, \quad B = \int_{\partial D_2}(f_2(\lambda)(\lambda - T)^{-1} d\lambda \]

respectively. Let \(C\) be as in (1). Then \(T|_{\mathcal{M}_1} = C|_{\mathcal{M}_1}\), and \(T|_{\mathcal{M}_2} = C^*|_{\mathcal{M}_2}\), \(\sigma(T|_{\mathcal{M}_1}) \subseteq \text{UHP}\), and \(\sigma(T|_{\mathcal{M}_2}) \subseteq \text{LHP}\).

PROOF. The only part that needs proof is \(T|_{\mathcal{M}_1} = C|_{\mathcal{M}_1}\) and \(T|_{\mathcal{M}_2} = C^*|_{\mathcal{M}_2}\). The rest is done in \([18]\). First note that
\[ \int_{\partial D_1} (C - \lambda)f_1(\lambda)(\lambda - T)^{-1} d\lambda = \int_{\partial D_1} (C - \lambda)f_1(\lambda)(\lambda - C)^{-1}(\lambda - C^*)^{-1}(\lambda - T^*) d\lambda = -\int_{\partial \Omega_1} f_1(\lambda)(\lambda - C^*)^{-1}(\lambda - T^*) d\lambda = 0. \]

But then

\[ 0 = \int_{\partial D_1} (C - \lambda)f_1(\lambda)(\lambda - T)^{-1} d\lambda = C \int_{\partial D_1} f_1(\lambda)(\lambda - T)^{-1} d\lambda - \int_{\partial D_1} \lambda f_1(\lambda)(\lambda - T)^{-1} d\lambda = CA - TA \quad \text{as desired.} \]

The proof that \((C^* - T)B = 0\) is similar. \(\square\)

6. Comments and more examples. While the results of [4], [5] and this paper have developed many basic properties of the class \(\theta\), numerous questions remain. For convenience, let \((Q)\) denote the class of quasinormals [2] and \((QA)\) denote operators of the form \(T_1 + T_2\) where \(T_1 \in (Q)\), \(T_1T_2 = T_2T_1\), and \(T_2\) is selfadjoint. Then \((Q) \subset (QA) \subset \theta\) and all inclusions are proper. An obvious problem is to determine what types of restrictions on operators in \(\theta\) force them to be in \((Q)\) or \((QA)\). In particular, are there \(T \in \theta\) which are subnormal and not in \((QA)\)?

It was shown in [4] that if \(T^*T - TT^*\) has a kernel, then operators in \(\theta\) have a block decomposition much like the operators in \((QA)\). If \(T \in \theta\) and \(T^*T - TT^*\) has rank one, then \(T \in (QA)\).

**Theorem 10.** Suppose that \(T \in \theta\) and \(T^*T - TT^*\) has rank one. Then \(T = [\lambda_1 + \lambda_2 S] \oplus N\) where \(\lambda_1\) is real, \(\lambda_2 > 0\), \(S\) is a unilateral shift of multiplicity one, and \(N\) is normal.

**Proof.** Suppose that \(T \in \theta\), \(T^*T - TT^*\) has rank one, and \(T\) is completely nonnormal. Then by [4] \(T\) has the scalar matrix,

\[
T = \begin{bmatrix}
a_1 & 0 & 0 \\
b_1 & a_2 & 0 \\
0 & b_2 & a_3 \\
& & \ddots
\end{bmatrix},
\]

all \(b_i\) are nonzero, and the \((1, 1)\) entry acts on the range of \([T^*T - TT^*]\). From (2) of [4] we have \(b\bar{a}_{i+1} = a_i\bar{b}_i + |a_{i+1}|^2 + |b_{i+1}|^2 = |b_i|^2 + |a_{i+1}|^2\). Also \(\bar{a}_i|b_i|^2 = |b_i|^2\lambda_i^2\) since \(T \in \theta\). Let \(\lambda_i = a_1\). Then \(\lambda_i\) is real and \(a_i = \lambda_i\) for all \(i\). Also \(|b_i|^2\) is independent of \(i\). Let \(\lambda_2 = |b_i|\) and recall that weighted shifts.
are unitarily equivalent if their weight sequences have the same moduli [10].

However, if $T^* T - TT^*$ has rank greater than one, the situation is different. We shall now construct a $T \in \theta$ such that $T$ is hyponormal, $T \not\in (Q4)$, and $T^* T - TT^*$ has rank two.

