HOPF INVARIANTS AND BROWDER'S WORK
ON THE KERVAIRE INVARIANT PROBLEM

BY

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Abstract. In this paper we calculate certain functional differentials in the
Adams spectral sequence converging to Wu cobordism whose values may be
thought of as Hopf invariants. These results are applied to reobtain Brow-
der's characterization: if \( q + 1 = 2^k \), there is a \( 2q \) dimensional manifold of
Kervaire invariant one if and only if \( h_2^k \) survives to \( E_\infty(S^0) \).

Introduction. In 1969 Browder \[2\] showed that there is a \( 2q \) dimensional
framed manifold with Kervaire invariant one if and only if \( q + 1 = 2^k \) and \( h_2^k \)
survives to \( E_\infty(S^0) \). More precisely in case \( q + 1 = 2^k \), if \((M^{2q}, F)\) is a
framed manifold whose Thom invariant is \( f \in \pi_{2q}(S^0) \), then Browder showed
that the Kervaire invariant, \( K(M^{2q}, F) \), is one if and only if \( f \) projects to \( h_k^2 \) in
\( E_\infty^{2k^2+1}(S^0) \). On the other hand, the present author \[3\] introduced homotopy
invariants \( I_1 \) and \( I_2 \) which are defined on certain subgroups of the homotopy
groups \( \pi_*(X) \) of a spectrum \( X \) and which take their values in the \( E_2 \) term of
an Adams spectral sequence for \( X \). For the mod 2 Adams spectral sequence
of \( S^0 \) and an element \( f \in \pi_{2q}(S^0) \), \( I_2(f) \) is always defined, and if \( q + 1 = 2^k \),
\( h_2^k \) survives if and only if there is \( f \in \pi_{2k^2+1-2}(S^0) \) so that \( I_2(f) = h_2^k \). Thus we
can restate Browder's theorem (at least in dimension \( q + 1 = 2^k \)) as follows.

Theorem 1 (Browder). If \((M^{2k^2+1-2}, F)\) is a framed manifold with Thom
invariant \( f \in \pi_{2k^2+1-2}(S^0) \), then

\[
I_2(f) = K(M^{2k^2+1-2}, F) h_2^k
\]

in \( E_2^{2k^2+1}(S^0) \).

It is the main purpose of this paper to show that the invariant \( I_2 \) not only
serves to give a convenient expression for Browder's theorem but in fact
serves as a vehicle for proving it.

Our efforts in this direction center on the mod 2 Adams spectral sequence
which converges to Wu cobordism. If \( MO(\nu_{q+1}) \) denotes the Thom spectrum
for Wu \((q + 1)\)-cobordism theory, the Wu manifolds \( (S^q \times S^q, \overline{q}) \) con-
structured by Browder [2, §5] have Thom invariant $\tilde{\omega} \in \pi_{2q}(MO(v_{q+1}))$. Then the main result of the paper explicitly determines $I_2(\tilde{\omega})$ as a subset of $E_2^{2q+1}(MO(v_{2q}))$ in case $q + 1 = 2^k$.

The paper is organized along the following lines.

In §1, after giving a preamble of notation and terminology basic to our work, we state the main theorem (Theorem 3) and deduce Theorem 1 from it.

§2 is devoted to setting up the homological algebra which is preliminary to the detailed study of $I_2(\tilde{\omega})$.

In §3 we prove Theorem 3 except for the verification of a geometric result (Lemma 8) whose details are given in §4.

1. Statement of main results. We begin by establishing the basic geometric and algebraic contexts for the paper. The geometric setting is the stable homotopy category constructed by Adams in [1]. A notation from that setting which shall be convenient is the following: if $f: X \to Y$ is a morphism, then its Puppe cofiber triangle is denoted by

$$
\begin{array}{c}
X \\
\downarrow f \\
Y
\end{array}
$$

$$
\begin{array}{c}
P(f) \\
\downarrow Q(f) \\
C(f)
\end{array}
$$

where the degree of $P(f) = 0$ and the degree of $Q(f) = -1$. It will also be convenient to name the objects and morphisms in the stable category which correspond to the Thom complexes arising in Browder's work [2]. Let $MO$ and $MO(v_{q+1})$ denote the spectra determined by $T\gamma_n$ and $T\gamma_n$ respectively and $\tilde{p}: MO(v_{q+1}) \to MO$ denote the morphism which corresponds to $T\gamma_n \to T\gamma_n$. Also if $H_q$ denotes the Eilenberg-Mac Lane space of type $(Z_2, q)$, let $j: H_q \to MO(v_{q+1})$ denote the morphism which corresponds to the fiber inclusion $j: H_q \to BO(v_{q+1})$, where $X$ denotes the suspension spectrum of the pointed CW complex $(X, \ast)$.

