STRUCTURE OF SUBALGEBRAS BETWEEN $L^{\infty}$
AND $H^{\infty}$

BY

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Abstract. Let $B$ be a closed subalgebra of $L^{\infty}$ of the unit circle which
contains $H^{\infty}$ properly. Let $C_B$ be the $C^*$-algebra generated by the inner
functions that are invertible in $B$. It is shown that the linear span $H^{\infty} + C_B$
is equal to $B$. Also, a closed subspace (called $VMO_B$) of $BMO$ (space of
functions of bounded mean oscillation) is identified to which $B$ bears the
same relation as $L^{\infty}$ does to $BMO$.

1. Introduction. Let $L^{\infty}$ denote the space of essentially bounded Lebesgue
measurable functions on the unit circle. Let $H^{\infty}$ denote the subspace of
functions in $L^{\infty}$ with vanishing negative Fourier coefficients. The algebras
studied here are the closed subalgebras of $L^{\infty}$ which contain $H^{\infty}$ properly.

The interest in such algebras originated, to a large extent, in a question
asked by R. G. Douglas [1]. Suppose $B$ is a closed subalgebra of $L^{\infty}$ which
contains $H^{\infty}$. Let $B_I$ be the closed subalgebra of $L^{\infty}$ generated by $H^{\infty}$ and the
complex conjugates of those inner functions (that is, unimodular $H^{\infty}$ func-
tions) that are invertible in $B$. Douglas asked: Is $B_I$ always equal to $B$? (If so,
we call $B$ a Douglas algebra.) The question has recently been answered

Another interesting question arose in the study of Douglas' problem. All
subalgebras of $L^{\infty}$ containing $H^{\infty}$ which have been investigated in detail share
certain structural characteristics. (For a survey of some of this work see [4].) Sarason [5] pointed out the phenomenon and asked, in particular, the
following question: Let $B$ be a closed algebra between $L^{\infty}$ and $H^{\infty}$. Let $C_B$ be
the $C^*$-algebra generated by the inner functions that are invertible in $B$ (which
is the same algebra generated by all quotients of inner functions invertible in
$B$). Is it true in general that $B = H^{\infty} + C_B$?

The largest such algebra is $L^{\infty}$. R. G. Douglas and W. Rudin [6] have
proved that $L^\infty$ is the closed algebra generated by quotients of inner functions in $H^\infty$. By a theorem of K. Hoffman and I. M. Singer [7], the smallest such subalgebra is the closed algebra generated by $H^\infty$ and $C$, where $C$ denotes the algebra of continuous functions on $\partial D$. D. Sarason [8] has proved that this closed algebra is in fact equal to $H^\infty + C$, the linear hull of $H^\infty$ and $C$. For $E$ an arbitrary subset of $\partial D$, let $L_E$ denote the set of functions in $L^\infty$ that are continuous on $E$. A. Davie, T. Gamelin and J. Garnett [9] have extended the preceding result by showing that the closed subalgebra of $L^\infty$ generated by $H^\infty$ and $L_E$ coincides with $H^\infty + L_E$, the linear hull of $H^\infty$ and $L_E$. In addition, they have proved that $L_E$ is the algebra generated by functions of the form $b_1 \bar{b}_2$, where $b_1$ and $b_2$ are Blaschke products whose zeros do not cluster at any point of $E$.

Sarason's question has been answered affirmatively for some other subalgebras of $L^\infty$ containing $H^\infty$ [10], [11]. In §2 below, I shall give an affirmative answer to this question in general.

In the examples mentioned above, a concrete description of the $C^*$-algebra $C_B$ in terms of properties of the original algebra $B$ has been given. The effort to get similar results for general $C_B$ has not been successful. However, Sarason [11] has studied a closed subspace $VMO$ (functions of vanishing mean oscillation) of $BMO$ (functions of bounded mean oscillation), and obtained results which, roughly speaking, tell that the relation of the closed algebra $H^\infty + C$ to the space $VMO$ is like the relation of $L^\infty$ to $BMO$. For example, if we let $QC$ (for quasi-continuous) denote the $C^*$-algebra $(H^\infty + C) \cap (H^\infty + C)$, then $VMO = QC + QC = C + \bar{C}$. (For a subspace $S$ of $L^\infty$, we let $\overline{S}$ denote the collection of complex conjugates of functions in $S$, and $\bar{S}$ denote the collection of harmonic conjugates of functions in $S$.) Using a method similar to Sarason's, S. Axler [12], T. Weight [13] and the author [10] have obtained parallel results for other specific subalgebras between $H^\infty$ and $L^\infty$. In §§3 and 4 below, I generalize these results and obtain, for any subalgebra $B$ between $L^\infty$ and $H^\infty$, a corresponding closed subspace $VMO_B$ of $BMO$. Theorems 6 and 7 say that $B$ bears the same relation to $VMO_B$ as $L^\infty$ does to $BMO$.

The following are some notations used in this paper. Let $L^1$ and $L^2$ denote Lebesgue spaces of integrable and square integrable functions with respect to normalized Lebesgue measure on the unit circle. Let $H^1$ and $H^2$ denote the corresponding Hardy spaces. Let $H^\infty_0$ and $H^\infty$ denote the functions in $H^1$ and $H^\infty$, respectively, with mean value 0. For a subset $S$ of $L^\infty$, let $H^\infty[S]$ denote the closed subalgebra of $L^\infty$ generated by $H^\infty$ and $S$. If $S$ consists of a single function $f$, denote $H^\infty[S]$ by $H^\infty[f]$. For a function $f$ in $L^\infty$, the norm $||f||$ is the essential supremum of $|f|$ on $\partial D$. For a subspace $S$ of $L^\infty$, the distance of $f$ to $S$ is given by $d(f, S) = \inf\{||f - g||, g \in S\}$. Finally, for a function $f$ in
Let $f(e^{i\theta})$ denote the harmonic extension of $f$ into the unit disk $D$ by means of Poisson's formula, i.e.,

$$f(e^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) P_r(\theta - t) \, dt,$$

where $P(r,t) = \frac{(1 - r^2)}{(1 - 2r \cos t + r^2)}$ is the Poisson kernel.

