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A NEW CHARACTERIZATION OF CESÀRO-PERRON INTEGRALS USING PEANO DERIVATES(1)

BY

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ABSTRACT. The Z_n-integrals are defined according to the method of Perron using Peano derivates. The properties of the integrals are given including the essential integration by parts theorem. The integrals are then shown to be equivalent to the Cesàro-Perron integrals of Burkill.

1. **Introduction.** It has been shown that the $C_n P$-integral of Burkill [2] is equivalent to the $C_n D$-integral of Sargent [7]. (The reader is directed to Verblunsky [8] for corrections to some of the important theorems.) In this paper we define a scale of integration equivalent to these. The method is that of Perron using Peano derivates. James [3] has also used Peano derivates to define an integral, the difference here being the use of a premajorant function as well as a majorant. The properties of the integral are all derived from properties of the generalized derivatives of the premajorant and preminorant functions.

Basic definitions are presented in §§2 and 3 with elementary properties in §4. The integration by parts theorem is in §5 and the proof of the equivalence with the $C_n P$-integral is in §6.

2. **Peano derivatives.**

DEFINITION 2.1. Let $F(x)$ be defined on an interval $[a, b]$. Let $x_0 \in (a, b)$. Let n be a natural number. If there are constants $\alpha_1, \ldots, \alpha_n$ depending on x_0 but not on h such that

$$F(x_0 + h) - F(x_0) - \sum_{k=1}^{n} \alpha_k \frac{h^k}{k!} = o(h^n) \quad \text{as } h \to 0 \tag{2.1}$$

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(1) This research was undertaken at Michigan State University under the direction of Jan Mařík and forms the basis of my thesis.

then α_n is called the generalized derivative or Peano derivative of F at x_0. This is denoted by $F_{(n)}(x_0)$. It is easily seen that if $F_{(n)}(x_0)$ exists then so do $F_{(k)}(x_0)$ $(1 \leqslant k \leqslant n)$ and then

$$F(x_0 + h) - F(x_0) - \sum_{k=1}^{n} \frac{h^k}{k!} F_{(k)}(x_0) = o(h^n) \quad \text{as } h \to 0.$$

In particular, $F_{(1)}(x_0) = F^{(1)}(x_0)$, the ordinary derivative. We also say that $F_{(0)}(x_0) = F(x_0)$ when F is continuous. By restricting h, say $h > 0$, we may also define one-sided generalized derivatives, denoted $F_{(n)+}(X_0)$ etc.

A function f defined and finite on an interval I will be called an nth exact Peano derivative (e.P.d.) on I provided that there is a (continuous) function F on I such that $F_{(n)}(x_0) = f(x_0)$ for each point x_0 in the interior of I and, in case I contains its end points, that the corresponding one-sided nth derivatives of F at these points equal f there. Similarly, if we say that F has an nth Peano derivative in I we mean that $F_{(n)}$ exists in the interior and the one-sided derivatives exist at the end points when these are in I.

DEFINITION 2.2. Let n be a natural number. Let F be defined in the interval $[a, b]$. Let $x_0 \in [a, b]$. If $n = 0$ we assume F is continuous at x_0. If $n > 0$ we assume $F_{(n-1)}(x_0)$ exists. Define $\theta_n(F, x_0, h)$ for h such that $x_0 + h \in (a, b)$ by

$$\frac{h^n}{n!}\theta_n(F, x_0, h) = F(x_0 + h) - \sum_{k=0}^{n-1} \frac{h^k}{k!} F_{(k)}(x_0). \tag{2.2}$$

Note that in case x_0 is a or b we agree that all these generalized derivatives are "one-sided". Define

$$\begin{aligned} \Delta_n F(x_0) &= \overline{F}_{(n)}(x_0) = \limsup_{h \to 0} \theta_n(F, x_0, h), \\ \delta_n F(x_0) &= \underline{F}_{(n)}(x_0) = \liminf_{h \to 0} \theta_n(F, x_0, h). \end{aligned} \tag{2.3}$$

Then $\Delta_n F(x_0)$ is called the nth upper generalized derivate of F at x_0 and $\delta_n F(x_0)$ is called the nth lower generalized derivate of F at x_0. It is clear that $F_{(n)}(x_0)$ exists if and only if $\Delta_n F(x_0) = \delta_n F(x_0)$ and both are finite. In this case $F_{(n)}(x_0)$ is the common value. But as $\Delta_n F$, $\delta_n F$ need not be finite we can say that $F_{(n)}(x_0)$ exists (in the finite or infinite sense) whenever $\Delta_n F(x_0) = \delta_n F(x_0)$. Again the "one-sided" derivates are easy modifications of (2.3) and, as above, when we speak about $\Delta_n F(x_0)$, etc., on a closed interval we shall mean it in the one-sided sense at end points. Let $\delta_{n,+} f(x)$ signify the lower right hand generalized derivate of f at x.

The next four propositions follow readily from the definitions.

