A RELATION BETWEEN TWO BIHARMONIC GREEN'S FUNCTIONS ON RIEMANNIAN MANIFOLDS

BY

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Abstract. The biharmonic Green's function γ whose values and Laplacian are identically zero on the boundary of a region and the biharmonic Green's function Τ whose values and normal derivative vanish on the boundary originated in the investigation of thin plates whose edges are simply supported or clamped, respectively. A relation between these two biharmonic Green's functions known for planar regions is extended to Riemannian manifolds thereby establishing that any Riemannian manifold for which γ exists must also carry T.

Introduction. In a paper by N. Aronszajn, the integral representation of Τ given by

\[ \Gamma(x, y) = \int_D g(x, \xi)g(y, \xi)\,d\xi - \int_D g(x, \xi)k(\xi, \eta)g(y, \eta)\,d\xi\,d\eta \]

is credited to S. Zaremba (see [1, p. 387]) where g is the harmonic Green's function, k is the reproducing kernel for the square integrable harmonic functions and D is a regular subregion of the plane. (For physical interpretations of γ and Τ alluded to in the abstract, see e.g. [2, Chapter IV, particularly pp. 236, 242]. An informative discussion relating k and Τ for plane regions is given in [3] and [4, pp. 265–272].) In the present paper, we note that in this representation of Τ, the first term is none other than γ, and the second term is the reproducing kernel K for the biharmonic potentials with square integrable Laplacians w.r.t. an appropriate inner product ( , ). Also, in extending this relation between γ and Τ to Riemannian manifolds it is more natural to consider it as a representation of γ. Explicitly, we prove

Theorem 1. On an arbitrary Riemannian manifold, if γ exists, then K and Τ also exist. Furthermore, K and Τ are orthogonal w.r.t. ( , ) and γ = K + Τ.

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1. Definitions. Let $R$ denote a Riemannian manifold, $\Delta$ its Laplace-Beltrami operator, and $g^\Omega_x$ the harmonic Green’s function for a regular subregion $\Omega \subset R$ with pole $x \in \Omega$. Expressed by $\chi^\Omega_x$ the biharmonic Green’s function of $\Omega$ satisfying the boundary conditions

$$\chi^\Omega_x = 0, \quad \Delta \chi^\Omega_x = 0 \quad \text{on } \partial \Omega,$$

and by $\Gamma^\Omega_x$ the biharmonic Green’s function of $\Omega$ satisfying

$$\Gamma^\Omega_x = 0, \quad \frac{\partial}{\partial \nu} \Gamma^\Omega_x = 0 \quad \text{on } \partial \Omega,$$

and where each biharmonic Green’s function has a fundamental singularity at $x$, i.e. $\Delta \chi^\Omega_x - g^\Omega_x$ and $\Delta \Gamma^\Omega_x - g^\Omega_x$ each can be extended to a function harmonic in all of $\Omega$. In the above, $\partial / \partial \nu$ refers to the normal derivative and $\partial$ is the boundary operator.

If $\{\Omega\}$ is an exhaustion of $R$ by regular subregions, the biharmonic Green’s functions $\chi_x$, $\Gamma_x$ of $R$ are said to exist provided the limits $\chi_x = \lim \chi^\Omega_x$ and $\Gamma_x = \lim \Gamma^\Omega_x$ as $\Omega \nearrow R$ exist and are finite on $R - \{x\}$. (Throughout this paper, if there is no reference to any region, it will be understood that the region shall be the entire manifold $R$, e.g. $\chi^R_x = \chi_x$, $\Gamma^R_x = \Gamma_x$.) If $\chi_x$ (similarly $\Gamma_x$) exists for all $x \in R$, we say that $R$ possesses the biharmonic Green’s function $\chi$ (respectively $\Gamma$). The family of Riemannian manifolds void of $\chi$ or $\Gamma$ is denoted by $O_\chi$ or $O_\Gamma$, respectively.