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2. Structure of subalgebras between $L^\infty$ and $H^\infty$. We will begin with a theorem of A. Bernard, J. Garnett and D. Marshall.

**Theorem 1** ([14, Theorem 2.1]). Let $f \in L^\infty$, $d(f, H^\infty) < 1$. Then there is unimodular function $u_0 \in L^\infty$ such that $u_0 \in f + H^\infty$ and $d(u_0, H^\infty_0) = 1$.

In [15] there is a sharper theorem: If $f \in L^\infty$, and if the coset $f + H^\infty$ contains two functions of norm $\leq 1$, then it contains a unimodular function. But the proof of Theorem 1 in [14] is much shorter, and gives in addition a unimodular function in the coset whose distance to $H^\infty_0$ is 1. We briefly indicate how to find the function here.

**Proof.** We may assume without loss of generality that $\|f\| < 1$. Write $a = \sup\{|f \, ud\theta/2\pi| : u \in f + H^\infty, \|u\| < 1\}$. Since $H^\infty$ is weak-star closed, a compactness argument shows there exists an extremal function $u_0$ such that $\int u_0 \, d\theta/2\pi = a$. It follows easily from the extremal property of $u_0$ that $d(u_0, H^\infty_0) = 1$ and $\|u_0\|_{L^\infty} = 1$. Now an argument using the duality between $L^\infty/H^\infty_0$ and $H^1$ can be applied to show that any function $u$ in $L^\infty$ with the properties $\|u\|_{L^\infty} \leq 1$, $d(u, H^\infty) < 1$ and $d(u, H^\infty_0) = 1$ is unimodular. In particular, $u_0$ is unimodular. (Details of the proof are in [14].)

The following fact was indicated to me in a discussion with D. Sarason.

**Theorem 2.** Let $f$ be a unimodular function in $L^\infty$. If $d(f, H^\infty) = 1$, and $d(f, H^\infty + C) < 1$, then $f \in H^\infty [f]$.

**Proof (Sarason).** For the given $f \in L^\infty$, let $T_f$ be the Toeplitz operator associated with $f$, i.e., $T_f$ is the bounded map from $H^2$ to $H^2$ defined by $T_f(h) = P(\varphi h)$, where $P$ is the orthogonal projection from $L^2$ to $H^2$.

Since $f$ is unimodular and $d(f, H^\infty) = 1$, $T_f$ is not left-invertible ([16, Lemma 1]). Since $d(f, H^\infty + C) < 1$, $T_f$ is left-Fredholm [17]. Thus the range of $T_f$ is a closed subspace of $H^2$, and the null space of $T_f$ is nontrivial. It then follows from a theorem of Coburn [18] that the null space of $T_f$ is trivial.
Hence $T_j$ is onto and so right invertible. Thus, $T_j$ is Fredholm, which implies that $f$ is in $H^\infty[f]$ by [19, 7: 33]. ([19] also lists all other results concerning the Toeplitz operator used here.)

The next lemma follows easily from the above theorems.

**Lemma 1.** Let $h \in H^\infty$, and let $b$ be an inner function such that $d(\overline{b}h, H^\infty) < 1$. Then there is an inner function $b_0$ in $h + bH^\infty$ with $\overline{b}_0 \in H^\infty[\overline{b}]$.

**Remarks.** (1) The existence of inner functions $b_0$ in $h + bH^\infty$ is proved in Satz 7 of Nevanlinna's work [20] in case $b$ is a Blaschke product. But it is not clear whether the $b_0$ constructed in his proof is invertible in $H^\infty[\overline{b}]$.

(2) If $b = z^n$, then Lemma 3 in slightly generalized form is contained in Carathéodory's classical theorem [21]: If $h \in H^\infty$, $||h||_\infty < 1$, then there exists a finite Blaschke product $u_0$ (with $\leq n$ zeros) whose power series expansion agrees with that of $h$ in the first $n$-coefficients.

**Proof of Lemma 1.** For the given $h \in H^\infty$ and inner function $b$, let $f = \overline{b}h$, so that $d(f, H^\infty) < 1$. Applying Theorem 1 to $f$, we get a unimodular function $u_0$ in $\overline{b}h + H^\infty$ with $d(u_0, H^\infty) = 1$. We claim that $u_0 \in H^\infty[\overline{b}]$. To see this, let $\varphi = u_0z$. Then $d(\overline{\varphi}, H^\infty) = d(u_0, H^\infty) = 1$ and $d(z\varphi, H^\infty) = d(u_0, H^\infty) = d(\overline{b}h, H^\infty) < 1$. So we can apply Theorem 2 to $\varphi$, concluding that $\varphi \in H^\infty[\varphi] = H^\infty[u_0z] = H^\infty[\overline{b}hz] \subset H^\infty[\overline{b}]$. So $\overline{u}_0 = \overline{z}\varphi \in H^\infty[\overline{b}]$. Let $b_0 = u_0b$, so that $b_0 \in h + u_0H^\infty$. Then $b_0$ is inner and $\overline{b}_0 = \overline{u}_0 \overline{b} \in H^\infty[\overline{b}]$.