PROPOSITION 2.3. *If $f^{(n)}(x)$ exists then $f_{(n)}(x)$ does as well, and they are equal.*

PROPOSITION 2.4. *Assume* $f_{(n)}(x)$ $(n \geqslant 1)$ *exists on* $[a,b]$ *and* $F(t) = \int_c^t f$ *where* $c \in (a,b)$. *Then*

$$F_{(k)}(x) = f_{(k-1)}(x) \qquad (k = 1, \ldots, n+1;\ x \in (a,b)).$$

COROLLARY 2.5. *Under the conditions of Proposition* 2.4,

$$\delta_{n+2} F(x) \geqslant \delta_{n+1} f(x).$$

PROOF. Let $M < \delta_{n+1} f(x)$. Then for all sufficiently small positive h,

$$\int_0^h \left[f(x+s) - f(x) - \sum_{k=1}^{n} \frac{s^k}{k!} f_{(k)}(x) \right] ds > \int_0^h \frac{s^{n+1}}{(n+1)!} M\, ds.$$

That is,

$$F(x+h) - F(x) - \sum_{k=1}^{n+1} \frac{h^k}{k!} F_{(k)}(x) > \frac{h^{n+2}}{(n+2)!} M,$$

thus $\delta_{n+2,+} F(x) > M$, and so

$$\delta_{n+2,+} F(x) \geqslant \delta_{n+1} f(x).$$

The rest is obvious. □

COROLLARY 2.6. *If* f *is defined in* $[a,b]$ *and if* $f_{(m)}$ *exists there and if*

$$F(x) = \int_a^x d\xi_1 \int_a^{\xi_1} d\xi_2 \cdots \int_a^{\xi_{k-1}} f(\xi_k)\, d\xi_k$$

for $x \in [a,b]$, *then*

$$f_j(x) = F_{k+j}(x) \qquad (0 \leqslant j \leqslant m;\ x \in [a,b])$$

and $\delta_{m+k+1} F \geqslant \delta_{m+1} f$.

PROPOSITION 2.7. *If* $\delta_n f(x) \geqslant 0$ *for every* x *in an interval* I *then* $f^{(n-1)}$ *exists and is nondecreasing on* I.

PROOF. This has been shown by James(2) [3] and Verblunsky [9]. □

PROPOSITION 2.8. *Let* f *and* g *be defined on an interval* I *such that* $f_{(n)} = g_{(n)}$ *on* I. *Then* f *and* g *differ by a polynomial of degree no more than* $n-1$.

PROOF. As $(f-g)_{(n)} = 0$ we have $(f-g)^{(n-1)}$ is a constant by Proposition 2.7. The rest is easy. □

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(2) The referee points out the errors in James' paper and mentions that Mukhopadhyay clears up the difficulties in a paper to appear in Pacific Journal of Mathematics.

3. The Z_n-integral.

DEFINITION 3.1. Let n be a natural number. Let M, f be defined in $[a,b]$. Then M is called an n-majorant of f in $[a,b]$ if there is a function P on $[a,b]$ such that

(1) $M = P_{(n)}$ on $[a,b]$,

(2) $\delta_{n+1} P(x) \geqslant f(x)$ for each $x \in [a,b]$.

(3) $\delta_{n+1} P(x) > -\infty$ for each $x \in [a,b]$.

The function P will be called a premajorant. Then n-minorants are defined similarly, replacing (2), (3) by

(2′) $\Delta_{n+1} P(x) \leqslant f(x)$ for each $x \in [a,b]$,

(3′) $\Delta_{n+1} P(x) < +\infty$ for each $x \in [a,b]$,

and then P is called a preminorant.

DEFINITION 3.2. Let f be defined on $[a,b]$. The upper Z_n-integral of f on $[a,b]$ is

$$(Z_n^*) \int_a^b f = \inf\{M(b) - M(a)\colon M \text{ is an } n\text{-majorant of } f \text{ on } [a,b]\}.$$

The lower Z_n-integral of f on $[a,b]$ is

$$(Z_{n*}) \int_a^b f = \sup\{m(b) - m(a)\colon m \text{ is a } n\text{-minorant of } f \text{ on } [a,b]\}.$$

(The infimum of an empty set is of course $+\infty$, etc.) If the upper and lower Z_n-integrals are finite and equal then we write $(Z_n) \int_a^b f$ for the common value and say that f is Z_n-integrable on $[a,b]$.

REMARK. Since we have restricted $n > 0$ it is easily seen that the premajorant P must be continuous. When we apply the definition to the case $n = 0$ we see that the conventions force P to be continuous. However, in this case we have exactly the definition of the Perron integral and here it is well known that the continuity of $M = P$ is of no consequence. (For example, see Saks [6, p. 247ff].) The Z_0-integral is then exactly the Perron integral.

PROPOSITION 3.3. *If M is an n-majorant of f and m is an n-minorant of f on $[a,b]$ then $M - m$ is nondecreasing.*

PROOF. Let P be an associated premajorant and p a preminorant of f. Then $\delta_{n+1}(P - p) \geqslant 0$ so that $M - m = (P - p)_{(n)}$ is nondecreasing. □

Note that because of Proposition 3.3, $(Z_n^*) \int_a^b f \geqslant (Z_{n*}) \int_a^b f$ for every function f.

The following simple result can be taken as the motivation for this integral.