Let $R^*$ denote the mod 2 Steenrod algebra and $R_*$ its dual, the latter supplied with the $Z_2$ base $\{\xi^R\}$ and the former with the dual base $\{Sq_i\}$ of Milnor [10]. Homology and cohomology will be taken with $Z_2$ coefficients and the basic algebraic context will be the category of $R^*$ comodules.

Having disposed of these preliminaries, we may now state our main results whose proofs are distributed over the next three sections.

**Theorem 2.** Let $a \in \pi_q(H_q) = Z_2$ be the generator. Then

$$\text{Ext}_{R_*}^{4q+1}(Z_2, H_\ast(MO(v_{q+1})))$$

is isomorphic to $Z_2$ generated by $I_1(\tilde{\omega})$. Further if $q + 1 = 2^k$ and $i: S^0 \to MO(v_{2q})$ is the morphism of spectra induced by $(pt. \to BO_n(v_{q+1}))$, then
Let $\tilde{q}: S^q \times S^q \to BO_n(v_{q+1})$ be the composition

$$S^q \times S^q \xrightarrow{\times a} H_q \times H_q \xrightarrow{\mu} H \xrightarrow{j} BO_n(v_{q+1})$$

where $\mu$ is the group structure map and where we have taken the liberty of letting "$a$" also denote the essential map $S^q \to H_q$. Browder [2] has shown that the Thom invariant corresponding to the Wu manifold $(S^q \times S^q, \tilde{q})$ is $j_\omega$, where $\omega \in \pi_2q(H_q) \cong \mathbb{Z}_2$ is the generator. Our principal result determines $I_2(j_\omega)$ for $q + 1 = 2^k$.

**Theorem 3.** If $q + 1 = 2^k$, then $I_1(j_\omega) = 0$, so that $I_2(j_\omega)$ is defined. In this case $I_2(j_\omega)$ is given by

$$(*) \quad h(\tilde{i}_\omega) = \{h_k \cdot I_1(\tilde{a})\}$$

in $\text{Ext}^2_{\mathbb{Z}_2}(\mathbb{Z}_2, H_*(MO(v_q)))/\text{im} d^2_2$, and this element is not equal to 0.

To illustrate the position of Theorem 3 relative to Browder’s work on the Kervaire invariant problem, we use it to prove Theorem 1.

Let $i: S^0 \to MO(v_{q+1})$ be as in Theorem 1. Browder has shown that there is a $2q$ dimensional framed manifold of Kervaire invariant one if and only if

$$\text{im} \{i_\#: \pi_{2q}(S^0) \to \pi_{2q}(MO(v_{q+1}))\} \neq 0.$$ 

For $q + 1 = 2^k$, we deduce from Theorem 3 that $\text{im} i_\# = 0$ if and only if $h_k^2$ fails to survive to $E_\infty$.

First suppose that $\text{im} i_\# = 0$. Then

$$P(i_\#: \pi_{2^k+1}(MO(v_{2^k})) \to \pi_{2^k+1}(C(i)))$$

is monic. Now

$$i_\#(h_k^2) = h_k \cdot i_\#(h_k) = h_k \cdot I_1(\tilde{a})$$

which survives since $\{h_k \cdot I_1(\tilde{a})\} = I_2(j_\omega) \neq 0$. Since $P(i_\#)$ is monic, $P(i_\#(g)) \neq 0$ for any $g \in \pi_{2^k+1}(MO(v_{2^k}))$ which projects to $\{h_k \cdot I_1(\tilde{a})\}$ in $E_\infty(MO(v_{2^k}))$. Thus by Lemma 3.4.1 of [8] $h_k^2$ is not even permanent.

Conversely suppose that $h_k^2$ fails to survive but that there is an element $f \in \pi_{2^k+1}(S^0)$ so that $i_\#(f) \neq 0$. By Theorem 7.2 of [2], $i_\# f = \tilde{a}_\omega$ and by naturality

$$I_2(j_\omega) = I_2(i_\# f) = i_\#(I_2(f)) = 0,$$

since $h_k^2$ fails to survive. But this contradicts Theorem 3, so $\text{im} i_\#$ must be 0.