**Corollary.** $O_\Gamma \subset O_\chi$.

2. The biharmonic Green’s function $\Gamma$. The class of parabolic manifolds (manifolds $R$ void of the harmonic Green’s function $g$, i.e. $g_x = \lim_{\Omega \nearrow R} g^\Omega_x$ is not finite for some $x \in R$) is customarily denoted by $O_G$. For $R \notin O_G$, we define a family $F$ of real valued functions on $R$ by

$$F = \left\{ f \left| \int_R |f(\xi)| g_x(\xi) d\xi \text{ is well defined and finite for all } x \in R \right. \right\},$$

and for $f \in F$ we define the function $Gf$ on $R$ by

$$Gf(x) = \int_R f(\xi) g_x(\xi) d\xi = \langle f, g_x \rangle.$$

The $G$-operator is an “inverse” for $\Delta$ in the following sense:

(i) If $f \in F$ and $Gf \in C^2(R)$, then $\Delta Gf = f$.

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(2) Subsequent to the writing of this paper, the author has been informed that although presently unavailable in the literature, two alternative proofs of the relation $O_\Gamma \subset O_\chi$ are known—both using entirely different methods from those presented here. Furthermore, it is known that $\phi < O_\Gamma < O_\chi$ (Chung-Nakai-Ralston-Sario).
(ii) If \( \varphi \in C^\infty_0 \), i.e. \( \varphi \) is \( C^\infty \) and has compact support in \( R \), then \( G\Delta \varphi = \varphi \).

(For the proof of (i) see e.g. Sario-Wang-Range [9], and for the proof of (ii) merely apply Green's identity to \( g \) and \( \varphi \).)

**Theorem 2.** If \( \gamma \) exists on \( R, x \in R \), then \( R \not\in O_G \) and

\[
\gamma(y) = \int_R g_x(\xi) g_y(\xi) \, d\xi \quad \text{for all } y \in R.
\]

**Proof.** By the Monotone Convergence Theorem, it suffices to show that for each regular subregion \( \Omega, x, y \in \Omega, \gamma^{\Omega}(y) = \int_{\Omega} g_x^{\Omega}(\xi) g_y^{\Omega}(\xi) \, d\xi \). Set \( f_x(y) = \int_\Omega g_x^{\Omega}(\xi) g_y^{\Omega}(\xi) \, d\xi \); then \( f_x = 0 \) on \( \partial \Omega \) since \( g_y^{\Omega} = 0 \) for \( y \in \partial \Omega \). Furthermore, \( \Delta f_x = g_x^{\Omega} \). To see this, we observe that for every \( \varphi \in C^\infty_0(\Omega) \),

\[
\langle g_x^{\Omega}, \varphi \rangle_\Omega = \langle g_x^{\Omega}, G^{\varphi} \rangle_\Omega = \langle f_x, \Delta \varphi \rangle_\Omega = \langle \Delta f_x, \varphi \rangle_\Omega.
\]

The first equality is just property (ii) satisfied by the \( G \)-operator; the second equality comes from an application of Fubini's Theorem, and the last equality utilizes Green's identity. From \( \Delta f_x = g_x^{\Omega} \), we see that \( f_x \) has a biharmonic singularity at \( x \), and \( \Delta f_x = 0 \) on \( \partial \Omega \). Hence, \( f_x \) satisfies the conditions that uniquely define \( \gamma \), i.e.

\[
\gamma^{\Omega}(y) = f_x(y) = \int_{\Omega} g_x^{\Omega}(\xi) g_y^{\Omega}(\xi) \, d\xi.
\]

**Corollary 1.** \( \gamma \) is positive and symmetric.

**Corollary 2.** If \( \gamma \) exists for some \( x \in R \), then \( \gamma \) exists for all \( x \in R \).

**Proof.** For an arbitrary \( y \in R \), we must show that \( \gamma_y < \infty \) assuming \( \gamma_x < \infty \) for some \( x \in R \). As just seen, the existence of \( \gamma \) for some \( x \) implies the existence of \( g_x \) for all \( x \). Let \( \Omega \) be a regular subregion containing both \( x \) and \( y \). For \( \xi \in \Omega \) and distinct from \( x \) and \( y \), let \( C_1(\xi) = \langle g_y, g_\xi \rangle_\Omega \langle g_x, g_\xi \rangle_\Omega \), \( m = \min g_x \) and \( M = \max g_y \) on \( \partial \Omega \). We then have

\[
\gamma_y(\xi) = \langle g_y, g_\xi \rangle_\Omega + \langle g_y, g_\xi \rangle_{R-\Omega} \\
\leq C_1(\xi) \langle g_x, g_\xi \rangle_\Omega + (M/m) \langle g_x, g_\xi \rangle_{R-\Omega} \leq C(\xi) \gamma_x(\xi)
\]

where \( C(\xi) = \max\{C_1(\xi), M/m\} < \infty \).