The following theorem is our main result.

**Theorem 3.** Let $B$ be any closed subalgebra of $L^\infty$ containing $H^\infty$. Let $C_B$ be the $C^*$-algebra generated by inner functions that are invertible in $B$. Then the linear span $H^\infty + C_B$ is a closed algebra, and is equal to $B$.

To prove Theorem 3, we first make a simple observation which is a direct consequence of Lemma 1 above.

**Remark.** If $B$ is a closed subalgebra of $L^\infty$ containing $H^\infty$, if $b$ is an inner function in $C_B$, and if $h$ is in $H^\infty$, then there exist functions $u \in C_B$ and $g \in H^\infty$ with $||u|| \leq 2||h||$, $||g|| \leq 3||h||$ and $\overline{b}h = u + g$.

**Proof of Theorem 3.** Since $B$ is a Douglas algebra, for any $f \in B$, there exists an inner function $b_1 \in C_B$, and an $h_1 \in H^\infty$ with $||f - \overline{b}_1h_1|| < \frac{1}{2}||f||$. Applying the above remark to the functions $h_1$ and $b_1$, we get $u_1 \in C_B$, $g_1 \in H^\infty$, with $||u_1|| \leq 2||h_1|| \leq 4||f||$, $||g_1|| \leq 3||h_1|| \leq 6||f||$, and $\overline{b}_1h_1 = u_1 + g_1$. Let $f_1 = f - \overline{b}_1h_1$; then $f_1 \in B$, and we can apply the above procedure to the function $f_1$, getting an inner function $b_2 \in C_B$, an $H^\infty$ function $h_2$, a $u_2 \in C_B$, and a $g_2 \in H^\infty$, with $||f_1 - \overline{b}_2h_2|| < \frac{1}{4}||f_1||$, $\overline{b}_2h_2 = u_2 + g_2$, $||u_2|| \leq 2||h_2|| \leq 4||f_1||$, and $||g_2|| \leq 3||h_2|| \leq 6||f_1||$. Iterating, we get $f_n \in B$, $u_n \in C_B$, $g_n \in H^\infty(f_0 = f) n = 1, 2, 3, \ldots$ such that
(1) 
\[ f_n = f_{n-1} - (u_n + g_n), \quad \|f_n\| \leq \frac{1}{2}\|f_{n-1}\| \]

(2) 
\[ \|u_n\| \leq 4\|f_{n-1}\|, \quad \|g_n\| \leq 6\|f_{n-1}\|. \]

Hence \( u = \sum_{n=1}^{\infty} u_n \) is in \( C_B \), \( g = \sum_{n=1}^{\infty} g_n \) is in \( H^\infty \), and \( f = \sum_{n=1}^{\infty} (u_n + g_n) = u + g \in H^\infty + C_B \). Since the linear span \( H^\infty + C_B \) is obviously contained in \( B \), we have proved the theorem.

For \( B \) a closed subalgebra of \( L^\infty \) containing \( H^\infty \), let \( Q_B = B \cap \overline{B} \). Then \( Q_B \) is the largest \( C^* \)-algebra contained in \( B \). As \( Q_B \) contains \( C_B \), the following statement follows immediately from Theorem 3.

**Corollary 1.** If \( B \) is any closed subalgebra of \( L^\infty \) containing \( H^\infty \), then \( B = H^\infty + Q_B \).

### 3. Descriptions of \( C^* \)-algebras \( C_B, Q_B \) for \( B = H^\infty[b] \)

Suppose \( b \) is an inner function and \( B = H^\infty[b] \) is the closed subalgebra of \( L^\infty \) generated by \( H^\infty \) and \( b \). This section concerns the \( C^* \)-algebras \( C_B \) and \( Q_B \) of \( B \). It turns out, though the general description of \( C_B \) and \( Q_B \) as subalgebras of \( L^\infty \) seem difficult to give, we can give a quite satisfactory description of the subspace of \( BMO \) corresponding to \( C_B \) and \( Q_B \).

For a function \( f \) in \( L^1 \) of \( \partial D \) and a subarc \( I \) of \( \partial D \), let \( f_I = (1/|I|) \int_I f(t) \, dt \), where \( dt \) denotes the Lebesgue measure on \( \partial D \) and \( |I| \) denotes the arc length of the arc \( I \). For each real number \( a \), \( 0 < a < 2\pi \), we define \( M_a(f) = \sup_{|I|<a} (1/|I|) \int_I |f - f_I| \, dt \), and we let \( \|f\|_* = \lim_{a \to 2\pi} M_a(f) \) and \( M_0(f) = \lim_{a \to 0} M_a(f) \). We say that the function \( f \) has bounded mean oscillation, or belongs to \( BMO \), if \( \|f\|_* < \infty \). We say the function \( f \) has vanishing mean oscillation, or belongs to \( VMO \), if \( M_0(f) = 0 \).

For a subarc \( I \) of \( \partial D \) with center \( e^{it} \) and measure \( 2\delta \), let \( R(I) = \{ re^{i\theta} \mid |\theta - t| < \delta, 1 - \delta < r < 1 \} \). A finite positive measure \( \mu \) on \( D \) is said to be a Carleson measure if there exists a constant \( C \) such that \( \mu(R(I)) \leq C |I| \) for all subarcs \( I \) of \( \partial D \). One of the main results of C. Fefferman and E. M. Stein [22, Theorem 3] is the following theorem, which gives several alternative descriptions of the space \( BMO \): (This is the special case of their theorem for the unit circle.)