PROPOSITION 3.4. *If $F_{(n+1)}$ exists and is finite on $[a,b]$ then $F_{(n+1)}$ is Z_n-integrable there and*

$$(Z_n) \int_a^b F_{(n+1)} = [F_{(n)}]_a^b.$$

PROOF. $F_{(n)}$ is at the same time an n-majorant and an n-minorant of $F_{(n+1)}$. □

We will frequently make use of the following in the later proofs.

PROPOSITION 3.5. *A function f defined on $[a, b]$ is Z_n-integrable there if and only if for each $\varepsilon > 0$ there is an n-majorant M and an n-minorant m such that*

$$M(b) - M(a) - (m(b) - m(a)) < \varepsilon.$$

Occasionally we shall use the more classical notation $\int_a^b f(t)\,dt$ or $(Z_n)\int_a^b f(t)\,dt$ instead of $(Z_n)\int_a^b f$.

4. Properties of the Z_n-integral.

PROPOSITION 4.1. *A Z_n-integrable function f on $[a, b]$ is finite almost everywhere on $[a, b]$.*

PROOF. Let M be an n-majorant and m an n-minorant of f on $[a, b]$. We may assume $M(a) = m(a) = 0$. Define $R(x) = M(x) - m(x)$. Then R is nondecreasing by Proposition 2.7. Let P, p be respectively a premajorant and preminorant of f associated with M, m. Then $\delta_{n+1}(P - p) \geqslant 0$ on $[a, b]$ so that $(P - p)^{(n)}$ exists and is nondecreasing by Proposition 2.7. Thus $(P - p)^{(n)} = (P - p)_{(n)} = R$ (Proposition 2.3). Suppose $f(x) = +\infty$. Then $\delta_{n+1} P(x) = +\infty$ and, as $\Delta_{n+1} p(x) < +\infty$, we have $(P - p)_{(n+1)}(x) = +\infty$. Similarly, if $f(x) = -\infty$, then $\Delta_{n+1} p(x) = -\infty$ and since $\delta_{n+1} p(x) > -\infty$ we have $(P - p)_{(n+1)}(x) = +\infty$. But $R' = (P - p)^{(n)'} = (P - p)^{(n+1)}$ exists and is finite a.e. in $[a, b]$ and so f is finite a.e. in $[a, b]$. □

PROPOSITION 4.2. *If f is Z_n-integrable on $[a, b]$ and if $c \in (a, b)$ then f is Z_n-integrable on each of $[a, c]$ and $[c, b]$. Moreover,*

$$(Z_n)\int_a^b f = (Z_n)\int_a^c f + (Z_n)\int_c^b f.$$

Furthermore, if f is integrable on the parts then it is integrable on the whole, as usual.

DEFINITION. Let f be Z_n-integrable in $[a, b]$. Then, if $x \in (a, b]$, we see from Proposition 4.2 that f is Z_n-integrable over $[a, x]$. We put $(Z_n)\int_a^a f = 0$ for any function. Then $(Z_n)\int_a^x f$ is defined for each $x \in [a, b]$. An indefinite Z_n-integral of f is any function of the form

$$F(x) = c + (Z_n)\int_a^x f$$

where c is constant.

PROPOSITION 4.3. *Let f be Z_n-integrable on $[a,b]$. Let F be any indefinite Z_n-integral of f on $[a,b]$. Then F is the uniform limit of a sequence of n-majorants on $[a,b]$ (similarly for n-minorants).*

PROOF. Let $\varepsilon > 0$. Let M, m be respectively an n-majorant and an n-minorant of f on $[a,b]$ such that $M(a) = m(a) = F(a)$ and $M(b) - m(b) < \varepsilon$. As $M - m$ is nondecreasing, $0 \leqslant M(x) - m(x) < \varepsilon$ for each $x \in [a,b]$. Since M, m are respectively an n-majorant and an n-minorant on every subinterval of $[a,b]$ we have $m(x) - m(a) \leqslant (Z_n) \int_a^x f \leqslant M(x) - M(a)$. Thus

$$0 \leqslant M(x) - M(a) - (Z_n) \int_a^x f = M(x) - F(x) < \varepsilon.$$

This rest is obvious. □

PROPOSITION 4.4. *Let F be an indefinite Z_n-integral of f on $[a,b]$. Let M be an n-majorant of f on $[a,b]$ such that $F(a) = M(a)$. Then $M - F$ is nondecreasing and continuous.*

PROOF. To show that $M - F$ is continuous, one applies the intermediate value property of derivatives. □

PROPOSITION 4.5. *Let F be an indefinite Z_n-integral of a function f on $[a,b]$. Then F is an nth e.P.d. on $[a,b]$, and so is Z_{n-1}-integrable.*

According to this theorem we can form the iterated integral

$$\int_a^x d\xi_0 \int_a^{\xi_0} d\xi_1 \cdots \int_a^{\xi_{n-1}} f(\xi_n) d\xi_n \tag{4.1}$$

whenever f is Z_n-integrable, where the innermost integral is a Z_n-integral, the next is a Z_{n-1}-integral, etc., and the outermost is a Z_0-integral. We shall use the symbol

$$\int_a^x f(\xi) d(\xi, n)$$

for the $(n+1)$-fold iterated integral in (4.1). Note also that $\int_a^x f(\xi) d(\xi, n)$ is a continuous function of x.

