A RELATION BETWEEN TWO BIHARMONIC GREEN'S FUNCTIONS ON RIEMANNIAN MANIFOLDS

BY

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ABSTRACT. The biharmonic Green's function γ whose values and Laplacian are identically zero on the boundary of a region and the biharmonic Green's function Γ whose values and normal derivative vanish on the boundary originated in the investigation of thin plates whose edges are simply supported or clamped, respectively. A relation between these two biharmonic Green's functions known for planar regions is extended to Riemannian manifolds thereby establishing that any Riemannian manifold for which γ exists must also carry Γ.

Introduction. In a paper by N. Aronszajn, the integral representation of Γ given by

$$\Gamma(x,y) = \int_D g(x,\xi)g(y,\xi)\,d\xi - \int_D g(x,\xi)k(\xi,\eta)g(y,\eta)\,d\xi\,d\eta$$

is credited to S. Zaremba (see [1, p. 387]) where g is the harmonic Green's function, k is the reproducing kernel for the square integrable harmonic functions and D is a regular subregion of the plane. (For physical interpretations of γ and Γ alluded to in the abstract, see e.g. [2, Chapter IV, particularly pp. 236, 242]. An informative discussion relating k and Γ for plane regions is given in [3] and [4, pp. 265–272].) In the present paper, we note that in this representation of Γ, the first term is none other than γ, and the second term is the reproducing kernel K for the biharmonic potentials with square integrable Laplacians w.r.t. an appropriate inner product ( , ). Also, in extending this relation between γ and Γ to Riemannian manifolds it is more natural to consider it as a representation of γ. Explicitly, we prove

Theorem 1. On an arbitrary Riemannian manifold, if γ exists, then K and Γ also exist. Furthermore, K and Γ are orthogonal w.r.t. ( , ) and γ = K + Γ.

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1. Definitions. Let $R$ denote a Riemannian manifold, $\Delta$ its Laplace-Beltrami operator, and $g^{\Omega}_{x}$ the harmonic Green's function for a regular subregion $\Omega \subset R$ with pole $x \in \Omega$. Expressed by $\chi^{\Omega}_{x}$ the biharmonic Green's function of $\Omega$ satisfying the boundary conditions

$$
\chi^{\Omega}_{x} = 0, \quad \Delta \chi^{\Omega}_{x} = 0 \quad \text{on } \partial \Omega,
$$

and by $\Gamma^{\Omega}_{x}$ the biharmonic Green's function of $\Omega$ satisfying

$$
\Gamma^{\Omega}_{x} = 0, \quad \frac{\partial}{\partial \nu} \Gamma^{\Omega}_{x} = 0 \quad \text{on } \partial \Omega,
$$

and where each biharmonic Green's function has a fundamental singularity at $x$, i.e. $\Delta \chi^{\Omega}_{x} - g^{\Omega}_{x}$ and $\Delta \Gamma^{\Omega}_{x} - g^{\Omega}_{x}$ each can be extended to a function harmonic in all of $\Omega$. In the above, $\partial/\partial \nu$ refers to the normal derivative and $\partial$ is the boundary operator.

If $\{\Omega\}$ is an exhaustion of $R$ by regular subregions, the biharmonic Green's functions $\chi_{x}, \Gamma_{x}$ of $R$ are said to exist provided the limits $\chi_{x} = \lim \chi^{\Omega}_{x}$ and $\Gamma_{x} = \lim \Gamma^{\Omega}_{x}$ as $\Omega \nearrow R$ exist and are finite on $R - \{x\}$. (Throughout this paper, if there is no reference to any region, it will be understood that the region shall be the entire manifold $R$, e.g. $\chi_{x} = \chi_{R}^{\Omega}$, $\Gamma_{x} = \Gamma_{R}^{\Omega}$.) If $\chi_{x}$ (similarly $\Gamma_{x}$) exists for all $x \in R$, we say that $R$ possesses the biharmonic Green's function $\chi$ (respectively $\Gamma$). The family of Riemannian manifolds void of $\chi$ or $\Gamma$ is denoted by $O_{\chi}$ or $O_{\Gamma}$, respectively.