**2. Homological algebra.** In this section we carry out the homological algebra which is preliminary to the proof of Theorem 3. For a spectrum $X$, let $E(X) = \{E_r(X), d_r\}$ denote the mod 2 Adams spectral sequence whose $E_2$
term is Ext$^\_\mu_\ast(Z_2, H_\ast(X))$ and whose limit is $\pi_\ast(X)$. Occasionally we shall regard $E$ as a functor on the stable category and denote induced maps on $E_1$ by a subscript "\ast\ast" and induced maps on (sub) quotients of $E_1$ by a subscript "\ast" (which convention is consistent with the above usage of sharped morphism letters for induced homomorphisms on Ext). In [3], it was shown that

$$E_1^\ast(X) = \left[ \bar{\mathcal{E}}_\ast \otimes H_\ast(X) \right],$$

with differential $d_1^\ast$ equal to

$$(f \otimes 1)(c \otimes 1)\psi_{\bar{\mathcal{E}}_\ast \otimes H_\ast(X)},$$

where $\bar{\mathcal{E}}_\ast = \mathcal{E}_\ast / \mathcal{E}_0$, $j: \mathcal{E}_\ast \to \bar{\mathcal{E}}_\ast$ is the quotient, $\bar{\mathcal{E}}_\ast$ denotes the $s$-fold tensor product of $\mathcal{E}_\ast$, $\mathcal{C}: \bar{\mathcal{E}}_\ast \to \bar{\mathcal{E}}_\ast$ is the canonical anti-automorphism and $\psi_{\bar{\mathcal{E}}_\ast \otimes H_\ast(X)}$ is the tensor product $\bar{\mathcal{E}}_\ast$ comodule structure map for $\bar{\mathcal{E}}_\ast \otimes H_\ast(X)$. In what follows, Ext$^\ast(Z_2, H_\ast(X))$ shall be regarded as a subquotient of $\bar{\mathcal{E}}_\ast \otimes H_\ast(X)$.

Let $\omega \in \pi_2q(H_q) = Z_2$ be the generator. Since $\omega$ is determined by the Hopf construction applied to

$$S^q \times S^q \xrightarrow{\alpha \times \alpha} H_q \times H_q \xrightarrow{\mu} H_q,$$

it is easy to see that $\omega_\ast$ is 0 on homology, so that $I_1(\omega)$ is defined. Its value is given in

**Proposition 4.**

$$I_1(\omega) = \left\{ \sum_{r=1}^{q+1} (q + 1 - r)\xi_1^r \otimes \tilde{g}(q + 1 - r) \right\}$$

in Ext$^{1,2q+1}(Z_2, H_\ast(X))$, where $\tilde{g}(s) \in H_{q+s}(H_q)$ is the element in basis dual to the base $\{Sq^s\}$ for $H^{q+s}(H_q)$ corresponding to $Sq^q$, $g$ is the generator of $H^q(H_q)$ and $(r, s)$ is the binomial coefficient $(r + s)! / r! s!$.

**Proof.** Let $x(q + 1) \in H_{2q+1}(C(\omega))$ be the element in the basis dual to the basis $\{x^s\}$ for $H^{2q+1}(C(\omega))$ corresponding to $Sq^{q+1}x$, where $x \in H^q(C(\omega))$ maps to $g \in H^q(H_q)$ by the map $P(\omega)^\ast$. Thus $x(q + 1)$ maps to $\xi \in H_{2q}(S^{2q})$ via $Q(\omega)^\ast$. Consequently the calculation of $I_1(\omega)$ amounts to the calculation of $d_1^\omega(\tilde{x}(q + 1))$, which in turn, amounts to the calculation of $\psi_C(\omega)(\tilde{x}(q + 1))$, where $\psi_C(\omega)$ is the $\mathcal{C}_\ast$ comodule structure map for $H_\ast(C(\omega))$.

To that end let $\varepsilon_q: H_q \to H_{q+1}$ be the obvious map of spectra of degree $+1$. Since $\varepsilon_0^q \omega = 0$, $\varepsilon_q$ extends to a map $\varepsilon_q^\ast: C(\omega) \to H_{q+1}$ of degree $+1$ for which $\varepsilon_q^\ast: H_j(H_{q+1}) \to H^{j-1}(C(\omega))$ is an isomorphism for $j < 2q + 1$ and a monomorphism for $j = 2q + 2$. Now let $y \in H^{1,j}(H_{q+1})$ be the generator; then $\varepsilon_q^\ast(y) = x$ so that $\varepsilon_q^\ast(Sq^{q+1}y) = Sq^{q+1}x$ and hence

$$\varepsilon_q^\ast(\tilde{x}(q + 1)) = \tilde{y}(q + 1),$$

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where $\tilde{y}(s)$ and $Sq^y$ are corresponding elements of the obvious dual bases. From basic properties,

$$\Psi_C(\omega)(x(q + 1)) = 1 \otimes x(q + 1) + S,$$

while

$$(1 \otimes \epsilon_{q^y})(\Psi_C(\omega)(x(q + 1))) = \Psi_{H_{q+1}}(\tilde{y}(q + 1)) = \sum_{r=0}^{q+1} (r, q + 1 - r)\xi^r_1 \otimes \tilde{y}(q + 1 - r)$$

since $H_*(H_{q+1})$ is an extended $\mathfrak{g}_*$ comodule in dimensions $< 2q + 2$. However

$$\Psi_C(\omega)(x(q + 1)) = \sum_{r=0}^{q+1} (r, q + 1 - r)\xi^r_1 \otimes \tilde{y}(q + 1 - r),$$

so

$$\Psi_C(\omega)(x(q + 1)) = \sum_{r=0}^{q+1} (r, q + 1 - r)\xi^r_1 \otimes \tilde{x}(q + 1 - r).$$