PROPOSITION 4.6. *Let $F(x)$ be an indefinite Z_n-integral of a function f on $[a,b]$. Let $G_{(n)} = F$ on $[a,b]$. Then $G_{(n+1)}(x) = f(x)$ for almost every $x \in [a,b]$.*

PROOF. The proof in Natanson [5] of the corresponding result about the Perron integral can be adapted. □

PROPOSITION 4.7. *Every Z_n-integrable function is measurable.*

PROOF. Let f be Z_n-integrable. Applying Proposition 4.6 let $G_{(n+1)} = f$ a.e. Then

$$f(x) = \lim_{k\to\infty}\left(k^{n+1}(n+1)!\left\{G\left(x+\frac{1}{k}\right) - \sum_{m=0}^{n} G_{(m)}(x)/k^m m!\right\}\right)$$

for almost every x. Thus f is the limit a.e. of a sequence of continuous functions (as G is continuous) and so is measurable. □

PROPOSITION 4.8. *The Z_0-integral is identical to the Perron integral. If f is Z_n-integrable on $[a,b]$ then f is also Z_{n+1}-integrable there and $(Z_n)\int_a f = (Z_{n+1})\int_a^b f$.*

PROOF. If M is an n-majorant then it is also an $(n+1)$-majorant. □

Recall also that the Perron integral is an extension of the finite Lebesgue integral.

PROPOSITION 4.9. *Let f be a nonnegative measurable function on $[a,b]$. Then*

$$(Z_n^*)\int_a^b f = (Z_{n*})\int_a^b f = (L)\int_a^b f.$$

(The last integral is the finite or infinite Lebesgue integral.) In particular, if $f \geqslant 0$ and if f is Z_n-integrable then f is Lebesgue integrable and $(Z_n)\int_a^b f = (L)\int_a^b f$.

PROOF. If $(L)\int_a^b f$ is finite then all follows from Proposition 4.8. If $(L)\int_a^b f = +\infty$ let $f_k = \min\{f,k\}$. Then f_k is Lebesgue integrable and it is easy to see that

$$(Z_{n*})\int_a^b f \geqslant (Z_{n*})\int_a^b f_k = (Z_n)\int_a^b f_k = (L)\int_a^b f_k \quad (k = 1,2,\ldots).$$

Then since $\lim(L)\int_a^b f_k = +\infty$, we also have $(Z_{n*})\int_a^b f = +\infty$. The rest follows at once. □

PROPOSITION 4.10. *Let f and g be Z_n-integrable in $[a,b]$. Let $h(x) = f(x) + g(x)$ whenever the right side has meaning. Then h is Z_n-integrable in $[a,b]$ and*

$$(Z_n)\int_a^b h = (Z_n)\int_a^b f + (Z_n)\int_a^b g.$$

Moreover, if $f(x) = k(x)$ a.e. in $[a,b]$, then k is Z_n-integrable and its integral equals that of f.

PROPOSITION 4.11. *Let f and P be defined in $[a,b]$ so that $P_{(n)}$ exists and so that*
(a) $\delta_{n+1}P(x) \geqslant f(x)$ *a.e. in* $[a,b]$,
(b) $\delta_{n+1}P(x) > -\infty$ *in* $[a,b]$.
Then $(Z_n^*)\int_a^b f \leqslant [P_{(n)}]_a^b$.

PROOF. Let $h(x) = f(x)$ when $\delta_{n+1}P(x) \geqslant f(x)$, $h(x) = -\infty$ otherwise. Then $P_{(n)}$ is an n-majorant for h and so $(Z_n^*)\int_a^b h \leqslant [P_{(n)}]_a^b$.

Define $k(x) = 0$ when $\delta_{n+1} P(x) \geqslant f(x)$, $k(x) = +\infty$ otherwise. Then $f(x) \leqslant h(x) + k(x)$ whenever the right side has meaning and so

$$(Z_n^*) \int_a^b f \leqslant (Z_n^*) \int_a^b h + 0 \leqslant [P_{(n)}]_a^b. \quad \square$$

From this theorem we see that we may enlarge the class of majorants to include those functions which satisfy property (2) of Definition 3.1 only almost everywhere. A similar result holds of course for minorants.

We note also that it is possible to relax requirement (3) of Definition 3.1 as well. We need only assume that the inequality holds outside of an arbitrary countable set. To show that the resulting class of premajorants leaves the integral unaffected one shows that for any premajorant function P, in the new sense, and any $\varepsilon > 0$ there is a continuous, nondecreasing function, F, such that $[F^{(n)}]_a^b < \varepsilon$, for which $P + F$ is a premajorant in the old sense and yet the function F contributes only ε to the upper integral. We omit details. We emphasize however that this relaxation of requirements leaves the *extent* of definition of the integral unchanged.

5. Integration by parts for the Z_n-integral.

Notation. In what follows let $BV[a,b]$ signify the class of functions which have finite variation on the interval $[a,b]$. Let $(R) \int_a^b f dg$, for $g \in BV[a,b]$and f continuous, be the Riemann-Stieltjes integral of f with respect to g in the sense that the infimum of the upper sums associated with f and the positive (negative) variation of g is the same as the supremum of the lower sums.

Fundamental to all that follows is the next theorem.

PROPOSITION 5.1. INTEGRATION BY PARTS.