**Corollary.** $O_{\Gamma} \subset O_{\chi}$ (2)

2. The biharmonic Green's function $\gamma$. The class of parabolic manifolds (manifolds $R$ void of the harmonic Green's function $g$, i.e. $g_{x} = \lim_{\Omega \nearrow R} g^{\Omega}_{x}$ is not finite for some $x \in R$) is customarily denoted by $O_{G}$. For $R \notin O_{G}$, we define a family $F$ of real valued functions on $R$ by

$$
F = \left\{ f \left| \int_{R} |f(\xi)| g_{x}(\xi) d\xi \text{ is well defined and finite for all } x \in R \right. \right\},
$$

and for $f \in F$ we define the function $Gf$ on $R$ by

$$
Gf(x) = \int_{R} f(\xi) g_{x}(\xi) d\xi = \langle f, g_{x} \rangle.
$$

The $G$-operator is an "inverse" for $\Delta$ in the following sense:

(i) If $f \in F$ and $Gf \in C^{2}(R)$, then $\Delta Gf = f$. 

(2) Subsequent to the writing of this paper, the author has been informed that although presently unavailable in the literature, two alternative proofs of the relation $O_{\Gamma} \subset O_{\chi}$ are known—both using entirely different methods from those presented here. Furthermore, it is known that $\phi < O_{\Gamma} < O_{\chi}$ (Chung-Nakai-Ralston-Sario).
(ii) If $\varphi \in C^\infty_0$, i.e. $\varphi$ is $C^\infty$ and has compact support in $R$, then $G\Delta \varphi = \varphi$. (For the proof of (i) see e.g. Sario-Wang-Range [9], and for the proof of (ii) merely apply Green's identity to $g$ and $\varphi$.)

**Theorem 2.** If $\gamma$ exists on $R$, $x \in R$, then $R \notin O_G$ and

$$
\gamma_y(y) = \int_R g_x(\xi) g_y(\xi) \, d\xi \quad \text{for all } y \in R.
$$

**Proof.** By the Monotone Convergence Theorem, it suffices to show that for each regular subregion $\Omega$, $x, y \in \Omega$, $\gamma_y(\Omega) = \int_\Omega g_x^\Omega(\xi) g_y^\Omega(\xi) \, d\xi$. Set $f_x(y) = \int_\Omega g_x^\Omega(\xi) g_y^\Omega(\xi) \, d\xi$; then $f_x = 0$ on $\partial \Omega$ since $g_y^\Omega = 0$ for $y \in \partial \Omega$. Furthermore, $\Delta f_x = g_x^\Omega$. To see this, we observe that for every $\varphi \in C^\infty_0(\Omega)$,

$$
\langle g_x^\Omega, \varphi \rangle_\Omega = \langle g_y^\Omega, \varphi \rangle_\Omega = \langle f_x, \Delta \varphi \rangle_\Omega = \langle \Delta f_x, \varphi \rangle_\Omega.
$$

The first equality is just property (ii) satisfied by the $G$-operator; the second equality comes from an application of Fubini's Theorem, and the last equality utilizes Green's identity. From $\Delta f_x = g_x^\Omega$, we see that $f_x$ has a biharmonic singularity at $x$, and $\Delta f_x = 0$ on $\partial \Omega$. Hence, $f_x$ satisfies the conditions that uniquely define $\gamma_y$, i.e.

$$
\gamma_y(\Omega) = \int_\Omega g_x^\Omega(\xi) g_y^\Omega(\xi) \, d\xi.
$$

**Corollary 1.** $\gamma$ is positive and symmetric.

**Corollary 2.** If $\gamma_x$ exists for some $x \in R$, then $\gamma_y$ exists for all $x \in R$.

**Proof.** For an arbitrary $y \in R$, we must show that $\gamma_y < \infty$ assuming $\gamma_x < \infty$ for some $x \in R$. As just seen, the existence of $\gamma_x$ for some $x$ implies the existence of $g_x$ for all $x$. Let $\Omega$ be a regular subregion containing both $x$ and $y$. For $\xi \in R$ and distinct from $x$ and $y$, let $C_1(\xi) = \langle g_y, g_x(\xi) \rangle_\Omega / \langle g_x, g_x(\xi) \rangle_\Omega$, $m = \min g_x$ and $M = \max g_y$ on $\partial \Omega$. We then have