3. **Square integrable harmonic functions.** Let \( HL^2(R) \) denote the square integrable harmonic functions on a Riemannian manifold \( R \), and let \( \|h\| = \langle h, h \rangle^{1/2} \) for \( h \in HL^2(R) \).

**Theorem 3.** For an arbitrary Riemannian manifold \( R \), \( HL^2(R) \) is a Hilbert space. Furthermore, there exists a positive function \( M \) on \( R \) satisfying
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(1) \[ |h| \leq M \|h\| \quad \text{for all } h \in H^2(R) \]

and for which \( M_E = \sup_{x \in E} M(x) < \infty \) for every compact \( E \subset R \).

**Proof.** We first consider the existence of \( M \) together with the finiteness of \( M_E \). Given compact \( E \), let \( \Omega \) be a regular subregion containing \( E \). For \( x \in E, c > 0 \), let \( A_c(x) \) be the annular region

\[
A_c(x) = \{ \xi \in \Omega | g^0_x(\xi) < c \} \quad \text{and} \quad M_1 = \sup_{x \in E, \xi \in A_c(x)} |\grad_x g^0_x(\xi)|.
\]

The finiteness of \( M_1 \) is a consequence of the continuity of \( g^0_x(\xi) \) and \( |\grad_x g^0_x(\xi)| \) on \( \Omega \times \Omega \)-diagonal and the fact that \( \sup_{x \in E, \xi \in A_c(x)} g^0_x(\xi) = c \). We think of \( A_c(x) \) as being composed of a collection of level surfaces \( \{S_d(x)\}_{0 < d < c} \) where \( S_d(x) = \{ \xi \in \Omega | g^0_x(\xi) = d \} \). If \( \alpha \) is a flow line joining \( S_{d_1} \) to \( S_{d_2}, 0 \leq d_1 < d_2 \leq c \), we have

\[
d_2 - d_1 = \int_\alpha |\grad_x g^0_x(\xi)| dL_\xi \leq M_1 (\text{length } \alpha)
\]

where \( dL_\xi \) refers to arc length. Hence, \( (d_2 - d_1)/M_1 \leq \text{length } \alpha \). From this along with

\[
|h(x)| \leq \int_{S_d} \left| \frac{\partial}{\partial \nu_\xi} g^0_x(\xi) \right| \cdot |h(\xi)| dS_\xi, \quad x \in E, 0 \leq d \leq c
\]

it follows that

\[
|h(x)| \leq \frac{c}{M_1} \int_{A_c(x)} |h(\xi)| dV_\xi.
\]

Here, \( dS_\xi \) is the surface element and \( dV_\xi \) is the volume element. Thus, by Schwarz we obtain

\[
|h(x)| \leq (M_1^2/c) \sqrt{\text{vol } \Omega} \|h\| \quad \text{for all } x \in E, h \in H^2(R).
\]

The existence of \( M \) and the finiteness of \( M_E \) is now clear.

To see that \( H^2 \) is a Hilbert space, let \( \{h_n\} \) be Cauchy in \( H^2 \). By the first part of this proof just completed,

\[
|h_n(x) - h_m(x)| \leq M_E \|h_n - h_m\|, \quad x \in E.
\]

Hence there exists \( h \) harmonic on \( R \) for which \( h_n \to h \) uniformly on compact subsets of \( R \). In particular, \( \|h - h_n\|_E \to 0 \) as \( n \to \infty \). Also, \( \{\|h_n\|\} \) is bounded.