Thus

$$d_1^0(\tilde{x}(q + 1)) = (j \otimes 1)(c \otimes 1)\Psi_C(\omega)(x(q + 1)) = \sum_{r=0}^{q+1} (r, q + 1 - r)\xi^r_1 \otimes \tilde{x}(q + 1 - r).$$

Since $P(\omega)_*(\tilde{g}(s)) = \tilde{x}(s)$ for $s < q$, we have

$$I_1(\omega) = \left\{ \sum_{r=1}^{q+1} (r, q + 1 - r)\xi^r_1 \otimes \tilde{g}(q + 1 - r) \right\}.$$

**COROLLARY 5.** If $q + 1 = 2^k$, then $I_1(\tilde{j}_\omega)$ is 0.

**Proof.** By naturality $I_1(\tilde{j}_\omega) = \tilde{j}_\omega(I_1(\omega))$. Now if $q + 1 = 2^k$, $I_1(\omega) = \{\xi_1^2 \otimes \tilde{g}\}$ and so $\tilde{j}_\omega I_1(\omega) = \{\xi_1^2 \otimes \tilde{j}_\omega(\tilde{g})\}$. But $\tilde{j}_\omega(\tilde{g}) = 0$.

Our next result serves to delimit the indeterminacy of $I_2(\tilde{j}_\omega)$.

**Proposition 6.** In the long exact Ext-sequence associated to

$$0 \to H_*(MO(v_{q+1})) \to H_*(C(\tilde{j}_\omega)) \to H_{*-1}(S^{2q}) \to 0,$$

the coboundary

$$\delta: \text{Ext}^{2q+1}_{H_*(Z_2)}(Z_2, H_*(S^{2q})) \to \text{Ext}^{2q+2}_{H_*(Z_2)}(Z_2, H_*(MO(v_{q+1})))$$

is given by
\[
\delta(h'_0) = h_0 \cdot I_1(j\omega),
\]
where \( h'_0 \) corresponds to \( h_0 \in \text{Ext}^1_d(Z_2, Z_2) \).

**Proof.** It suffices to show that \( \delta(h'_0) = h_0 \cdot I_1(\omega) \) in the Ext-sequence for \( \omega: S^{2q} \to H_q \) and then apply naturality. To get this latter formula, we calculate \( \delta \) by means of the cobar differential.

On the one hand,
\[
1 \otimes Q(\omega)_* : \mathcal{C}_* \otimes H_* C(\omega) \to \mathcal{C}_* \otimes H_*(S^{2q})
\]
maps \([\xi_1, \bar{x}(q + 1)]\) to \([\xi_1, \bar{x}]\) which represents \( h'_0 \). On the other hand,
\[
(1 \otimes P(\omega)_*): \mathcal{C}_*^2 \otimes H_*(H_q) \to \mathcal{C}_*^2 \otimes H_*(C(\omega))
\]
maps
\[
[\xi_1] \cdot \left( \sum_{r=1}^{q+1} (r, q + 1 - r)[\xi_1 \bar{g}(q + 1 - r)] \right)
\]

\[
= \sum_{r=1}^{q+1} (r, q + 1 - r)[\xi_1 \bar{g}(q + 1 - r)]
\]
to
\[
\sum_{r=1}^{q+1} (r, q + 1 - r)[\xi_1 \bar{x}(q + 1 - r)] = d^1_1[\xi_1 \bar{x}(q + 1)].
\]
Consequently, \( \delta(h'_0) = h_0 I_1(\omega) \) according to the definition of \( \delta \) and the product in \( \text{Ext}^d_\mathcal{C} \).

We close this section with a proof of Theorem 2. To facilitate matters we set down some notation and isolate a lemma, which appears to be useful in computing certain Ext groups. First, following Liulevicius [6], [7], we identify \( H_* (MO) \) with the extended comodule
\[
\mathcal{C}_* \otimes Z_2[u_2, u_4, u_5, \ldots, u_\lambda, \ldots]
\]
(as comodules), where degree \( u_\lambda = \lambda \neq 2^i - 1 \) and \( u_0 \) is the unit of the polynomial ring. If \( U \in H^0(MO) \) is the Thom class, then its dual in \( H_0(MO) \) is \( 1 \otimes u_0 \).