$$
\gamma_y(\xi) = \langle g_y, g_x(\xi) \rangle_\Omega + \langle g_y, g_y(\xi) \rangle_{R-\Omega} \\
\leq C_1(\xi) \langle g_x, g_x(\xi) \rangle_\Omega + (M/m) \langle g_x, g_y(\xi) \rangle_{R-\Omega} \leq C(\xi) \gamma_y(\xi)
$$

where $C(\xi) = \max\{C_1(\xi), M/m\} < \infty$.

3. **Square integrable harmonic functions.** Let $HL^2(R)$ denote the square integrable harmonic functions on a Riemannian manifold $R$, and let $\|h\| = \langle h, h \rangle^{1/2}$ for $h \in HL^2(R)$.

**Theorem 3.** For an arbitrary Riemannian manifold $R$, $HL^2(R)$ is a Hilbert space. Furthermore, there exists a positive function $M$ on $R$ satisfying
(1) \[ |h| \leq M\|h\| \text{ for all } h \in HL^2(R) \]

and for which \( M_E = \sup_{x \in E} M(x) < \infty \) for every compact \( E \subset R \).

**Proof.** We first consider the existence of \( M \) together with the finiteness of \( M_E \). Given compact \( E \), let \( \Omega \) be a regular subregion containing \( E \). For \( x \in E \), \( c > 0 \), let \( A_c(x) \) be the annular region

\[ A_c(x) = \{ \xi \in \Omega | g_x^\Omega(\xi) < c \} \quad \text{and} \quad M_1 = \sup_{x \in E, \xi \in A_c(x)} |\text{grad}_\xi g_x^\Omega(\xi)|. \]

The finiteness of \( M_1 \) is a consequence of the continuity of \( g_x^\Omega(\xi) \) and \( |\text{grad}_\xi g_x^\Omega(\xi)| \) on \( \Omega \times \Omega \)-diagonal and the fact that

\[ \sup_{x \in E, \xi \in A_c(x)} g_x^\Omega(\xi) = c. \]

We think of \( A_c(x) \) as being composed of a collection of level surfaces \( \{S_d(x)\}_{0 < d < c} \) where \( S_d(x) = \{ \xi \in \Omega | g_x^\Omega(\xi) = d \} \). If \( \alpha \) is a flow line joining \( S_{d_1}(x) \) to \( S_{d_2}(x) \), \( 0 \leq d_1 < d_2 \leq c \), we have

\[ d_2 - d_1 = \int_{\alpha} |\text{grad}_\xi g_x^\Omega(\xi)| dL_\xi \leq M_1 (\text{length } \alpha) \]

where \( dL_\xi \) refers to arc length. Hence, \( (d_2 - d_1)/M_1 \leq \text{length } \alpha \). From this along with

\[ |h(x)| \leq \int_{S_d} \left| \frac{\partial}{\partial \nu_\xi} g_x^\Omega(\xi) \right| \cdot |h(\xi)| dS_\xi, \quad x \in E, \ 0 \leq d \leq c, \]

it follows that

\[ \frac{|h(x)|}{M_1} \leq M_1 \int_{A_c(x)} |h(\xi)| dV_\xi. \]

Here, \( dS_\xi \) is the surface element and \( dV_\xi \) is the volume element. Thus, by Schwarz we obtain

\[ |h(x)| \leq (M_1^2/c)\sqrt{\text{vol } \Omega} \|h\| \text{ for all } x \in E, \ h \in HL^2. \]

The existence of \( M \) and the finiteness of \( M_E \) is now clear.