**Theorem 4 (Fefferman and Stein).** For a function \( f \) defined on \( \partial D \), the following conditions are equivalent:

1. \( f \in BMO \).
2. \( f = u + \tilde{v} \) for some functions \( u, v \in L^\infty \), where \( \tilde{v} \) denotes the harmonic conjugates of \( v \).
3. \( f \in L^1 \) and the measure \( \mu \) on \( D \) defined by
   \[ d\mu = (1 - r)|\nabla f(re^{i\theta})|^2 r \, dr \, d\theta \]
   is a Carleson measure, where
Furthermore, if \( C = \sup_{|I| < 2\pi} \left( \frac{1}{|I|} \right) \mu(R(I)) \), then there exists an absolute constant \( A_1 \) with \( C \leq A_1 \|f\|_p^p \).

In [11], D. Sarason studied the space \( VMO \), and gave several alternative descriptions of the space. One of his results is the following theorem:

**Theorem 5 (Sarason).** For a function \( f \) defined on \( \partial D \) and in \( BMO \), the following conditions are equivalent:

1. \( f \in VMO \).
2. \( f = u + v \) for some functions \( u, v \in C \).

Sarason's work suggests that we look at the subspace \( C_B + \overline{C}_B \) in \( BMO \). As we shall see later, this subspace is a closed subspace of \( BMO \), and there are several alternative descriptions of the space analogous to those of \( BMO \) in the Fefferman-Stein theorem.

We begin with some notations and definitions.

For each \( 0 < \delta < 1 \), let \( G_\delta \) be the region \( \{ f \in D| |b(z)| > 1 - \delta \} \). For each \( \delta > 0 \) and real number \( a_0 > 1 \), let \( \mathcal{S}(\theta, a_0) \) denote the collections of all subarcs \( I \) of the circle of the form \( I = \{ e^{it} | t - \theta | < a(1 - r) \} \) for some point \( re^{i\theta} \in G_\delta \), and some real number \( 1 < a < a_0 \) with \( a(1 - r) < \pi \). For each \( \varepsilon > 0 \), let \( a_\varepsilon \) be the smallest integer \( 2^n \) such that \( 1 + 2^n / 2^n \leq \varepsilon \). We define \( VMO_B \) to be the space of all functions \( f \) in \( BMO \) with the following property: For every \( \varepsilon > 0 \), there exists a \( \delta > 0 \), such that \( (1/|I|) \int_I |f - f_I| < \varepsilon \) whenever \( I \in \mathcal{S}(\theta, a_\varepsilon) \).

The following theorem is our main result.

**Theorem 6.** For a function \( f \) defined on \( \partial D \) and in \( BMO \), the following conditions are equivalent:

1. \( f \in VMO_B \).
2. \( f = u + v \) for some functions \( u, v \) in \( Q_B = B \cap \overline{B} \).
3. (a) \( f = u + \tilde{v} \) for some functions \( u, v \) in \( C_B \).
3. (b) Given \( \varepsilon > 0 \), there exists some \( \delta > 0 \) such that the measure \( \mu_\delta \) on \( D \) defined by \( d\mu_\delta = \chi_{G_\delta} (1 - r) |\nabla f|^2 r \, dr \, d\theta \) is a Carleson measure with \( \mu_\delta(R(I)) \leq \varepsilon |I| \) for all subarcs \( I \) of \( \partial D \).

The equivalence between conditions (1) and (2) in the above theorem have been verified for some specific Douglas algebras by S. Axler [12], T. Weight [13] and the author [10]. Condition (3) has been established in the special case \( B = H^\infty + C \) by D. Stegenga [23].

Let \( L_B \) denote the collection of functions in \( BMO \) satisfying condition (3) above. To prove the above theorem, we first quote two results from [3].
Lemma 2 [3, Theorem 6]. Suppose \( f \in L_B \). There is an absolute constant \( C \) such that for every \( \epsilon > 0 \),

\[
\left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{it} k(t) \, dt \right| \leq C \epsilon^{1/2} \|f\|_e \|k\|_e
\]

for all \( k \in H^1_0 \cap H^\infty_0 \), when the positive integer \( n \) is sufficiently large.

Lemma 3 [3, Theorem 8]. \( L_B \cap L^\infty = Q_B (= B \cap \overline{B}) \).

Remark. In [3], Lemma 2 was proved for functions \( f \in L_B \cap L^\infty \), but the same proof, with the constant \( \|f\| \) replaced by \( \|f\|_e \) whenever it appears, will work for functions \( f \in L_B \).

Lemma 4 (John-Nirenberg [24, Lemma 1]). Suppose \( f \) is a function in \( BMO \) and \( I \) is a subarc of \( \partial D \). For each \( s > 0 \), let \( \lambda(s) \) be the set of points on \( I \) where \( |f - f_I| > s \), and let \( |\lambda(s)| \) denote the measure of \( \lambda(s) \). Then there exist constants \( A, \alpha, \text{ and } s_0 \) (independent of \( f \)) such that

\[
|\lambda(s)| \leq \frac{A}{\|f\|_e} \left( \int_I |f - f_I| \, dt \right) e^{-\alpha s/\|f\|_e}
\]

for all \( s \geq \|f\|_e s_0 \).

The estimates in the following lemmas will be used to establish the equivalence of (1) and (2), (3) in Theorem 6. The proofs presented here are only slight modifications of those in [22].

Lemma 5. Let \( f \in BMO \) and \( 1/|I| \int_I |f - f_I| \, dt \leq \epsilon \). Then

\[
(1/|I|) \int_I |f - f_I|^2 \, dt \leq C_1 \|f\|_e
\]

where \( C_1 \) is an absolute constant.