**Example 3.** Let $T$ be given by

$$T = \begin{bmatrix} A_1 & 0 & 0 & 0 \\ B_1 & A_2 & 0 & 0 \\ 0 & B_2 & A_3 & 0 \\ & & & \end{bmatrix}$$

on countably many copies of a two dimensional Hilbert space. Let

$$A_i = \begin{bmatrix} 0 & e_i \\ f_i & 0 \end{bmatrix}, \quad B_i = \begin{bmatrix} \delta_i & 0 \\ 0 & \gamma_i \end{bmatrix},$$

where $e_i, f_i, \delta_i, \gamma_i$ are real scalars. Then $T^* T - TT^*$ has matrix $\text{Diag}(D, 0, 0, \ldots)$ if and only if

$$A_i^* A_i + B_i^* B_i - A_i A_i^* = D, \quad i \geq 2,$$

and

$$B_i^* A_{i+1} = A_i B_i^*, \quad i \geq 1.$$  

If (6), (7), (8) are satisfied, then $T \in \theta$ if $A_i^* D$ is hermitian [4]. Take $0 < \alpha < 1$ and $c = (2 + 2\alpha)^{-1/2}$. Set $e_1 = ca, f_1 = c$, and

$$\delta_1 = \gamma_1 = (1 + |c|^2(\alpha^2 - 1))^{1/2} = (\alpha + |c|^2(1 - \alpha^2))^{1/2}.$$  

Equation (6) gives $D = [1, 0]$. Equation (7) becomes

$$e_{i+1} = e_i \gamma_i / \delta_i, \quad f_{i+1} = f_i \delta_i / \gamma_i, \quad i > 1,$$

while (8) is

$$\delta_{i+1}^2 = \delta_i^2 + e_{i+1}^2 - f_{i+1}^2, \quad \gamma_{i+1}^2 = \gamma_i^2 + f_{i+1}^2 - e_{i+1}^2.$$  

Note that given $e_i, f_i, \delta_i, \gamma_i$, then $e_{i+1}, f_{i+1}$ are determined by (9). Then (10), if consistent, gives a unique positive $\delta_{i+1}, \gamma_{i+1}$. A straightforward computation yields that $e_1 = e_7, f_1 = f_7, \delta_1 = \delta_7, \gamma_1 = \gamma_7$. Thus the sequences $A_i, B_i$, defined by (9), (10), our initial conditions and the requirement $\delta_i, \gamma_i > 0$, are well defined and bounded. Furthermore, $A_i^* D$ is hermitian so $T \in \theta$. But
For the convenience of the reader interested in studying this example more carefully we give the $B_i, A_i$, explicitly. As noted, $A_{i+6} = A_i, B_{i+6} = B_i$. The blocks are

\[
B_1 = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} \sqrt{\alpha} & 0 \\ 0 & 1 \end{bmatrix}, \quad B_3 = \begin{bmatrix} \sqrt{\alpha} & 0 \\ 0 & 1 \end{bmatrix},
\]

\[
B_4 = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_1 \end{bmatrix}, \quad B_5 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\alpha} \end{bmatrix}, \quad B_6 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\alpha} \end{bmatrix},
\]

and

\[
A_1 = \begin{bmatrix} 0 & c\alpha \\ c & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & c\alpha \\ c & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & c\sqrt{\alpha} \\ c\sqrt{\alpha} & 0 \end{bmatrix},
\]

\[
A_4 = \begin{bmatrix} 0 & c \\ c\alpha & 0 \end{bmatrix}, \quad A_5 = \begin{bmatrix} 0 & c \\ c\alpha & 0 \end{bmatrix}, \quad A_6 = \begin{bmatrix} 0 & c\sqrt{\alpha} \\ c\sqrt{\alpha} & 0 \end{bmatrix}.
\]

In Example 1, the two components of $\sigma(T)$ were not spectral sets since the projections obtained by integrating the resolvent were not hermitian. Hence $\sigma(T)$ was not a spectral set of $T$.

**Example 4.** Let $\{T_i\}$ be a family of operators in $\theta$ constructed as in Example 1 such that $\bigcup_i \sigma(T_i)$ is dense in the unit disc. Let $T = \Sigma \oplus T_i$. Each $T_i$ has norm no greater than one. So $T$ is a contraction such that $\sigma(T)$ is the unit disc. Thus $\sigma(T)$ is a spectral set for $T$ [15, p. 441]. Note that $T$ is nonhyponormal and completely nonnormal.

However, if $T \in \theta$, $\sigma(T)$ is the unit disc and $\sigma(C)$ is contained in the unit circle, then $T$ is an isometry since $T^* T = C^* C = I$. For a related result see [7].