To see that \( HL^2 \) is a Hilbert space, let \( \{h_n\} \) be Cauchy in \( HL^2 \). By the first part of this proof just completed,

\[ |h_n(x) - h_m(x)| \leq M_g \|h_n - h_m\|, \quad x \in E. \]

Hence there exists \( h \) harmonic on \( R \) for which \( h_n \rightarrow h \) uniformly on compact subsets of \( R \). In particular, \( \|h - h_n\|_E \rightarrow 0 \) as \( n \rightarrow \infty \). Also, \( \{|h_n|\} \) is bounded

\[ (3) \text{ The reader might find it enlightening to compare the first part of this proof with an inequality given in [3, p. 503].} \]
since \( \{h_n\} \) is Cauchy. We conclude that \( h \in H^{1,2} \) from the inequality
\[
\|h\|_E \leq \|h - h_n\|_E + \|h_n\|
\]
by taking the limit as \( n \to \infty \) and then the supremum over all compact \( E \subset R \). To see that \( h_n \to h \) in norm, we consider the inequality
\[
\|h - h_n\| \leq \|h - h_n\|_E + \|h_N - h_n\| + \|h\|_{R-E} + \|h\|_{R-E}.
\]
Regarding the R.H.S., choose \( N \) sufficiently large so that the second term is \( \leq \epsilon/4 \) for all \( n \geq N \), then choose \( E \) so large that the sum of the last two terms \( \leq \epsilon/2 \), and finally take \( n \geq N \) and large enough that the first term \( \leq \epsilon/4 \).

We restate Theorem 3 in an equivalent form.

**Theorem 3'.** For an arbitrary Riemannian manifold \( R \), \( H^{1,2}(R) \) is a Hilbert space, and there exists a unique symmetric reproducing kernel \( k \in H^{1,2}(R) \) satisfying \( h = \langle h, k \rangle \) for all \( h \in H^{1,2}(R) \). Also, \( k_E = \sup_{x \in E} k_x(x) < \infty \) for each compact \( E \subset R \).

That Theorem 3' implies Theorem 3 is clear. Conversely, the existence of \( k_x \) is assured by the Riesz representation theorem for bounded functionals defined on a Hilbert space, and by (1) which says, for every \( x \in R \), evaluation is a bounded functional on \( H^{1,2} \). That \( k_E \) is finite is seen by substituting \( k_x \) into (1) thereby obtaining \( k_E \leq M^2_E \). The symmetry and uniqueness of \( k \) is confirmed in the usual manner.

**Lemma 1.** \( k_x(y) \) is continuous on \( R \times R \).

**Proof.** Fix \( x_0, y_0 \in R \) and consider the inequality
\[
|k_y(x) - k_{y_0}(x_0)| \leq |k_y(x) - k_{y_0}(x)| + |k_{y_0}(x) - k_{y_0}(x_0)|, \quad x, y \in R.
\]
On the R.H.S., the second term offers no difficulty since \( k_{y_0} \) is continuous, in fact harmonic, and we direct our attention to the first term.

Let \( U, V \) be regular subregions of \( R \). By Schwarz \( |k_x(y)|^2 \leq k_x(x)k_y(y) \) so that by Theorem 3' \( k \) is bounded on \( U \times V \). Consequently, there is no harm in assuming that \( k \) is positive on \( U \times V \). Let \( U_1 \) be a regular subregion whose closure \( \overline{U}_1 \) is contained in \( U \). For \( x \in \overline{U}_1, y, y_0 \in V \), we have
\[
k_y(x) - k_{y_0}(x) = \int_{\partial U} (k_y(\xi) - k_{y_0}(\xi)) \frac{\partial}{\partial \xi} g^U_\xi(\xi) \, dS_\xi.
\]
By the continuity of \( \partial g^U_\xi(\xi)/\partial \xi \) on \( \overline{U}_1 \times \partial U \),
\[
|k_y(x) - k_{y_0}(x)| \leq \text{const} \int_{\partial U} |k_y(\xi) - k_{y_0}(\xi)| \, dS_\xi, \quad x \in \overline{U}_1.
\]
By Harnack's inequality there exists $c > 0$ such that

$$0 < k_y(\xi) = k_\xi(y) < ck_\xi(y_0) = ck_y(\xi)$$

for all $y \in V_1 \subset V$, $\xi \in \partial U$. Therefore,

$$|k_y(\xi) - k_{y_0}(\xi)| \leq (c + 1)k_{y_0}(\xi),$$

where the R.H.S. is integrable over the $\partial U$. Hence the Lebesgue Dominated Convergence Theorem applies to (2), and the proof is herewith complete.