Proof. The lemma follows easily from Lemma 4 above, and the observation that if we let \( I_0 = \|f\|_e s_0 \) (\( s_0 \) as in Lemma 4), then the condition \( (1/|I|) \int_I |f - f_I| \, dt \leq \epsilon \) implies that \( |\lambda(I_0)| I_0 \leq \epsilon |I| \).

Lemma 6. Let \( f \) be a function in \( VMO_B \), and let \( \epsilon > 0 \). Let \( \delta \) be as in the definition of \( VMO_B \). Suppose \( z_0 = r_0 e^{i\theta_0} \in G_0, r_0 > 1/2 \). Let \( I_0 = \{e^{it} \mid t - \theta_0 \leq 1 - r_0 \} \). Then

\[
A(I_0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t) - f_{I_0} P(r_0, \theta_0 - i) \, dt \leq C_2 \epsilon,
\]

where \( C_2 \) is a constant depending only on \( \|f\|_e \).

Proof. Let \( I_n \) be the arc with the same center as \( I_0 \) and measure \( 2^n |I_0| \). Let \( N \) be the integer such that \( 2^N = 2^N \). We carry out the proof for the case
\[ 2^N |I_0| < \pi. \] (The same proof works in the contrary case, with a slight change in the constant \( C_2 \).) We have

\[
A(I_0) = \frac{1}{2\pi} \sum_{n=0}^{N} \int_{I_n \setminus I_{n-1}} |f(t) - f_{I_0}| |P(r_0, \theta_0 - t)| dt
\]

\[
+ \frac{1}{2\pi} \int_{\partial D \setminus I_N} |f(t) - f_{I_0}| |P(r_0, \theta_0 - t)| dt.
\]

The estimate

\[
|f_{n+1} - f_n| \leq \frac{1}{|I_{n-1}|} \int_{I_n} |f - f_n| dt \leq \frac{|I_n|}{|I_{n-1}|} \int_{I_n} |f - f_n| dt \leq 2\varepsilon,
\]

valid for \( n = 0, 1, \ldots, N \) gives

\[
|f_{I_0} - f_n| \leq \sum_{k=1}^{n} |f_{k+1} - f_k| \leq 2n\varepsilon, \quad n = 0, 1, 2, \ldots, N,
\]

which, together with an elementary estimate of \( P(r_0, \theta_0 - t) \), yields

\[
\int_{I_n \setminus I_{n-1}} |f(t) - f_{I_0}| |P(r_0, \theta_0 - t)| dt
\]

\[
\leq \frac{\pi^2}{|I_0|} \frac{1}{2^{2n-1}} \left( \int_{I_n} (|f(t) - f_n| + |f_{I_0} - f_n|) dt \right)
\]

\[
= \frac{\pi^2}{2^{2n-1}} \frac{1}{|I_n|} \int_{I_n} (|f(t) - f_n| + 2n\varepsilon) dt
\]

\[
\leq \frac{\pi^2}{2^{2n-1}} \frac{1 + 2n}{2^{2n-1}} \varepsilon, \quad n = 0, 1, \ldots, N.
\]

Hence

\[
\sum_{n=0}^{N} \frac{1}{2\pi} \int_{I_n \setminus I_{n-1}} |f(t) - f_{I_0}| |P(r_0, \theta_0 - t)| dt \leq \left( \sum_{n=0}^{N} \frac{1 + 2n}{2^n} \right) \varepsilon \pi.
\]

Using similar estimates we get, for \( N_1 \) the largest integer such that \( 2^{N_1} |I| < 2\pi \),

\[
\frac{1}{2\pi} \int_{\partial D \setminus I_{N_1}} |f(t) - f_{I_0}| |P(r_0, \theta_0 - t)| dt
\]

\[
= \frac{1}{2\pi} \sum_{n=N_1}^{N} \int_{I_n \setminus I_{n-1}} |f(t) - f_{I_0}| |P(r_0, \theta_0 - t)| dt
\]

\[
+ \frac{1}{2\pi} \int_{\partial D \setminus I_{N_1}} |f(t) - f_{I_0}| |P(r_0, \theta_0 - t)| dt
\]

\[
\leq \pi \sum_{n=N_1}^{\infty} \frac{1 + 2n}{2^n} ||f||_* + \frac{1 + 2N}{2^N} ||f||_*
\]

\[
\leq \varepsilon \pi ||f||_* \quad (by \ definition \ of \ a_\varepsilon).
\]
Hence $A(l_0) \leq C_2 \varepsilon$ for $C_2 = (\sum_{n=0}^{\infty} (1 + 2n) / 2^n + \|f\|_\infty) \pi$.

**Proof of Theorem 6.** We will first establish the equivalence between (2)(a) and (3).

(2)(a) $\Rightarrow$ (3). Suppose $f = u + \bar{v}$ for some $u, v \in C_B$. Since $C_B \subseteq Q_B$, by Lemma 3, $u \in L_B$. Since the space $L_B$ is closed under harmonic conjugation, the same reasoning gives that $\bar{v} \in L_B$. Hence $f$ is in $L_B$.