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(3) The reader might find it enlightening to compare the first part of this proof with an inequality given in [3, p. 503].
since \( \{h_n\} \) is Cauchy. We conclude that \( h \in HL^2 \) from the inequality
\[
\|h\|_E \leq \|h - h_n\|_E + \|h_n\|
\]
by taking the limit as \( n \to \infty \) and then the supremum over all compact \( E \subset R \). To see that \( h_n \to h \) in norm, we consider the inequality
\[
\|h - h_n\| \leq \|h - h_n\|_E + \|h_N - h_N\| + \|h_N\|_{R-E} + \|h\|_{R-E}.
\]
Regarding the R.H.S., choose \( N \) sufficiently large so that the second term is \( \leq \epsilon/4 \) for all \( n \geq N \), then choose \( E \) so large that the sum of the last two terms \( \leq \epsilon/2 \), and finally take \( n > N \) and large enough that the first term \( \leq \epsilon/4 \).

We restate Theorem 3 in an equivalent form.

**Theorem 3'**. For an arbitrary Riemannian manifold \( R \), \( HL^2(R) \) is a Hilbert space, and there exists a unique symmetric reproducing kernel \( k \in HL^2(R) \) satisfying \( h = \langle h, k \rangle \) for all \( h \in HL^2(R) \). Also, \( k_E = \sup_{x \in E} k_x(x) < \infty \) for each compact \( E \subset R \).

That Theorem 3' implies Theorem 3 is clear. Conversely, the existence of \( k_x \) is assured by the Riesz representation theorem for bounded functionals defined on a Hilbert space, and by (1) which says, for every \( x \in R \), evaluation is a bounded functional on \( HL^2 \). That \( k_E \) is finite is seen by substituting \( k_x \) into (1) thereby obtaining \( k_E \leq M_E^2 \). The symmetry and uniqueness of \( k \) is confirmed in the usual manner.

**Lemma 1.** \( k_x(y) \) is continuous on \( R \times R \).

**Proof.** Fix \( x_0, y_0 \in R \) and consider the inequality
\[
|k_y(x) - k_{y_0}(x_0)| \leq |k_y(x) - k_{y_0}(x)| + |k_{y_0}(x) - k_{y_0}(x_0)|, \quad x, y \in R.
\]

On the R.H.S., the second term offers no difficulty since \( k_{y_0} \) is continuous, in fact harmonic, and we direct our attention to the first term.

Let \( U, V \) be regular subregions of \( R \). By Schwarz \( |k_x(y)|^2 \leq k_x(k_y) \) so that by Theorem 3' \( k \) is bounded on \( U \times V \). Consequently, there is no harm in assuming that \( k \) is positive on \( U \times V \). Let \( U_1 \) be a regular subregion whose closure \( \overline{U_1} \) is contained in \( U \). For \( x \in \overline{U_1}, y, y_0 \in V \), we have
\[
k_y(x) - k_{y_0}(x) = \int_{\partial U} (k_y(\xi) - k_{y_0}(\xi)) \frac{\partial}{\partial \xi} g^U_x(\xi) dS_\xi.
\]
By the continuity of \( \frac{\partial g^U_x(\xi)}{\partial \xi} \) on \( \overline{U_1} \times \partial U \),
\[
|k_y(x) - k_{y_0}(x)| \leq \text{const} \int_{\partial U} |k_y(\xi) - k_{y_0}(\xi)| dS_\xi, \quad x \in U_1.
\]
By Harnack's inequality there exists $c > 0$ such that

$$0 < k_y(\xi) = k_x(y) < ck_\xi(y_0) = c k_y(\xi)$$

for all $y \in V_1 \subset V, \xi \in \partial U$. Therefore,

$$|k_y(\xi) - k_y(\xi)| \leq (c + 1) k_y(\xi),$$

where the R.H.S. is integrable over the $\partial U$. Hence the Lebesgue Dominated Convergence Theorem applies to (2), and the proof is herewith complete.

Let $\Omega$ denote a regular subregion of $R$ and $\Omega'$ another regular subregion or possibly $\Omega' = R$.