(a) *Let f be Perron integrable in $[a,b]$, $F(x) = (Z_0) \int_a^x f$. Let $G \in BV[a,b]$. Then*

$$(Z_0) \int_a^b fG + (R) \int_a^b F dG = [FG]_a^b.$$

(b) *Let $n \geqslant 1$. Let f be Z_n-integrable on $[a,b]$. Let $F(x) = (Z_n) \int_a^x f$. Let G and γ be defined on $[a,b]$ such that $\gamma \in BV[a,b]$ and $G^{(n-1)}$ is an indefinite integral of γ. Then fG is Z_n-integrable and*

$$(Z_n) \int_a^b fG + (Z_{n-1}) \int_a^b FG' = [FG]_a^b.$$

Part (a) is well known (Saks [6, p. 244f]). The proof of (b) is due largely to Mařík. Later we shall show that the strong conditions put on the function G are really necessary. Before presenting the proof we need some preliminaries.

LEMMA 5.2. *Let φ, ψ be functions on $[a,b]$. Let q be a natural number. Let $\varphi^{(q)}$ be continuous on $[a,b]$ and let $\psi^{(q-1)}$ be absolutely continuous on $[a,b]$. Let R be an indefinite integral of $\varphi\psi^{(q)}$. Set*

$$V = \sum_{j=0}^{q-1} (-1)^j \varphi^{(q-j-1)}\psi^{(j)} + (-1)^q R.$$

Then $V' = \varphi^{(q)}\psi$.

PROOF.

$$\begin{aligned}\int_a^x \varphi^{(q)}\psi &= [\varphi^{(q-1)}\psi]_a^x - \int_a^x \varphi^{(q-1)}\psi' = \cdots \\ &= [\varphi^{(q-1)}\psi - \varphi^{(q-2)}\psi' + \cdots + (-1)^{q-1}\varphi\psi^{(q-1)}]_a^x + (-1)^q \int_a^x \varphi\psi^{(q)} \\ &= V(x) - V(a). \quad \square\end{aligned}$$

DEFINITION 5.3. Let $n > 1$ be an integer. Let M be a continuous function on $[a,b]$. Let $\gamma \in BV[a,b]$. Let $g^{(n-2)}$ be an indefinite integral of γ. Let $K_0, \ldots, K_n$ be functions on $[a,b]$ with the following properties:

(a) $K_k^{(k)} = Mg^{(k)}$ $(k = 0, 1, \ldots, n-2)$,

(b) $K_{n-1}^{(n-2)}$ is an indefinite integral of $M\gamma$,

(c) there is a number c such that $K_n^{(n-2)}$ is an indefinite integral of $c + \int_a^x M d\gamma$ $(x \in [a,b])$. Then we say that $K_0, \ldots, K_n$ have property ω with respect to M, g on $[a,b]$.

LEMMA 5.4. *Let a, b, n, g, γ be as above. Let M be a function on $[a,b]$ such that $M^{(n)}$ is continuous on $[a,b]$. Let $K_0, \ldots, K_n$ have property ω with respect to M, g. Set*

$$S = \sum_{k=0}^{n} (-1)^k \binom{n}{k} K_k.$$

Then $S^{(n)} = M^{(n)}g$.

PROOF. We first show that

$$\begin{aligned}S^{(r)} &= \sum_{j=0}^{r} (-1)^j \binom{n+j-r-1}{j} M^{(r-j)}g^{(j)} \\ &\quad + \sum_{j=r+1}^{n} (-1)^j \binom{n}{j} K_j^{(r)}\end{aligned} \tag{5.1}$$

for $r = 0, \ldots, n-2$. The relation (5.1) is obvious for $r = 0$. If it is true for some r $(0 \leqslant r < n-2)$, then $S^{(r+1)} = A + B + C + D + E$ where

$$A = \sum_{j=0}^{r} (-1)^j \binom{n+j-r-1}{j} M^{(r+1-j)} g^{(j)},$$

$$B = \sum_{j=0}^{r-1} (-1)^j \binom{n+j-r-1}{j} M^{(r-j)} g^{(j+1)},$$

$$C = (-1)^r \binom{n-1}{r} M g^{(r+1)},$$

$$D = (-1)^{r+1} \binom{n}{r+1} K_{r+1}^{(r+1)},$$

$$E = \sum_{j=r+2}^{n} (-1)^j \binom{n}{j} K_j^{(r+1)}.$$

We can rewrite B and D as

$$B = -\sum_{j=1}^{r} (-1)^j \binom{n+j-r-2}{j-1} M^{(r+1-j)} g^{(j)},$$

$$D = (-1)^{r+1} \binom{n}{r+1} M \cdot g^{(r+1)}.$$

Then, applying the relation $\binom{p+1}{q+1} - \binom{p}{q} = \binom{p}{q+1}$, we get

$$A + B = \sum_{j=0}^{r} (-1)^j \binom{n-j-r-2}{j} M^{(r+1-j)} g^{(j)}$$

and

$$C + D = (-1)^{r+1} \binom{n-1}{r+1} M g^{(r+1)},$$

which completes the induction and proves (5.1). If we put $r = n-2$ into (5.1) we see

$$\begin{aligned} S^{(n-2)} &= \sum_{j=0}^{n-2} (-1)^j (j+1) M^{(n-2-j)} g^{(j)} \\ &\quad + (-1)^{n-1} n K_{n-1}^{(n-2)} + (-1)^n K_n^{(n-2)}. \end{aligned} \tag{5.2}$$