**Lemma 7.** Suppose given a commutative ladder
\[
\begin{array}{c}
C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2 \\
\downarrow f_0 \quad \quad \quad \downarrow f_1 \quad \quad \quad \downarrow f_2 \\
C'^0 \xrightarrow{d'^0} C'^1 \xrightarrow{d'^1} C'^2
\end{array}
\]
of $K$-modules ($K$ a commutative ring with unit) in which $d^1 d^0 = 0$, $\text{im} d^0 = \ker d^1$ and $f_1$ and $f_2$ are isomorphisms. Then there is a short exact sequence

$$0 \rightarrow \ker d^0 \rightarrow \frac{C^0}{\ker d^0 \cap \text{im} f_0} \rightarrow \frac{C^1}{\text{im} f_0} \rightarrow \ker d^1 \rightarrow 0$$

where $\alpha$ is induced by inclusion and $\beta$ by $f_1^{-1} d^0$.

**Proof.** Clearly $\ker d^1 / \text{im} d^0$ is isomorphic to $\text{im} d^0 / \ker d^0 (\text{im} f_0)$ via $f_1$. But projection of $C^0 / \text{im} f_0$ into this last group has kernel $(\ker d^0 + \text{im} f_0) / \text{im} f_0$ which is isomorphic to $\ker d^0 / \ker d^0 \cap \text{im} f_0$.

In our application, $\text{im} f_0 \supset \ker d^0$ so that $\ker d^1 / \text{im} d^0 = \text{coker} f_0$ via $f_1$.

**Proof of Theorem 2.** To apply Lemma 7 to this situation, for $\varepsilon = 0, 1, 2$ choose

$$C^\varepsilon = E_1^{\varepsilon \cdot q + 1} (MO(v_{q+1})), \quad C''^\varepsilon = E_1^{\varepsilon \cdot q + 1} (MO),$$

$f_1$ to be the maps induced by $\bar{p}: MO(v_{q+1}) \rightarrow MO$ and $d^\varepsilon$ and $d'^\varepsilon$ to be the appropriate differentials. Recalling that $U$ is the generator of $H^0(MO)$, we have $\ker \bar{p}^\varepsilon \subset \text{im}(d^0)^\varepsilon$ since $\ker \bar{p}^\varepsilon$ is generated by

$$cS_q^{q+1} U = (\xi^q)^* (\xi^q)^* (\xi^q)^* (\xi^q)^*.$$
Adams filtration of $\tilde{\omega}$ is 2. Thus $I_2(\tilde{\omega}) \neq 0$.

It remains for us to verify the formula (*) of Theorem 3. The outline for this part of the proof goes as follows. First we rewrite the map $\tilde{\omega}$ in a convenient form listing the necessary properties of this rewrite. These are summarized in Lemma 8 below. The various parts of the lemma are then applied to establish (*) and finally (in §4) Lemma 8 itself is verified.

**Lemma 8.** There is a morphism of spectra
\[ \alpha: H_q \wedge \text{MO}(v_{q+1}) \to \text{MO}(v_{q+1}) \]
with the following properties:

(a) the Thom invariant $\tilde{\omega}$ of $(S^q \times S^q, q)$ is equal to
\[ \alpha(a \wedge 1)(1 \wedge \tilde{\omega}) : S^{2q} \to \text{MO}(v_{q+1}), \]
(b) $\alpha(a \wedge 1)(1 \wedge i): S^{q} \to S^{q} \wedge S^{0} \to \text{MO}(v_{q+1})$ is equal to $\tilde{\omega}$,
(c) $\alpha(a \wedge 1)_* : H_* (\text{MO}(v_{2q} \wedge \text{MO}(v_{2q})) \to H_* (\text{MO}(v_{2q}))$ is 0 in dimensions $< 2^{k+1} - 2$.

According to the definition of $I_2$, the formula (*) will follow if we show that
\[ h_k \cdot I_1(\tilde{\omega}) \in (P(\tilde{\omega}))_*^{-1} d_2^0 (Q(\tilde{\omega}))_*^{-1} (i). \]

What we shall arrange is the location of an element
\[ z \in (Q(\tilde{\omega}))_*^{-1} (i) \subset E_{2}^{0,2^{k+1}-1} (C(\tilde{\omega})) \]
so that
\[ (P(\tilde{\omega}))_* (h_k \cdot I_1(\tilde{\omega})) = d_2^{0,2^{k+1}-1} (z). \]