Let $\Omega$ denote a regular subregion of $R$ and $\Omega'$ another regular subregion or possibly $\Omega' = R$.

**Lemma 2.** For every $x \in \Omega \subset \Omega'$,

$$0 \leq \|k_x^\Omega - k_x^{\Omega'}\|_\Omega^2 \leq k_x^\Omega(x) - k_x^{\Omega'}(x).$$

**Proof.** Expand $\langle k_x^\Omega - k_x^{\Omega'}, k_x^\Omega - k_x^{\Omega'} \rangle_\Omega$ and employ the reproducing properties of $k_x^\Omega$ and $k_x^{\Omega'}$.

**Remark.** Taking $\Omega'$ to be $R$, we obtain as an immediate consequence of Lemma 2 that

$$\|k_x^\Omega\|_\Omega^2 = k_x^\Omega(x) \leq k_x^\Omega(x) = \|k_x\|^2.$$

Thus, $k_x^\Omega \to k_x$ in $L^2$ norm which together with (1) of Theorem 3 says that the convergence is also uniform on compacta.

**4. A reproducing kernel for biharmonic potentials with square integrable Laplacians.** If $R \notin O_x$, then $HL^2 \subset F$. To see this, recall that in the proof of Corollary 2 in §2, for fixed $x, y \in \Omega$, $x \neq y$, $g_y \leq (M/m)g_x$ on $R - \Omega$ so that

$$||g_y||_{R-\Omega} \leq (M/m)\langle g_y, g_x \rangle_{R-\Omega} < \chi_x(y) < \infty.$$  

Hence, for $h \in HL^2(R)$,

$$\int_R |h(\xi)||g_y(\xi)| d\xi = \langle |h|, g_y \rangle_{\Omega} + \langle |h|, g_y \rangle_{R-\Omega}$$

$$\leq \langle |h|, g_y \rangle_{\Omega} + \|h\| \|g_y\|_{R-\Omega} < \infty.$$

If $R \notin O_x$, by the biharmonic potentials with square integrable Laplacians, we mean $GHL^2 = \{Gh|h \in HL^2\}$. We define an inner product $(\cdot, \cdot)$ on $GHL^2$ by

$$(u, v) = \langle \Delta u, \Delta v \rangle, \quad u, v \in GHL^2$$

and denote the induced norm by $\|\|$.  

Theorem 4. If $R \not\in \mathcal{O}_y$, then $GHL^2(R)$ is a Hilbert space, and there exists a positive function $M^R$ such that $|u| \leq M^R \|u\|$ for all $u \in GHL^2(R)$.

Proof. That $GHL^2$ is a Hilbert space is easily seen from the fact that $H^2$ is a Hilbert space.

For $x \in R$ and $c > 0$, let $U = \{\xi \in R|g_x(\xi) > c\}$. For $u \in GHL^2$, apply Green's identity to $\chi^U_x$ and $h - \Delta u$ thereby obtaining

$$G_u h(x) = \int_{\partial U} h(\xi) \frac{\partial \gamma^U_x(\xi)}{\partial \xi} dS_\xi.$$

From this representation together with the reasoning as given in the first part of Theorem 3, it follows that there exists $m(x) > 0$ such that $|G_u h(x)| \leq m^U(x) \|h\|_U$ for all $u \in GHL^2$. Note that $G_u h(x) = \langle h, g_x - c \rangle_U$ since $g^U_x = g_x - c$ on $U$. Consequently, $|\langle h, g_x - c \rangle_U| \leq m^U(x) \|h\|_U$. Therefore, we have

$$|u(x)| \leq |\langle h, g_x - c \rangle_U| + |\langle h, c \rangle_U| + |\langle h, g_x \rangle_{R-U}|$$

$$\leq m^U(x) \|h\|_U + c \sqrt{\text{Vol } U} \|h\|_U + \|g_x\|_{R-U} \|h\|_{R-U}$$

$$\leq M^R(x) \|u\|_U$$

where

$$(4) \quad M^R(x) = \max\{m^U(x) + c \sqrt{\text{Vol } U}, \|g_x\|_{R-U}\}$$

is finite and independent of $u$.