Set

$$R = K_{n-1}^{(n-2)}, \qquad T_p = \sum_{j=p}^{n-2} (-1)^j M^{(n-2-j)} g^{(j)} + (-1)^{n-1} R$$

$$(p = 0, 1, \ldots, n-2).$$

Set $Z = R - K_n^{(n-2)}$. Then

$$\sum_{p=0}^{n-2} T_p = \sum_{j=0}^{n-2} (-1)^j (j+1) M^{(n-2-j)} g^{(j)} + (-1)^{n-1}(n-1)R,$$

so that

$$S^{(n-2)} = \sum_{p=0}^{n-2} T_p + (-1)^{n-1} Z. \tag{5.3}$$

We also have

$$T_p = \sum_{i=0}^{n-2-p} (-1)^{i+p} M^{(n-2-i-p)} g^{(i+p)} + (-1)^{n-1} R.$$

If we put $\varphi = (-1)^p M$, $\psi = g^{(p)}$, $q = n - 1 - p$, then we get from Lemma 5.2 that

$$T'_p = (-1)^p M^{(n-1-p)} g^{(p)}. \tag{5.4}$$

Further,

$$\begin{aligned} Z(x) &= \left(c_1 + \int_a^x M\gamma\right) - c_2 - \int_a^x \left(c_3 + \int_a^t M\,d\gamma\right) dt \\ &= c_4 + \int_a^x \left(-c_3 + M(t)\gamma(t) - \int_a^t M\,d\gamma\right) dt \\ &= c_4 + \int_a^x \left(c_5 + \int_a^t M'\gamma\right) dt \quad \text{(the } c_j\text{'s are constant).} \end{aligned}$$

Thus

$$Z'(x) = c_5 + \int_a^x M'\gamma. \tag{5.5}$$

Now from (5.3) and (5.4) we get

$$S^{(n-1)} = \sum_{p=0}^{n-2} (-1)^p M^{(n-1-p)} g^{(p)} + (-1)^{n-1} Z', \tag{5.6}$$

and, from (5.5), Z' is an indefinite integral of $M'\gamma$. If we put $\varphi = M'$, $q = n - 1$, $\psi = g$ we have $\psi^{(q-1)} = g^{(n-2)}$ on $[a,b]$ and $\psi^{(q)} = \gamma$ a.e. so the formula $S^{(n)} = M^{(n)} g$ follows at once from (5.6) and Lemma 5.2. □

LEMMA 5.5. *Let $n > 1$ be an integer. Let M, g, γ be functions defined on $I = [a,b]$. Let $M_{(n)}$ exist on I and let $\gamma \in BV(I)$. Let $g^{(n-2)}$ be an indefinite integral of γ. Then there is a function S such that*

$$S_{(n)} = M_{(n)} g \quad \text{on } I. \tag{5.7}$$

For any such function S and for any $x \in I$ for which $g'(x)$ is finite we have

$$\theta_{n+1}(S,x,h) = M_{(n)}(x)g'(x) + g(x)\theta_{n+1}(M,x,h) + o(1). \tag{5.8}$$

Note. If $n > 2$ then $g'(x)$ is finite for every x. If $n = 2$ then g' is finite except possibly on a countable set and even here we have, for $h > 0$ (respectively $h < 0$), that formula (5.8) holds if we replace $g'(x)$ by $g'^{+}(x)$ ($g'^{-}(x)$), the right- (left-) hand derivative.

PROOF. Let $K_0, \ldots, K_n$ have property ω with respect to M, g. Let $S = \sum_{k=0}^{n} (-1)^k \binom{n}{k} K_k$. Let $x \in I$. We may assume $x = 0$. Let P be a polynomial of degree $\leqslant n$ such that the function $\overline{M} = M - P$ fulfills

$$\overline{M}(t) = o(t^n) \quad \text{as } t \to 0. \tag{5.9}$$

Let $\overline{K}_k$ be functions such that $\overline{K}_k^{(k)} = \overline{M}g^{(k)}$ $(k = 0, \ldots, n-2)$,

$$\overline{K}_{n-1}^{(n-2)}(t) = \int_0^t \overline{M}\gamma, \quad \overline{K}_n^{(n-2)}(t) = \int_0^t \left(\int_0^s \overline{M}d\gamma\right) ds,$$

and $\overline{K}_k^{(j)}(0) = 0$ whenever $0 \leqslant j \leqslant k-1$ and $k \leqslant n-1$ or $0 \leqslant j \leqslant n-3$ and $k = n$. Obviously $\overline{K}_k(t) = o(t^{n+k})$ $(k = 0, \ldots, n-1)$ and $\overline{K}_n(t) = o(t^{2n-1})$. If we put $\overline{S} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \overline{K}_k$ we have

$$\overline{S}(t) = \overline{M}(t)g(t) + o(t^{n+1}). \tag{5.10}$$

Set $H_k = K_k - \overline{K}_k$. It is easy to see that $H_0, \ldots, H_n$ have property ω with respect to P, g. Define