We record some data derived from Lemma 8 to be used in finding the element $z$. From part (a) there is a map $A : C(1 \wedge \tilde{\omega}) \to C(\tilde{\omega})$ so that
\[
\begin{array}{cccc}
S^{2k-1} \wedge \text{MO}(v_{2k}) & \xrightarrow{P(1 \wedge \tilde{\omega})} & C(1 \wedge \tilde{\omega}) \\
\alpha(a \wedge 1) & & A \\
\text{MO}(v_{2k}) & \xrightarrow{P(\tilde{\omega})} & C(\tilde{\omega})
\end{array}
\]
commutes. We note that $P(1 \wedge \tilde{\omega})_*$ and $P(\tilde{\omega})_*$ are isomorphisms on homology in dimensions $< 2^{k+1} - 2$; consequently $A_*$, as well as $[\alpha(a \wedge 1)]_*$, is 0 in dimensions $< 2^{k+1} - 2$. Now let $w \in H_{2k}(C(\tilde{\omega}))$ be the element defined in the proof of Theorem 2, $\omega \in H_{2k+1}(C(1 \wedge \tilde{\omega}))$ its $(2^k - 1)$-fold suspension. Set $z = A_*(\omega w)$. It is easy to see that not only is $Q(\tilde{\omega})_*(z) = i \in \text{MO}(v_{q+1})$.
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\[ H_{2k+1-2}(S^{2k+1-2}) \] but also that \( z \in E_2^{0,2k+1-1}(C(j_\omega)) = \ker d_1^{0,2k+1-1}. \) Equally easy to see is that

\[ P(1 \wedge j_\omega) \zeta_i^{2k} \otimes \sigma(u) = d_1^0(\sigma w) \]

in \( E_1^{1,2k+1}(C(1 \wedge \bar{j}_a)) \), where \( u \in H_0(MO(v_{2k})) \) corresponds to \( 1 \otimes u_0 \) via \( P_\ast \) and \( \sigma \), as above, denotes \((2^k - 1)\)-fold suspension. Furthermore

\[ [\alpha(a \wedge 1)](\zeta_i^{2k} \otimes \sigma u) = 0 \quad \text{and} \quad A_\ast(d_1^0\sigma w) = 0. \]

Consequently the functional differentials \((d_1^1)_{a(a \wedge 1)} \) and \((d_1^1)_A \) are defined on \( \zeta_i^{2k} \otimes \sigma u \) and \( d_1^0(\sigma w) \) respectively and, by naturality, their values are related by

\[ P(j_\omega) \#(d_1^1)_{a(a \wedge 1)}(\zeta_i^{2k} \otimes \sigma u) = (d_1^1)_A(d_1^0(\sigma w)) \]

in the quotient of \( E_1^{2,2k+1}(C(j_\omega)) \) by the total indeterminacy. The key point here is this. Since \( \alpha(a \wedge 1) \) and \( A \) induce the zero homomorphism on homology in dimensions < \( 2^k + 1 - 2 \), they likewise induce the zero homomorphism on \( E_1^{2,2k+1}. \) Consequently \((d_1^1)_{a(a \wedge 1)} \) and \((d_1^1)_A \) take their values in \( E_1^{2,2k+1}(MO(v_{2k})) / \text{im} \ d_1^1 \) and \( E_1^{2,2k+1}(C(j_\omega)) / \text{im} \ d_1^1 \) respectively, and so in \((\dagger)\)

\[ P(j_\omega) \# \text{ is, after restriction, the homomorphism on } E_2 = \text{Ext} \text{ induced by } P(j_\omega). \]

On the one hand, now, it is a routine calculation, using part (b) of Lemma 8, to show that

\[ (d_1^1)_{a(a \wedge 1)}(\zeta_i^{2k} \otimes \sigma u) = \{ \zeta_i^{2k} \otimes \xi_1^{2k} \otimes \sigma u \} \]

which of course is \( h_k \cdot I_1(j_\omega) \). On the other hand, by the second Peterson-Stein formula for Adams differentials [5],

\[ (d_1^1)_A(d_1^0(\sigma w)) = d_2^0(A_\ast(\sigma w)) = d_2^0(z) \]

in \( E_1^{2,2k+1}(C(j_\omega)) / \text{im} \ d_1^1 \). Thus \( I_2(j_\omega) = (h_k \cdot I_1(j_\omega)) \) in

\[ \text{Ext}_Z^{2,2k+1}(Z_2, H_\ast(MO(v_{2k}))) / \text{im} \ d_2^{2,2k+1-1} \]

and the proof of Theorem 3 is complete subject to the verification of Lemma 8.