(3) $\Rightarrow$ (2)(a). Suppose $f$ is in $L_B$. Since $L_B$ is closed under complex conjugation, we may assume without loss of generality that $f$ is real-valued. By Theorem 4, there exist $u, v$ in $L^\infty$ such that $f = u + \bar{v}$. Let $g = u + iv$. Then $g \in L^\infty$. We have $\int_\pi^\pi (f - g) h dt = \int_\pi^\pi (\bar{v} - iv) h dt = 0$ for all $h \in H^1 \cap H^\infty$. Applying Lemma 2 to $f$, we get, for any $\varepsilon, \|(1/2\pi) \int_\pi^\pi gb^n k dt\| < C\varepsilon^{1/2} \|k\| \|f\|_\infty$, for all $k \in H_0^1 \cap H^\infty$ and for $n$ sufficiently large. Since the quotient space $L^\infty / H^\infty$ is the dual of the space $H_0^1$, this implies that $d(gb^n, H^\infty)$ (which equals the norm of the functional that $gb^n$ induces on $H_0^1$) tends to 0 when $n$ tends to $\infty$. Hence $g \in B$. Applying Theorem 3, we can write $g = r + h$ for some $r \in C_B, h \in H^\infty$. Thus $f - \bar{f} = g - i\bar{g} = r - i\bar{r} \in C_B + \tilde{C}_B$. Taking complex conjugates on both sides of the above equality, we get $f + \bar{f} = \bar{r} + i\bar{r} \in C_B + \tilde{C}_B$. Hence $f \in C_B + \tilde{C}_B$. The proof is complete.

It now follows easily that (2) and (3) are equivalent, since (2) $\Rightarrow$ (3) by the same reasoning as (2)(a) $\Rightarrow$ (3), and (3) $\Rightarrow$ (2)(a) $\Rightarrow$ (2).

The equivalence between (1) and (2), (3) follows from the following three assertions:

(a) $B \cap \overline{B} \subseteq VMO_B$.
(b) If $u \in VMO_B \cap L^\infty$ then $\bar{u} \in VMO_B$.
(c) $VMO_B \subseteq L_B$.

**Proof of (a).** Since $B \cap \overline{B}$ is a $C^*$-algebra, and so is spanned by its unimodular functions, and since $VMO_B \cap L^\infty$ is a closed subspace of $L^\infty$, it suffices to show every unimodular function in $B \cap \overline{B}$ is in $VMO_B$.

For fixed $\varepsilon > 0$, and $f$ a unimodular function invertible in $B \cap \overline{B}$, let $\varepsilon_1$ be any positive number less than $[2(4\pi(1 + a^2) + 1)]^{-1}\varepsilon$, and choose $\delta > 0$ so that $|f(z)| \geq 1 - \varepsilon_1^3$ for all points $z$ in the region $G_\delta$. (The existence of such $\delta$ is asserted in the proof of Theorem 7 in [3].) Then apply the same type of estimate as in the proof of Theorem 3 in [11], we can show that whenever $I$ is a subarc of the form $\{e^{it} \mid |t - \theta| \leq a(1 - r)\}$ in $\mathcal{G}(\delta, a_\varepsilon)$, we have

$$\frac{1}{|I|} \int_I |f(e^{it}) - f_I| dt \leq 2\left(\frac{4\pi(1 + a^2)}{a} + 1\right)\varepsilon_1 \leq 2(4\pi(1 + a^2) + 1)\varepsilon_1 < \varepsilon.$$

Hence $f$ is in $VMO_B$. 

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Proof of (b). Let \( u \) be a function in \( VMO_B \), and assume without loss of generality that \( u \) is real valued. Given \( \varepsilon > 0 \), let \( \eta_1, \eta_2 \) be two numbers satisfying \( 2\eta_2^2 < 2\eta_1^2 < \eta_2 \). Choose \( \delta > 0 \) corresponding to \( \eta_2 \) for \( u \) as in the definition of \( VMO_B \), i.e.,

\[
\sup_{I \in \mathcal{I}(\delta,\eta_2)} \frac{1}{|I|} \int_I |u - u_I| \, dt < \eta_2.
\]

Fix a subarc \( I \in \mathcal{I}(\delta,\eta_2) \), we will show that \( (1/|I|) \int_I (\bar{u} - \bar{u}_I) \, dt \leq \varepsilon \). Let \( J \) be the subarc with the same center as \( I \) and twice the dimensions. Let \( u_1 = \chi_J(u - u_J), u_2 = \chi_{3D \setminus J}(u - u_J) \); then we have

\[
\frac{1}{|I|} \int_I |\bar{u} - \bar{u}_1| \, dt \leq \frac{1}{|I|} \int_I |\bar{u}_1 - (\bar{u}_I)_J| \, dt.
\]

Since \( \|\bar{u}_1\|_2 \leq \|u_1\|_2 \),

\[
\frac{1}{|I|} \int_I |\bar{u}_1|^2 \, dt \leq \frac{1}{|I|} \int_I |\bar{u}_I|^2 \, dt \leq \frac{1}{|I|} \int_I |u_1|^2 \, dt = \frac{1}{|I|} \int_J |u - u_J|^2 \, dt \leq 2\varepsilon_2 C_1 \|u\|_\infty.
\]

The last step of the above inequality follows since \( J \in \mathcal{I}(\delta,\eta_2) \), so we can apply Lemma 5. Thus

\[
\frac{1}{|I|} \int_I |u_1 - (\bar{u}_I)_J|^2 \, dt \leq 2 \left( \frac{1}{|I|} \int_I |\bar{u}_1|^2 \, dt + |(u_1)_J|^2 \right) \leq 2 \cdot 4\varepsilon_2 C_1 \|u\|_\infty = 8\varepsilon_2 C_1 \|u\|_\infty,
\]

so

\[
\frac{1}{|I|} \int_I |u_1 - (\bar{u}_I)_J| \, dt \leq (8\varepsilon_2 C_1 \|u\|_\infty)^{1/2}.
\]