$$V = \sum_{k=0}^{n} (-1)^k \binom{n}{k} H_k = S - \overline{S}.$$

By Lemma 5.4 we have $V^{(n)} = P^{(n)}g$. Let G satisfy $G^{(n)} = g$. Let $\alpha = M_{(n)}(0)$. Obviously $P^{(n)} = \alpha$ so that $V^{(n)} = \alpha g = \alpha G^{(n)}$. Thus there is a polynomial Q of degree $< n$ such that $S - \overline{S} = V = \alpha G + Q$ or

$$S = Q + \alpha G + \overline{S}. \tag{5.11}$$

There is a polynomial Q_1 of degree $< n$ such that

$$G(t) = Q_1(t) + \frac{t^n}{n!}g(0) + o(t^n). \tag{5.12}$$

It follows from (5.9)–(5.12) that $S(t) = Q_2(t) + (t^n/n!)\alpha g(0) + o(t^n)$, where $Q_2 = Q + \alpha Q_1$ is a polynomial of degree $< n$. Thus $S_{(n)}(0) = \alpha g(0)$. If $g'(0)$ is finite we have $g(t) = g(0) + o(t)$ and it follows from (5.9) and (5.10) that

$$(5.13)\quad \overline{S}(t) = g(0)\overline{M}(t) + o(t^{n+1}) = g(0)\frac{t^{n+1}}{(n+1)!}\theta_{n+1}(M,0,t) + o(t^{n+1}).$$

Further,

$$G(t) = Q_1(t) + \frac{t^n}{n!}g(0) + \frac{t^{n+1}}{(n+1)!}g'(0) + o(t^{n+1}),$$

so that from (5.11) and (5.13)

$$S(t) = Q_2(t) + \frac{t^n}{n!}\alpha g(0) + \frac{t^{n+1}}{(n+1)!}(\alpha g'(0) + g(0)\theta_{n+1}(M,0,t)) + o(t^{n+1})$$

which completes the proof. □

PROPOSITION 5.6. *Let n be an integer, $n > 1$. Let f, G, γ be functions on $[a,b]$ such that f is Z_n-integrable on $[a,b]$, $\gamma \in BV[a,b]$ and $G^{(n-2)}$ is an indefinite integral of γ. Then for any indefinite Z_n-integral F of f we have*

$$(Z_n)\int_a^b (FG' + fG) = [FG]_a^b.$$

PROOF. Assume first that G, G'^+ and G'^- are positive functions. (If $n > 2$ there is no need to consider G'^+ and G'^- separately as G' exists in this case.) Let U be an n-majorant of f such that $U(a) = F(a)$. By Lemma 5.5 there is a function S such that

$$S_{(n)} = P_{(n)}G = UG.$$

Moreover, by formula (5.8) and the note following Lemma 5.5

$$\delta_{n+1}S(x) \geqslant \min\{U(x)G'^+(x) + G(x)\delta_{n+1}P(x), U(x)G'^-(x) + G(x)\delta_{n+1}P(x)\}.$$

Then $\delta_{n+1}S(x) > -\infty$ for every $x \in [a,b]$ and

$$\delta_{n+1}S(x) \geqslant F(x)G'(x) + G(x)f(x)$$

whenever $G'(x)$ exists (that is, except for a countable set when $n = 2$, and everywhere if $n > 2$). Applying Proposition 4.11 we see that

$$(Z_n^*)\int_a^b (FG' + fG) \leqslant [UG]_a^b$$

and so

$$(Z_n^*)\int_a^b (FG' + fG) \leqslant [FG]_a^b.$$

A similar consideration for n-minorants shows that

$$(Z_{n*})\int_a^b (FG' + fG) \geqslant [FG]_a^b$$

and so the theorem is proved for this case.

Returning to the general case we see easily that G'^+ is bounded on $[a,b)$ and G'^- is bounded on $(a,b]$. Namely, if $n > 2$ then G' is continuous and if $n = 2$ then G is an indefinite integral of a function of bounded variation. Thus we can find a linear function G_2 on $[a,b]$ such that $G_2 > 0$ on $[a,b]$, $G_2' > 0$ on $[a,b]$ and that the function $G_1 = G + G_2$ satisfies the conditions of the special case considered above. Let $\gamma_2 = G_2^{(n-1)}$ and $\gamma_1 = \gamma + \gamma_2$. Then $\gamma_j \in BV[a,b]$ and $G_j^{(n-2)}$ is an indefinite integral of γ_j ($j = 1,2$). Then we may apply the first part of the proof to each G_j, so

$$(Z_n)\int_a^b (FG_j' + fG_j) = [FG_j]_a^b \qquad (j = 1,2),$$

from which the assertion follows at once by linearity. □

The analogues of Lemma 5.5 and Proposition 5.6 for the case $n = 1$ are slightly different.