4. Proof of Lemma 8. We begin with the construction of \( \alpha. \) For \( n \gg 2q, \) let

\[ \alpha_n: H_q \times BO_n(v_{q + 1}) \rightarrow BO_n(v_{q + 1}) \]

be the principal fiber space structure map. Since \( \text{proj}_3(1 \times n) = \pi_\alpha_n, \) the bundles \( \alpha_n^* (\gamma_n) \) and \( \alpha_n^* (\gamma_n \oplus \theta_1) \) are isomorphic to \((H_q \times R^0) \times \gamma_n \) and \((H_q \times R^0) \times (\gamma_n \oplus \theta_1) \) respectively. Thus, by passing to Thom complexes, we have maps \( \tilde{\alpha}_n: (H_q)_+ \wedge MO(n) \rightarrow MO(n) \) and commutative diagrams
These diagrams, via the handicrafted smash products $X \wedge_{\ast} Y$ [1, p. 49], give a map of spectra

$$\alpha: (H_q)_+ \wedge MO(v_{q+1}) \to MO(v_{q+1}).$$

Additionally, there is an isomorphism of spectra $X \vee S = X_+$, where $X$ and $X_+$ are the suspension spectra of the pointed spaces $(X, x_0)$, $x_0 \in X$ and $(X_+, +)$, $+ \in X$. Thus $\alpha$ may be regarded as being defined on $(H_q \vee S) \wedge MO(v_{q+1})$. The map $\alpha$, now, is the composite of $\alpha$ with the inclusion

$$H_q \wedge MO(v_{q+1}) \to (H_q \vee S^0) \wedge MO(v_{q+1}).$$

Property (a) now follows routinely from the definition of the Thom invariant and the fact that $\tilde{q}: S^q \times S^q \to BO_n(v_{q+1})$ is equal to $\alpha_n(a \wedge 1) \cdot (1 \wedge \tilde{a})$. Property (b) follows from commutative diagrams on the level of Thom complexes arising from commutative diagrams

$$S^q \times pt \to H_q \xrightarrow{j} BO_n(v_{q+1}) \to BO_n(v_{q+1}).$$

Turning now to part (c), we note that $\alpha_*$ factors through $\alpha_n$ which, by the Thom isomorphism is

$$\alpha_n: H_*(H_q \times BO_n(v_{q+1})) \to H_*(BO_n(v_{q+1})).$$

Thus it suffices to prove

**Proposition 9.** Let $\tilde{g} \in H_{2k-1}^*(H_{2k-1})$ be the generator. Then

$$\alpha_n(\tilde{g} \otimes c'): H_m(BO_n(v_{2k}')) \to H_{m+2k-1}(BO_n(v_{2k}'))$$

is $0$ for $m < 2k - 1$.

**Proof.** It suffices to show that, on cohomology, $\alpha_n^*(c)$ does not contain a term of the form $g \otimes c'$ with $c' \in H^{l-2k+1}(BO_n(v_{2k}))$ when

$$c \in H^l(BO_n(v_{2k})) \text{ and } l < 2k+1 - 2.$$
To show this we appeal to Theorem 7.4 in the work of J.-P. Meyer on functional operations and principal coactions [9]. Since \( \{\text{Sq}^g\} \) is a transgressive \( \mathbb{Z}_2 \)-base for \( H_*(H_{2k-1}) \) in dimensions \( < 2^{k+1} - 2 \) and \( \tau(\text{Sq}^g) = \text{Sq}^Rv^*_2 \), it suffices to show

**Theorem 10.** There are no relations of the form

\[
C \cdot v^*_2 + \sum_{R \neq (0,0,\ldots)} C_R \cdot \text{Sq}^Rv^*_2 = 0
\]

in \( H^l(BO(n)) \) for \( l < 2^{k+1} - 1 \). Here \( C \) and \( C_R \) are elements of \( H^*(BO(n)) \) subject only to the restrictions that \( C \neq 0 \), \( \dim \text{Sq}^R > 0 \) and

\[
\dim C_R + \dim \text{Sq}^R = l = \dim C + 2^k.
\]

Before giving the proof, we need some technical data concerning the representation of \( H^*(BO(n)) \) as the full symmetric algebra of \( \mathbb{Z}_2[t_1, \ldots, t_n] \).

Let \( \pi(m) = \{\phi | \phi \text{ is a partition of } m\} \). If \( \phi = (c_1, \ldots, c_r) \in \pi(m) \), we define the length of \( \phi \), \( l(\phi) \), to be \( r \) and the degree of \( \phi \), \( n(\phi) \), to be \( c_1 + \cdots + c_r \). Corresponding to the partition \( \phi = (c_1, \ldots, c_r) \) let \( s(\phi) \) be the “smallest” symmetric polynomial of \( \mathbb{Z}_2[t_1, \ldots, t_n] \) containing the monomial \( t_1^{c_1} \cdots t_n^{c_r} \). As is well known the set \( \{s(\phi)\} \) forms a \( \mathbb{Z}_2 \)-base for \( H^*(BO(n)) \). In each dimension \( m \) we order the base \( \{s(\phi) | \phi \in \pi(m)\} \) by the order induced from the following order of \( \pi(m) \). Let \( \succ \) be any total order of \( \pi(m) \) which is consistent with

(i) \( \omega' \succ \omega \) if \( \omega' \) refines \( \omega \) [11, p. 183],

(ii) if \( \omega \) and \( \omega' \) are incompatible with respect to refinement, then \( \omega' \succ \omega \) if \( l(\omega') > l(\omega) \).