To estimate \( (1/|I|) \int_I |\bar{u}_2 - (\bar{u}_I)_J| \, dt \), we let \( f = u_2 + \bar{u}_2 \), i.e.

\[
f(z) = \frac{1}{2\pi} \int_{\partial D \setminus J} \frac{e^{it} + z}{e^{it} - z} (u - u_J) \, dt.
\]

This formula holds on \( I \) as well as in \( D \). Differentiating \( f \), we get

\[
f'(z) = \frac{1}{\pi} \int_{\partial D \setminus J} \frac{e^{it}}{(e^{it} - z)^2} (u(t) - u_J) \, dt.
\]

Let \( e^{i\theta_0} \) be the center of \( I \), and \( z_0 = (1 - \frac{1}{2}|I|)e^{i\theta_0} \); then for \( e^{is} \in I \), we have
\[ |f'(e^{it})| \leq \frac{1}{\pi} \int_{\partial D \setminus J} \frac{|u(i) - u_j|}{|e^{it} - e^{it_j}|^2} \, dt \]
\[ \leq \frac{C_3}{\pi} \int_{\partial D \setminus J} \frac{|u(i) - u_j|}{|e^{it} - z_0|^2} \, dt \quad \text{where } C_3 = \left(1 + \frac{\pi^2}{2}\right)^2 \]
\[ \leq \frac{C_3}{\pi} \left( \int_{\partial D \setminus J} \frac{|u(i) - u_j|}{|e^{it} - z_0|^2} \, dt + \frac{2}{|I|} |u_I - u_J| \right) \]
\[ \leq \frac{C_3}{\pi} \left( \int_{-\pi}^\pi \frac{|u - u_j|}{|e^{it} - z_0|^2} \, dt + \varepsilon_2 \frac{1}{|I|} \right). \]

Since each interval with the same center as \( I \) and with length \( \leq a \varepsilon_1 |I| \) is in \( a \varepsilon_2 \), we can apply the same method as in Lemma 6 to estimate \( S_1 = (1/2\pi) \sum_{\lambda \in S} \frac{|u - u_l|}{|e^{it} - z_0|^2} \, dt \) and get
\[ S_1 \leq (2/|I|) (C_2 \varepsilon_2 + \varepsilon_1 \pi \|u\|_{L^2}). \]

Hence
\[ |f'(e^{it})| \leq \frac{1}{|I|} \frac{2C_3}{\pi} (C_2 \varepsilon_2 + \varepsilon_1 \pi \|u\|_{L^2} + 4\varepsilon_2) \]
\[ = \frac{1}{|I|} C_4 \quad \text{for some constant } C_4 \text{(depends on } \varepsilon_1, \varepsilon_2). \]

Hence the oscillation of \( f \) over \( I \) does not exceed \( C_4 \). And the same applies to \( \text{Im} f = \tilde{u}_2 \), i.e.
\[(7) \quad \frac{1}{|I|} \int_I |\tilde{u}_2 - (\tilde{u}_2)_I| \, dt \leq C_4.\]

From (5), (6), (7) we get \((1/|I|) \int_I |u - u_I| \, dt \leq \varepsilon \) if \( \varepsilon_1, \varepsilon_2 \) are chosen sufficiently small. This concludes the proof of (b).

**Proof of (c).** The proof given here is a slight modification of the proof of Lemma 4 in [22]. Suppose \( f \) is a function in \( \text{VMO}_B \). Given \( \varepsilon > 0 \), choose \( \varepsilon_1 \) smaller then \( \varepsilon \), to be fixed later, and let \( \delta \) correspond to \( \varepsilon_1 \) for \( f \) as in the definition of \( \text{VMO}_B \). Let \( z_0 = r_0 e^{i\theta_0} \in G_0 \) with \( r_0 > \frac{1}{2} \), and let \( S(\theta_0, r_0) \) be the region \( \{re^{i\theta} \ | \ |\theta - \theta_0| \leq 4(1 - r_0), r_0 \leq r < 1\} \). Then as indicated in Theorem 6 in [3], to show \( f \) is in \( L_B \), it suffices to show
\[ \int_{S(\theta_0, r_0)} (1 - r) \left| \nabla f \right|^2 r \, dr \, d\theta \leq \varepsilon(1 - r_0). \]

To do this, let \( J = \{e^{it} \ | \ |t - \theta_0| \leq 5(1 - r_0)\} \), \( I = \{e^{it} \ | \ |t - \theta_0| \leq 1 - r_0\} \), \( f_1 = \chi_J(f - f_J), f_2 = \chi_{\partial D - J}(f - f_J) \). Then
\[ \iint_{S(\theta_0, r_0)} (1 - r)|\nabla f_1|^2 r \, dr \, d\theta \leq \iint_D (1 - r)|\nabla f_1|^2 r \, dr \, d\theta \]
\[ \leq \iint_D |\nabla f_1|^2 r \log \frac{1}{r} \, dr \, d\theta = \frac{1}{2} \int_J |f - f_j|^2 \, dt \leq \frac{|f|}{2} \epsilon_1 C_1 ||f||_* . \]

The last inequality follows if we assume \( \alpha_e > \delta \) (which is so when \( \epsilon_1 \) is small), so that \( J \in \Phi(\alpha_e) \), enabling us to apply Lemma 5. Moreover,

\[ |\nabla f_2(re^{i\theta})| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |\nabla P(r, \theta) - i|f_2(e^{i\theta})| \, dt \]
\[ = \frac{1}{\pi} \int_{\partial D \setminus J} \frac{1}{|e^{it} - re^{i\theta}|^2} |f(e^{i\theta}) - f_j| \, dt . \]