**Lemma 2.** For every $x \in \Omega \subset \Omega'$,

$$0 \leq \|k_x^\Omega - k_x^{\Omega'}\|_\Omega^2 \leq k_x^\Omega(\chi) - k_x^{\Omega'}(\chi).$$

**Proof.** Expand $\langle k_x^\Omega - k_x^{\Omega'}, k_x^\Omega - k_x^{\Omega'} \rangle_\Omega$ and employ the reproducing properties of $k_x^\Omega$ and $k_x^{\Omega'}$.

**Remark.** Taking $\Omega'$ to be $R$, we obtain as an immediate consequence of Lemma 2 that

$$\|k_x^\Omega\|_\Omega^2 = k_x^\Omega(\chi) = \|k_x\|_\Omega^2.$$ 

Thus, $k_x^\Omega \rightarrow k_x$ in $L^2$ norm which together with (1) of Theorem 3 says that the convergence is also uniform on compacta.

4. A reproducing kernel for biharmonic potentials with square integrable Laplacians. If $R \in O, then H^2 \subset F$. To see this, recall that in the proof of Corollary 2 in §2, for fixed $x, y \in \Omega, x \neq y, g_y \leq (M/m)g_x$ on $R - \Omega$ so that

$$\|g_y\|_{R-\Omega} \leq (M/m) \langle g_y, g_x \rangle_{R-\Omega} < \chi(x) < \infty.$$ 

Hence, for $h \in H^2(R)$,

$$\int_R |h(\xi)|g_y(\xi) d\xi = \langle h, g_y \rangle_{\Omega} + \langle h, g_y \rangle_{R-\Omega} \leq \langle h, g_y \rangle_{\Omega} + \|h\| \|g_y\|_{R-\Omega} < \infty.$$ 

If $R \in O$, by the biharmonic potentials with square integrable Laplacians, we mean $GHL^2 = \{gh| h \in H^2\}$. We define an inner product $(\cdot, \cdot)$ on $GHL^2$ by

$$(u, v) = \langle \Delta u, \Delta v \rangle$$

and denote the induced norm by $\|\|$. 

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**Theorem 4.** If \( R \not\in \mathcal{O}_y \), then \( GHL^2(R) \) is a Hilbert space, and there exists a positive function \( M^R \) such that \(|u| \leq M^R \|u\|\) for all \( u \in GHL^2(R) \).

**Proof.** That \( GHL^2 \) is a Hilbert space is easily seen from the fact that \( HL^2 \) is a Hilbert space.

For \( x \in R \) and \( c > 0 \), let \( U = \{ \xi \in R | g_x(\xi) > c \} \). For \( u \in GHL^2 \), apply Green's identity to \( \gamma^U_x \) and \( h = \Delta u \) thereby obtaining

\[
G_U h(x) = \int_{\partial U} h(\xi) \frac{\partial}{\partial \xi} \gamma^U_x(\xi) \, dS_\xi.
\]

From this representation together with the reasoning as given in the first part of Theorem 3, it follows that there exists \( m(x) > 0 \) such that \(|G_U h(x)| \leq m^U(x) \|h\|_U\) for all \( u \in GHL^2 \). Note that \( G_U h(x) = \langle h, g_x - c \gamma_U \rangle \) since \( g^U_x = g_x - c \) on \( U \). Consequently, \(|h, g_x - c \gamma_U| \leq m^U(x) \|h\|_U\). Therefore, we have

\[
|u(x)| \leq |\langle h, g_x - c \gamma_U \rangle| + |\langle h, \gamma_U \rangle| + |\langle h, g_x \gamma_{R-U} \rangle|
\leq m^U(x) \|h\|_U + c \sqrt{\text{vol} \, U} \|h\|_U + \|g_x \gamma_{R-U} \|h\|_{R-U}
\leq M^R(x) \|u\|
\]

where

\[
M^R(x) = \max\{m^U(x) + c \sqrt{\text{vol} \, U} \|g_x \gamma_{R-U}\}\}
\]

is finite and independent of \( u \).

**Theorem 4'.** If \( R \not\in \mathcal{O}_y \), then \( GHL^2(R) \) is a Hilbert space and there exists a unique symmetric reproducing kernel \( K \in GHL^2(R) \) such that \( u = \langle u, K \rangle \) for all \( u \in GHL^2(R) \).