If $T \in (Q)$, then $T^n \in (Q)$ for all positive integers $n$. Which other operators in $\theta$ have powers also in $\theta$? As a partial answer we note that

**Proposition 7.** If $T \in (QA)$, then $T^2 \in \theta$ if and only if $T \in (Q)$.

**Proof.** Using the canonical form for $(Q)$ given in [2] it is easy to reduce the problem to showing that $(\alpha + S)^2 \not\in \theta$ for all real $\alpha \neq 0$ where $S$ is a unilateral shift. It suffices to show that $T = 2\alpha S + S^2 \not\in \theta$. But

\[
T^* T = (S^* + 2\alpha)(S + 2\alpha) = 4\alpha^2 + I + 2\alpha(S + S^*),
\]

\[
T + T^* = 2\alpha(S + S^*) + S^2 + S^{*2}.
\]

Thus $T \in \theta$ if $S^* + S$ and $S^2 + S^{*2}$ do not commute. But
\[(S^* + S)(S^2 + S^*^2) - (S^2 + S^*^2)(S^* + S)\]
\[= (S^* + S)(S^3 + S^*) - (S^2 + S^*^2)(1 + S^2)\]
\[= S^2 + S^*^2 + S^4 + SS^* - S^2 - S^*^2 - S^4 - 1 = SS^* - 1 \neq 0.\]

Thus \(T \notin \theta.\)

Note that if \(T\) is a weighted bilateral shift with positive weights whose smallest period is \(k\), then \(T^{nk} \in \theta, T^m \notin \theta\) for all \(m \neq nk, \) where \(n > 0.\)

The structure of the spectral measure of \(C\) and the structure of \(T\) are, of course, related. It was shown in [5] that if \(T \in \theta\) is completely nonnormal, then \(F(\mathbb{N}) = 0.\) Since eigenspaces of \(T + T^*\) reduce \(T\) if \(T\) is hyponormal [11], we have

**Proposition 8.** If \(T \in \theta, T\) is hyponormal, and \(T\) is completely nonnormal, then \(F(L) = 0\) for any vertical line \(L.\)

**Example 5.** Let \(\Delta\) be the boundary of \(\{x + iy: |x| < 1, |y| < 1\}\) equipped with linear Lebesgue measure. Let \(\mathcal{M}_1 = H^2(\Delta), C\) be the operator of multiplication by \(z + 2i\) and define \(T\) as in Example 1. Then \(T\) is completely nonnormal, \(T \in \theta, \sigma(C)\) is a square centered at \(2i,\) and \(F(\{z: \Re z = 1\}) \neq 0.\)

Consideration of the shift shows that one can have \(T \in \theta, \sigma_p(T^*) \neq \emptyset,\) and \(\sigma_p(C) = \emptyset.\) The converse is not possible.

**Proposition 9.** If \(T \in \theta\) and \(\lambda \in \sigma_p(C),\) then at least one of the following must hold:

(a) \(\lambda\) is a reducing eigenvalue of \(T,\)

(b) \(\overline{\lambda}\) is a reducing eigenvalue of \(T,\)

(c) \(\lambda, \overline{\lambda}\) are both eigenvalues of \(T^*.\)

**Proof.** Suppose that \(T \in \theta\) and \(C\phi = \lambda\phi.\) Then

\[(\lambda - T^*)(\lambda - T)\phi = (\lambda - C^*)(\lambda - C)\phi = 0,\]

and

\[(\overline{\lambda} - T^*)(\overline{\lambda} - T)\phi = (\overline{\lambda} - C)(\overline{\lambda} - C^*)\phi = 0. \qed\]

The next example shows that (c) of Proposition 9 is actually possible. It is based on an operator first constructed by Sarason [10, Problem 156].

**Example 6.** Let \(\mathcal{K}_0\) be a one-dimensional Hilbert space, \(g \in \mathcal{K}_0\) of norm one. Let \(\mathcal{K}\) be the orthogonal sum of \(L^2\) of the circle and \(\mathcal{K}_0.\) Let \(\tilde{S} = \mathcal{M}_z \oplus 0,\) where \(\mathcal{M}_z\) is multiplication by \(z\) in \(L^2.\) Let \(\mathcal{M}_1\) be the \(\tilde{S}\) invariant subspace generated by \(1 \oplus g\) and \(zH^2.\) \(\tilde{S}\) is the minimal normal dilation of \(\mathcal{M}_1.\) Let \(C = \tilde{S} + 2i\) and define \(T\) as in Theorem 3. Then \(T \in \theta.\) \(T\) is completely nonnormal by Theorem 5, so \(\sigma_p(T) = \emptyset.\) But \(2i \in \sigma_p(C)\) since \(0 \in \sigma_p(\tilde{S}).\)
Note that in Example 6, $\partial \sigma(C) \nsubseteq \partial \sigma(T)$. Since $S^{|M_1}$ and $S^{|M_2}$ are both unitarily equivalent to a unilateral shift we have that the $T$ of Example 1 is similar to the $T$ of Example 6. However, the $C$ of Example 1 has no point spectrum and hence is not similar to the $C$ of Example 6.

REFERENCES