Hence if \( re^{i\theta} \in S(\theta_0, r_0) \), we have

\[ |\nabla f_2(re^{i\theta})| \leq \frac{1}{\pi} C_5 \int_{-\pi}^{\pi} \frac{|f(e^{it}) - f_j|}{|e^{it} - z_0|^2} \, dt \quad \left( C_5 = 2\left( 1 + \frac{17\pi^2}{2} \right) \right) \]
\[ \leq \frac{1}{\pi} C_5 \int_{-\pi}^{\pi} \frac{|f(e^{it}) - f_j|}{|e^{it} - z_0|^2} \, dt + \frac{2\pi}{1 - r_0^2} |f_j - f_j| \]
\[ \leq \frac{4}{|J|} C_5 (2\pi C_2 \epsilon_1 + 5\epsilon_1) \quad \text{(by Lemma 6)} \]
\[ = C_6 \frac{1}{|J|} \epsilon_1 \quad \text{for some constant} \ C_6 . \]

Thus

\[ \iint_{S(\theta_0, r_0)} (1 - r)|\nabla f_2|^2 r \, dr \, d\theta \leq \frac{1}{2} C_5^2 \epsilon_1 \frac{1}{|J|^2} (1 - r_0) \, dr \, d\theta \]
\[ = \frac{1}{2} C_6^2 \epsilon_1 (1 - r_0) . \]

Since \( |\nabla|^2 \leq 2(|\nabla f_1|^2 + |\nabla f_2|^2) \), from inequalities (8) and (9), we get the desired conclusion \( \iint_{S(\theta_0, r_0)} (1 - r)|\nabla f|^2 r \, dr \, d\theta \leq \epsilon(1 - r_0) \) if we choose a suitable \( \epsilon_1 \). Thus the proof of the theorem is complete.

4. Description of \( \mathbb{C}^* \)-algebras \( C_B, Q_B \) for \( H^\infty \subset B \subset L^\infty \). Suppose \( B \) is a closed subalgebra of \( L^\infty \) which contains \( H^\infty \). Then, as stated in §1, \( B \) is a Douglas algebra. Suppose \( B \) is generated by \( H^\infty \) and a collection \( \{b_\lambda\} \) of conjugates of inner functions, where \( \lambda \) runs over some index set \( E \). For each \( \lambda \in E \), let \( B_\lambda \) be the closed subalgebra \( H^\infty[b_\lambda] \). For each finite subset \( F \subset E \), let \( B_F \) be the closed algebra generated by \( B_\lambda \), for all \( \lambda \in F \), and let \( b_F \) be the inner function \( \prod_{\lambda \in F} b_\lambda \). Then it is easy to see that \( B_F = H^\infty[b_F] \). For each \( \delta > 0 \), let \( G_\delta(b_F) = \{ z \in D \mid |b_F(z)| \geq 1 - \delta \} \). For each finite subset \( F \subset E \),
each $\delta > 0$, and each real number $a_0$, let $\mathcal{G}_{\delta, a_0}(F) = \{I | I = \{e^{it} | t - \theta| < a(1 - r), \text{with } re^{i\theta} \in G_{\delta}(b_F), a(1 - r) < \pi\}$, where $1 \leq a \leq a_0$. For each $\varepsilon > 0$, let $a_\varepsilon$ be the same integer as in §3. We define $\text{VMO}_B$ to be the space of all functions $f$ in $\text{BMO}$ with the following properties:

For every $\varepsilon > 0$, there exists some $\delta > 0$, and some finite subset $F$ of $E$, such that $(1/|I|) \int_I |f - f_I| \, dt < \varepsilon$ whenever $I \in \mathcal{G}_{\delta, a_\varepsilon}(F)$. The following theorem is parallel to Theorem 6:

**Theorem 7.** Let $B$, $b_\lambda$, $E$, $F$, $\delta$, $b_F$, $G_\delta(b_F)$, $\text{VMO}_B$ be defined as above. For a function $f$ defined on $\partial D$ and in $\text{BMO}$, the following conditions are equivalent:

1. $f \in \text{VMO}_B$.
2. $f = u + \tilde{v}$ for some functions $u, v$ in $Q_B = B \cap \overline{B}$.
3. Given $\varepsilon > 0$, there exists some $\delta > 0$ and some finite subset $F$ of $E$, such that the measure $\mu_\delta(F)$ on $D$ defined by $d\mu_\delta(F) = \chi_{G_\delta(b_F)}(1 - r)\sqrt{\int f^2 r \, dr \, d\theta}$ is a Carleson measure, and $\mu_\delta(F)(R(I)) < \varepsilon|I|$ for all subarcs $I$ of $\partial D$.

In view of Theorem 4 in [3] (which says that the collection of functions in $L^\infty$ which satisfy condition (3) in Theorem 7 above is the $C^*$-algebra $Q_B$) and the argument used to establish Theorem 7 in [3], it is easy to see the same proof of Theorem 6 which pertains to the special case $B = H^\infty[\overline{B}]$, also work for general algebra $B$ after some slight modification. We will skip the details here.

The following corollary is an immediate consequence of Lemma 3 (which holds also for general $B$) and Theorem 7. It gives some information about the boundary behavior of functions in the subalgebra $Q_B$, and hence answers another question proposed by D. Sarason in [5].

**Corollary.** $Q_B = \text{VMO}_B \cap L^\infty = (C_B + \tilde{C}_B) \cap L^\infty$.

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