Theorem 4'. If $R \not\in \mathcal{O}_y$, then $GHL^2(R)$ is a Hilbert space and there exists a unique symmetric reproducing kernel $K \in GHL^2(R)$ such that $u = \langle u, K \rangle$ for all $u \in GHL^2(R)$.

Theorem 5. If $R \not\in \mathcal{O}_y$, then

$$K_x(y) = \int_{R \times R} g_x(\xi) k_x(\eta) g_y(\eta) d\xi d\eta, \quad x, y \in R.$$

Proof. Define $h_x$ on $R$ by $h_x(\xi) = Gk_x(\xi)$; then we claim that $h_x = \Delta K_x$.

To establish our claim, it suffices to show that $\langle \phi, h_x \rangle = \langle \phi, \Delta K_x \rangle$ for all $\phi \in C^\infty_0$. Since $H^2$ is a closed subspace of $L^2$, there exist unique $\varphi_1 \in H^2$, $\varphi_2 \in (H^2)^\perp$ such that $\varphi = \varphi_1 + \varphi_2$. Here $(H^2)^\perp$ denotes the orthogonal complement of $H^2$ in $L^2$. Therefore,
\[ \langle \varphi, h_x \rangle = \int_R \varphi(\xi)G_k\xi(x) \, d\xi = \int_R \varphi(\xi) \left( \int_R k_\eta(\eta)g_x(\eta) \, d\eta \right) \, d\xi = \int_R \left( \int_R \varphi(\xi)k_\eta(\xi) \, d\xi \right)g_x(\eta) \, d\eta = \int_R \varphi(\eta)g_x(\eta) \, d\eta = G\varphi(\eta)(x). \]

The first equality is just the definition of \( h_x \), the second and last equalities come from the definition of the \( G \)-operator, the third equality is Fubini, and the fourth equality uses the reproducing property of \( k_\eta \) and the fact that \( \varphi_2 \) and \( k_\eta \) are orthogonal. On the other hand, by the orthogonality of \( \varphi_2 \) and \( \Delta K_x \), by property (i) of \( \S 2 \), by the definition of \( ( , ) \), and by the reproducing property of \( K_x \), we have

\[ \langle \varphi, \Delta K_x \rangle = \langle \varphi_1, \Delta K_x \rangle = \langle \Delta G\varphi_1, \Delta K_x \rangle = (G\varphi_1, K_x) = G\varphi_1(x), \]

which completes the proof of our claim.

Since \( K_x \in GHL^2 \) there exists \( h \in HL^2 \) such that \( K_x = Gh \). However, \( h_x = \Delta K_x = h \) so that \( K_x = Gh_x \) which when written out is the R.H.S. of our theorem.

5. Convergence of reproducing kernels for potentials. In this section, we prove the following theorem.

**Theorem 6.** For \( R \not\in O_\gamma \), \( K^\Omega \rightarrow K \) pointwise and in norm \( || || \) as \( \Omega \nearrow R \).

**Lemma 3.** If \( R \not\in O_\gamma \), then for every \( x \in R \) \( ||g_x - g_x^\Omega|| \leq 0 \) as \( \Omega \nearrow R \).

**Proof.** By (3) at the beginning of \( \S 4 \), \( R \not\in O_\gamma \) guarantees that \( ||g_x||_{R-\Omega} < \epsilon/3 \) for given \( \epsilon > 0 \) and sufficiently large \( \Omega \). Having chosen such an \( \Omega \), choose \( c > 0 \) so small that \( \Omega \subset \Omega' \) where \( \Omega' = \{ \xi \in R|g_x(\xi) > c \} \) and \( c\sqrt{\text{vol} \Omega} < \epsilon/3 \). Consider the inequality

\[ ||g_x - g_x^\Omega|| < ||g_x||_{R-\Omega} + ||g_x^\Omega||_{R-\Omega} + ||g_x - g_x^\Omega||_\Omega. \]

Since \( ||g_x^\Omega||_{R-\Omega} < ||g_x||_{R-\Omega} < \epsilon/3 \), the sum of the first two terms on the R.H.S. is \( < 2\epsilon/3 \). Furthermore, \( g_x - g_x^\Omega \) is harmonic on \( \Omega' \) and \( = c \) on \( \partial\Omega' \) so that \( g_x - g_x^\Omega = c \) throughout \( \Omega' \). Therefore, the last term \( = c\sqrt{\text{vol} \Omega} < \epsilon/3 \) which completes the proof.