**Theorem 5.** If \( R \not\in \mathcal{O}_y \), then

\[
K_x(y) = \int_{R \times R} g_x(\xi) k_\xi(\eta) g_y(\eta) \, d\xi \, d\eta, \quad x, y \in R.
\]

**Proof.** Define \( h_x \) on \( R \) by \( h_x(\xi) = Gk_\xi(x) \); then we claim that \( h_x = \Delta K_x \).

To establish our claim, it suffices to show that \( \langle \varphi, h_x \rangle = \langle \varphi, \Delta K_x \rangle \) for all \( \varphi \in C_0^\infty \). Since \( HL^2 \) is a closed subspace of \( L^2 \), there exist unique \( \varphi_1 \in HL^2 \), \( \varphi_2 \in (HL^2)^\perp \) such that \( \varphi = \varphi_1 + \varphi_2 \). Here \( (HL^2)^\perp \) denotes the orthogonal complement of \( HL^2 \) in \( L^2 \). Therefore,
\[ \langle \varphi, h_x \rangle = \int_R \varphi(\xi) Gk_\xi(x) \, d\xi = \int_R \varphi(\xi) \left( \int_R k_\xi(\eta) g_x(\eta) \, d\eta \right) \, d\xi = \int_R \left( \int_R \varphi(\xi) k_\eta(\xi) \, d\xi \right) g_x(\eta) \, d\eta = \int_R \varphi_1(\eta) g_x(\eta) \, d\eta = G \varphi_1(x). \]

The first equality is just the definition of \( h_x \), the second and last equalities come from the definition of the \( G \)-operator, the third equality is Fubini, and the fourth equality uses the reproducing property of \( k_\eta \) and the fact that \( \varphi_2 \) and \( k_\eta \) are orthogonal. On the other hand, by the orthogonality of \( \varphi_2 \) and \( \Delta K_x \), by property (i) of \( §2 \), by the definition of \( \langle , \rangle \), and by the reproducing property of \( K_x \), we have

\[ \langle \varphi, \Delta K_x \rangle = \langle \varphi_1, \Delta K_x \rangle = \langle \Delta G \varphi_1, K_x \rangle = (G \varphi_1, K_x) = G \varphi_1(x), \]

which completes the proof of our claim.

Since \( K_x \in GHL^2 \) there exists \( h \in HL^2 \) such that \( K_x = Gh \). However, \( h_x = \Delta K_x = h \) so that \( K_x = Gh_x \) which when written out is the R.H.S. of our theorem.

5. Convergence of reproducing kernels for potentials. In this section, we prove the following theorem.

**Theorem 6.** For \( R \notin O_\gamma, K^\Omega \to K \) pointwise and in norm \( |||| \) as \( \Omega \nearrow R \).

**Lemma 3.** If \( R \notin O_\gamma \), then for every \( x \in R \) \( \|g_x - g_x^\Omega\| \searrow 0 \) as \( \Omega \nearrow R \).

**Proof.** By (3) at the beginning of \( §4 \), \( R \notin O_\gamma \) guarantees that \( \|g_x\|_{R-\Omega} < \epsilon/3 \) for given \( \epsilon > 0 \) and sufficiently large \( \Omega \). Having chosen such an \( \Omega \), choose \( c > 0 \) so small that \( \Omega \subset \Omega' \) where \( \Omega' = \{ \xi \in R | g_x(\xi) > c \} \) and \( c \sqrt{\text{vol } \Omega} < \epsilon/3 \). Consider the inequality

\[ \|g_x - g_x^\Omega\| \leq \|g_x\|_{R-\Omega} + \|g_x^\Omega\|_{R-\Omega} + \|g_x - g_x^\Omega\|_\Omega. \]

Since \( \|g_x^\Omega\|_{R-\Omega} < \|g_x\|_{R-\Omega} < \epsilon/3 \), the sum of the first two terms on the R.H.S. is \( < 2\epsilon/3 \). Furthermore, \( g_x - g_x^\Omega \) is harmonic on \( \Omega' \) and \( = c \) on \( \partial \Omega' \) so that \( g_x - g_x^\Omega = c \) throughout \( \Omega' \). Therefore, the last term \( = c \sqrt{\text{vol } \Omega} < \epsilon/3 \) which completes the proof.