Relative to the refinement notion, we note the following two points. First, if \( 2\omega \) denotes the partition obtained from \( \omega \) by termwise multiplication by 2, then \( 2\omega' \) refines \( 2\omega \) if and only if \( \omega' \) refines \( \omega \). Second, if \( s(\omega) \) occurs in \( s(\phi) \cdot s(1^{2^k}) \) then \( \phi \cdot 1^{2^k} \) refines \( \omega \). (Of course \( n \), the number of indeterminates, \( > l(\phi) + 2^k \).

Now we express \( v^*_2 \), the \( 2^k \) Wu class of \( BO(n) \), in terms of the base \( \{s(\phi)\} \).

**Proposition 11.** Let

\[
\mathbb{S} = \{\rho \in \pi(2^k) | \rho = (1^{a_1}, (2^2 - 1)^{a_2}, \ldots)\}
\]

Then

\[
v^*_2 = \sum_{\rho \in \mathbb{S}} s(\rho).
\]

**Proof.** Since \( c\text{Sq}^{2^k} = \sum_{\rho \in \mathbb{S}} \text{Sq}^R(\rho) \) [10], \( v^*_2 U = \sum U s(\rho) \), where \( U \in H^*(MO(n)) \) is the Thom class and \( R(\rho) = (a_1, a_2, \ldots) \). Thus the proof of Thom’s theorem (II. 10) [13, p. 42] implies \( v^*_2 = \sum_{\rho \in \mathbb{S}} s(\rho) \).
**Corollary 12.** If \( s(\phi) \) is the largest basis element occurring in \( C \in H^p(BO(n)) \) with \( p < 2^k - 1 \), then \( s(\phi \cdot 1^{2^k}) \) is the largest basis element occurring in \( C \cdot v_{2^k} \).

We need one final observation concerning how certain basis elements multiply.

**Lemma 13.** If \( \phi, \psi \) and \( \rho' \) are partitions subject to the following conditions:

(i) \( n(\phi) < 2^k - 1 \) and \( n(\psi) < n(\phi) \);

(ii) \( n(\rho') > 2^k \) and \( l(\rho') < 2^k \);

(iii) \( n(\psi) + n(\rho') = n(\phi) + 2^k \).

Then \( s(\psi)s(\rho') \) does not contain a basis element \( s(\sigma) \) for which \( \sigma \) refines \( \phi \cdot 1^{2^k} \).

The proof of Lemma 13 is a bookkeeping exercise involving the length and degree conditions and is left to the reader.

**Proof of Theorem 10.** Set

\[
\exp = C_{v_{2^k}} + \sum_{R \neq (0,0, \ldots)} C_R Sq^R_{v_{2^k}} \in H^I(BO(n)),
\]

\( l < 2^{k+1} - 1 \). Corresponding to this element \exp, we construct a group \( G \) and a homomorphism \( \Phi: H^I(BO(n)) \to G \) so that \( \Phi(\exp) \neq 0 \).

Let \( s(\phi) \) be the largest term occurring in \( C \) with \( \phi = (c_1, \ldots, c_r) \). Following Stong [12], let

\[
R_{P_{2\phi \cdot 1^{2^k}}} = \left( \bigotimes_{i=1}^r R_{P_{2q_i}} \right) \times \bigotimes_{j=1}^{2^k} (R_{P_{2^j}})
\]

and let \( \Phi \) be the projection of Stong’s map \( \Omega \) into \( H^*(R_{P_{2\phi \cdot 1^{2^k}}}) \). Relative to \( \{s(\omega)\} \), \( \exp \) is equal to \( s(\phi \cdot 1^{2^k}) \) plus a sum of terms \( s(\sigma) \) for which \( \sigma \) does not refine \( \phi \cdot 1^{2^k} \). But Stong has shown that for such \( \sigma \), \( \Phi(s(\sigma)) = 0 \), while \( \Phi s(\phi \cdot 1^{2^k}) \) equals the top dimensional cohomology class of \( R_{P_{2\phi \cdot 1^{2^k}}} \). Thus \( \Phi(\exp) \neq 0 \).

**References**


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