Regarding functions which up to now were considered to be defined only on some subregion \( \Omega \) of a Riemannian manifold \( R \), we shall find it convenient to henceforth consider them to be defined on all of \( R \) by making them \( = 0 \) on the complement of \( \Omega \). In particular, by setting \( g^\Omega = 0 \) outside of \( \Omega \), we have also extended \( G_\Omega \) to be an operator on \( F(R) \)-explicitly, \( G_\Omega f(x) = 0, x \in R - \Omega, f \in F(R) \). Not only will our notation fail to distinguish between a function defined on \( \Omega \) and its trivial extension, it will continue to ignore the distinction between a function and its restriction.
Recall from the proof of Theorem 5, the function $h_x$ given by $h_x(\xi) = Gk_\xi(x)$, and similarly define $h_x^\Omega$ by $h_x^\Omega(\xi) = G_\Omega k_\xi^\Omega(x)$. Also define $h_{\Omega,x}$ by $h_{\Omega,x}(\xi) = G_\Omega k_\xi(x)$. Considerations at the beginning of §4 assure that these functions are well defined. That $h_x \in L^2$ is clear since $h_x = \Delta K_x \in L^2$ and similarly for $h_x^\Omega$. To show $h_{\Omega,x}$ is square integrable, we need only show that $h_x - h_{\Omega,x} \in L^2$. We note that $f = g_x - g_x^\Omega \in L^2$ by Lemma 3 and that $h_x(\xi) - h_{\Omega,x}(\xi) = \langle f, k_\xi \rangle$. Since $f = f_1 + f_2$ with $f_1 \in HL^2$ and $f_2 \in (HL^2)^\perp$, we see that $\langle f, k_\xi \rangle = f_1(\xi)$. Therefore we conclude,

$$\|h_x - h_{\Omega,x}\| = \|f_1\| \leq \|f_1\| + \|f_2\| = \|f\| = \|g_x - g_x^\Omega\| < \infty.$$  

We have proven:

**Lemma 4.** Given $R \not\subset \Omega$, then $h_x, h_{\Omega,x} \in HL^2(R)$, and $\|h_x - h_{\Omega,x}\| \to 0$ as $\Omega \nearrow R$. In fact, $\|h_x - h_{\Omega,x}\| \leq \|g_x - g_x^\Omega\|$.

**Lemma 5.** For $R \not\subset \Omega$, $\|h_x - h_x^\Omega\| \to 0$ as $\Omega \nearrow R$.

**Proof.** Since

$$\|h_x - h_x^\Omega\| \leq \|h_x - h_{\Omega,x}\| + \|h_{\Omega,x} - h_x^\Omega\|,$$

by Lemma 4 we need only show that $\|h_{\Omega,x} - h_x^\Omega\| \to 0$ as $\Omega \nearrow R$ for every compact $E$. By the definitions of $h_{\Omega,x}$ and $h_x^\Omega$, the linearity of the $G_\Omega$-operator, Theorem 4, and Lemma 2, we have for all $\Omega \supset E$.

$$\int_E (h_{\Omega,x}(\xi) - h_x^\Omega(\xi))^2 d\xi = \int_E (G_\Omega k_\xi(x) - G_\Omega k_\xi^\Omega(x))^2 d\xi$$

$$= \int_E [G_\Omega (k_\xi - k_\xi^\Omega)]^2(x) d\xi \leq (M(x))^2 \int_E \|k_\xi - k_\xi^\Omega\|_\Omega d\xi$$

$$\leq (M(x))^2 \int_E (k_\xi(\xi) - k_\xi(\xi))^2 d\xi.$$  

By (4) in §4, we see that $M(x) < M(x) < \infty$. Also, by Lemmas 1 and 2, $k_\xi^\Omega(x)$ and $k_\xi(x)$ are measurable, in fact continuous, and $k_\xi^\Omega(\xi) \sim k_\xi(\xi)$ on $E$ so that the Monotone Convergence Theorem assures the last expression to $0$ as $\Omega \nearrow R$.