Regarding functions which up to now were considered to be defined only on some subregion \( \Omega \) of a Riemannian manifold \( R \), we shall find it convenient to henceforth consider them to be defined on all of \( R \) by making them \( = 0 \) on the complement of \( \Omega \). In particular, by setting \( g^\Omega = 0 \) outside of \( \Omega \), we have also extended \( G_\Omega \) to be an operator on \( F(R) \)-explicitly, \( G_\Omega f(x) = 0, x \in R - \Omega, f \in F(R) \). Not only will our notation fail to distinguish between a function defined on \( \Omega \) and its trivial extension, it will continue to ignore the distinction between a function and its restriction.
Recall from the proof of Theorem 5, the function $h_x$ given by $h_x(\xi) = G_k(x)$, and similarly define $h_\Omega^Q$ by $h_\Omega^Q(\xi) = G_\Omega k_\Omega^Q(x)$. Also define $h_{\Omega,x}$ by $h_{\Omega,x}(\xi) = G_\Omega k_{\Omega,x}(x)$. Considerations at the beginning of §4 assure that these functions are well defined. That $h_x \in L^2$ is clear since $h_x = \Delta_x \in L^2$ and similarly for $h_\Omega^Q$. To show $h_{\Omega,x}$ is square integrable, we need only show that $h_x - h_{\Omega,x} \in L^2$. We note that $f = g_x - g_\Omega^Q \in L^2$ by Lemma 3 and that $h_x(\xi) - h_{\Omega,x}(\xi) = \langle f, k_\xi \rangle$. Since $f = f_1 + f_2$ with $f_1 \in H\Omega^2$ and $f_2 \in (\Omega^2 L^2)^1$, we see that $\langle f, k_\xi \rangle = f_1(\xi)$. Therefore we conclude,

$$\|h_x - h_{\Omega,x}\| = \|f_1\| \leq \|f_1\| + \|f_2\| = \|f\| = \|g_x - g_\Omega^Q\| < \infty.$$  

We have proven:

**Lemma 4.** Given $R \in \Omega$, then $h_x, h_{\Omega,x} \in H\Omega^2(R)$, and $\|h_x - h_{\Omega,x}\| \to 0$ as $\Omega \nearrow R$. In fact, $\|h_x - h_{\Omega,x}\| \leq \|g_x - g_\Omega^Q\|$.

**Lemma 5.** For $R \in \Omega$, $\|h_x - h_\Omega^Q\| \to 0$ as $\Omega \nearrow R$.

**Proof.** Since

$$\|h_x - h_{\Omega,x}\| < \|h_x - h_{\Omega,x}\| + \|h_{\Omega,x} - h_\Omega^Q\|,$$

by Lemma 4 we need only show that $\|h_{\Omega,x} - h_\Omega^Q\| \to 0$ as $\Omega \nearrow R$ for every compact $E$. By the definitions of $h_{\Omega,x}$ and $h_\Omega^Q$, the linearity of the $G_\Omega$-operator, Theorem 4, and Lemma 2, we have for all $\Omega \supset E$,

$$\int_E (h_{\Omega,x}(\xi) - h_\Omega^Q(\xi))^2 d\xi = \int_E (G_\Omega k_{\Omega,x}(x) - G_\Omega k_\Omega^Q(x))^2 d\xi$$

$$= \int_E [G_\Omega(k_{\Omega} - k_\Omega^Q)(x)]^2 d\xi \leq (M^Q(x))^2 \int_E \|k_{\Omega} - k_\Omega^Q\|_\Omega^Q d\xi$$

$$< (M^Q(x))^2 \int_E (k_{\Omega}(\xi) - k_\Omega^Q(\xi)) d\xi.$$  

By (4) in §4, we see that $M^Q(x) \leq M^R(x) < \infty$. Also, by Lemmas 1 and 2, $k_{\Omega}(\xi)$ and $k_\Omega^Q(\xi)$ are measurable, in fact continuous, and $k_{\Omega}(\xi) \nearrow k_\Omega^Q(\xi)$ on $E$ so that the Monotone Convergence Theorem assures the last expression $\to 0$ as $\Omega \nearrow R$.