**Completion of the proof of Theorem 6.** Subtracting and adding $\langle K_x^\Omega, K_x \rangle$ and by Schwarz, we obtain

$$|K_x(\xi) - K_x^\Omega(\xi)| \leq \|K_\xi\| \cdot \|K_x - K_x^\Omega\| + \|K_x^\Omega\| \cdot \|K_\xi - K_x^\Omega\|,$$

so that we need only show $K_x^\Omega \to K_x$ in norm. However, $\|K_x - K_x^\Omega\| = \|h_x - h_x^\Omega\| \to 0$ as $\Omega \nearrow R$.  

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6. Completion of the proof of Theorem 1. We first show that $\Gamma_x^\Omega = \gamma_x^\Omega - K_x^\Omega$
for each regular subregion $\Omega$, $x \in \Omega$. Since $K_x^\Omega$ is biharmonic on $\Omega$ and $\gamma_x^\Omega$ has
a biharmonic singularity at $x \in \Omega$, surely $\gamma_x^\Omega - K_x^\Omega$ is biharmonic on $\Omega - \{x\}$
and possesses a biharmonic singularity at $x$. It is also clear that $\gamma_x^\Omega - K_x^\Omega = 0$
on $\partial \Omega$ since each term $= 0$ on $\partial \Omega$. Using this together with Green's identity, we have

$$(5) \quad \int_{\partial \Omega} h(\xi) \frac{\partial}{\partial \xi_x} (\gamma_x^\Omega(\xi) - K_x^\Omega(\xi)) dS_\xi = -\int_\Omega h(\xi) \Delta (\gamma_x^\Omega(\xi) - K_x^\Omega(\xi)) dV_\xi$$

where $h$ is harmonic. Since $\int_\Omega h(\xi) \Delta \gamma_x^\Omega(\xi) dV_\xi = G_\Omega h(x)$ and $\int_\Omega h(\xi) \Delta K_x^\Omega(\xi) = G_\Omega h(x)$, we conclude that the R.H.S. of (5) $= 0$. Applying Green's identity
to the function $= 1$ and $\gamma_x^\Omega - K_x^\Omega$, we see that $\int_{\partial \Omega} (\partial/\partial \xi_x)(\gamma_x^\Omega(\xi) - K_x^\Omega(\xi)) dS_\xi = 0$. Hence there exists a harmonic solution to the boundary value problem
$h = (\partial/\partial \nu)(\gamma_x^\Omega - K_x^\Omega)$ on $\partial \Omega$. Substituting this solution into (5), we see that
$(\partial/\partial \nu)(\gamma_x^\Omega - K_x^\Omega) = 0$ on $\partial \Omega$, thereby verifying that $\Gamma_x^\Omega = \gamma_x^\Omega - K_x^\Omega$. Hence, if
$\gamma$ exists, by Theorem 6 $K$ exists, and $\Gamma = \lim_{\Omega \to R} (\gamma - K^\Omega) = \gamma - K$.

Lastly, $K$ and $\Gamma$ are orthogonal since

$$(K_x, \Gamma_x) = \int_R \Delta K_x(\xi)(\Delta \gamma_x(\xi) - \Delta K_x(\xi)) d\xi = K_x(x) - K_x(x) = 0.$$

In closing, I would like to hint at other applications of the methods presented. From Theorem 2, it is immediate that the existence of a positive quasiharmonic function implies the existence of $\gamma$ [5]. On the other hand, it is clear that the existence of $\gamma$ assures that the biharmonic functions with square integrable Laplacians possess Riesz representations [6], [9]. Since Theorem 3' guarantees that $k$ always exists, one can define a span whose vanishing is equivalent to the nonexistence of nonzero square integrable harmonic functions [8]. Also, $K$ may be found useful in formulating and solving a biharmonic interpolation problem similar to one known for harmonic functions [4, pp. 275–280], [7].

**Bibliography**


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