**Completion of the proof of Theorem 6.** Subtracting and adding $\langle k_\Omega^Q, K_x \rangle$ and by Schwarz, we obtain

$$|K_x(\xi) - K_\Omega^Q(\xi)| \leq \|K_\xi\| \cdot \|K_x - K_\Omega^Q\| + \|K_\Omega^Q\| \cdot \|K_x - K_\Omega^Q\|,$$

so that we need only show $K_x \nearrow K_\Omega^Q$ in norm. However, $\|K_x - K_\Omega^Q\| = \|h_x - h_\Omega^Q\| \to 0$ as $\Omega \nearrow R$. 

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6. Completion of the proof of Theorem 1. We first show that \( \Gamma_x^\Omega = \gamma_x^\Omega - K_x^\Omega \) for each regular subregion \( \Omega, x \in \Omega \). Since \( K_x^\Omega \) is biharmonic on \( \Omega \) and \( \gamma_x^\Omega \) has a biharmonic singularity at \( x \in \Omega \), surely \( \gamma_x^\Omega - K_x^\Omega \) is biharmonic on \( \Omega \setminus \{x\} \) and possesses a biharmonic singularity at \( x \). It is also clear that \( \gamma_x^\Omega - K_x^\Omega = 0 \) on \( \partial \Omega \) since each term = 0 on \( \partial \Omega \). Using this together with Green's identity, we have

\[
\int_{\partial \Omega} h(\xi) \frac{\partial}{\partial \xi} (\gamma_x^\Omega(\xi) - K_x^\Omega(\xi)) dS_\xi = -\int_{\Omega} h(\xi) \Delta (\gamma_x^\Omega(\xi) - K_x^\Omega(\xi)) dV_\xi
\]

where \( h \) is harmonic. Since \( \int_{\Omega} h(\xi) \Delta \gamma_x^\Omega(\xi) dV_\xi = G_\Omega h(x) \) and \( \int_{\Omega} h(\xi) \Delta K_x^\Omega(\xi) = G_\Omega h(x) \), we conclude that the R.H.S. of (5) = 0. Applying Green's identity to the function = 1 and \( \gamma_x^\Omega - K_x^\Omega \), we see that \( \int_{\partial \Omega} (\partial/\partial \xi)(\gamma_x^\Omega(\xi) - K_x^\Omega(\xi)) dS_\xi = 0 \). Hence there exists a harmonic solution to the boundary value problem \( h = (\partial/\partial \nu)(\gamma_x^\Omega - K_x^\Omega) \) on \( \partial \Omega \). Substituting this solution into (5), we see that \( (\partial/\partial \nu)(\gamma_x^\Omega - K_x^\Omega) = 0 \) on \( \partial \Omega \), thereby verifying that \( \Gamma_x^\Omega = \gamma_x^\Omega - K_x^\Omega \). Hence, if \( \gamma \) exists, by Theorem 6 \( K \) exists, and \( \Gamma = \lim_{\Delta \to 0} (\gamma - K) = y - K \).

Lastly, \( K \) and \( \Gamma \) are orthogonal since

\[
(K_x, \Gamma_x) = \int_R \Delta K_x(\xi)(\Delta \gamma_x(\xi) - \Delta K_x(\xi)) d\xi = K_x(x) - K_x(x) = 0.
\]

In closing, I would like to hint at other applications of the methods presented. From Theorem 2, it is immediate that the existence of a positive quasiharmonic function implies the existence of \( \gamma \) [5]. On the other hand, it is clear that the existence of \( \gamma \) assures that the biharmonic functions with square integrable Laplacians possess Riesz representations [6], [9]. Since Theorem 3' guarantees that \( k \) always exists, one can define a span whose vanishing is equivalent to the nonexistence of nonzero square integrable harmonic functions [8]. Also, \( K \) may be found useful in formulating and solving a biharmonic interpolation problem similar to one known for harmonic functions [4, pp. 275–280], [7].

**BIBLIOGRAPHY**


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