LEMMA 5.7. *Let M, g be functions on* $[0,1]$. *Let M be Z_0-integrable, let $g \in BV[0,1]$ and let* $\operatorname{var}(g,[0,h]) = O(h)$. *Let P be an indefinite Z_0-integral of M and let S be an indefinite Z_0-integral of Mg. Assume $P'(0)$ is finite. Set*

$$T(h) = \frac{2}{h^2}\int_0^h (g(t) - g(0))\,dt, \quad \lambda = \limsup_{h\to 0} h^{-1} \operatorname{var}(g,[0,h]).$$

Then $\lambda < \infty$ and

$$-\lambda \leqslant \delta_1 g(0) \leqslant \liminf_{h\to 0} T(h), \quad \limsup_{h\to 0} T(h) \leqslant \Delta_1 g(0) \leqslant \lambda, \tag{5.14}$$

and

$$\begin{aligned} S(h) = S(0) &+ hg(0)P'(0) \\ &+ h^2(T(h)P'(0) + g(0)\theta_2(P,0,h))/2 + o(h^2). \end{aligned} \tag{5.15}$$

PROOF. The inequalities $\Delta_1 g(0) \leqslant \lambda < +\infty$ are obvious. Let $\beta > \Delta_1 g(0)$. Then for sufficiently small h we have $g(h) - g(0) < h\beta$ and $T(h) \leqslant (2/h^2)\int_0^h t\beta\,dt = \beta$ so that $\limsup_{h\to 0} T(h) \leqslant \Delta_1 g(0)$. This proves (5.14).

If $M \equiv 1$ then $P'(0) = 1$ and $\theta_2(P,0,h) \equiv 0$,

$$S(h) - S(0) = \int_0^h g = hg(0) + \int_0^h (g(t) - g(0))\,dt = hg(0) + \frac{h^2}{2}T(h)$$

so that (5.15) holds in this case. If $g \equiv 1$ then $T(h) \equiv 0$, $S(h) - S(0) = P(h) - P(0) = hP'(0) + h^2\theta_2(P,0,h)/2$ so that (5.15) holds in this case also. If we hold either M or g fixed in formula (5.15) it is easy to see that the formula is linear with respect to the other so that we may assume $P'(0) = g(0) = 0$. Let $K(x) = P(x) - P(0)$. Then $K(t) = o(t)$, $g(t) = O(t)$ and since $\operatorname{var}(g,[0,h]) = O(h)$ we have $\int_0^h K\,dg = o(h^2)$ so that $S(h) - S(0) = \int_0^h Mg = K(h)g(h) - \int_0^h K\,dg = o(h^2)$ which completes the proof. □

PROPOSITION 5.8. *Let f, G be functions on $[a,b]$. Let f be Z_1-integrable on $[a,b]$. Let G be continuous and in $BV[a,b]$. Suppose* $\operatorname{var}(G,[x-h,x+h]) = O(h)$ *for each $x \in (a,b)$. Let F be an indefinite Z_1-integral of f. Then*

$$(Z_1)\int_a^b (FG' + fG) = [FG]_a^b.$$

PROOF. If G is constant the conclusion is obvious so applying a simple linearity argument we may assume that $G > 0$. Let $\varepsilon > 0$. Let M be a 1-majorant of f on $[a,b]$ such that $M(a) = F(a)$ and $M(b) < F(b) + \varepsilon$. Let S be an indefinite Z_0-integral of MG and let $P' = M$ on $[a,b]$. Using Proposition 5.1 (a) and the well-known mean value theorem for the Riemann-Stieltjes integral we see that $S' = MG$ on $[a,b]$. Moreover, from Lemma 5.7 we have that $\delta_2 S(x) > -\infty$ for every x and that $\delta_2 S(x) = G'(x)M(x) + G(x)\delta_2 P(x)$ whenever $G'(x)$ exists. Thus

$$\begin{aligned}\delta_2 S(x) &\geqslant G'(x)M(x) + G(x)f(x)\\ &\geqslant G'(x)F(x) + G(x)f(x) - \varepsilon|G'(x)| \quad \text{a.e. in } [a,b].\end{aligned}$$

Thus

$$\begin{aligned}-\varepsilon\int_a^b |G'| + (Z_1^*)\int_a^b (FG' + fG) &= (Z_1^*)\int_a^b (-\varepsilon|G'| + FG' + fG)\\ &\leqslant [S']_a^b = [MG]_a^b \leqslant [FG]_a^b + \varepsilon[G]_a^b.\end{aligned}$$

It follows that $(Z_1^*)\int_a^b (FG' + fG) \leqslant [FG]_a^b$. We can show similarly that $(Z_{1*})\int_a^b (FG' + fG) \geqslant [FG]_a^b$ so that the theorem is proved. □

PROOF OF PROPOSITION 5.1 (b). Let $n = 1$, and suppose that F, f, G and γ satisfy the conditions of the theorem. Then F, f and G satisfy the conditions of Proposition 5.8, so that $(Z_1)\int_a^b (FG' + fG) = [FG]_a^b$. Moreover, by Proposition 4.5, F is Z_0-integrable and $\gamma \in BV[a,b]$ so that $F\gamma$ is Z_0-integrable and

$$(Z_0)\int_a^b F\gamma = (Z_0)\int_a^b FG',$$

by Proposition 4.10, which in turn is $(Z_1)\int_a^b FG'$ by Proposition 4.8. Thus

$$[FG]_a^b = (Z_1)\int_a^b (FG' + fG) = (Z_0)\int_a^b FG' + (Z_1)\int_a^b fG$$

as was to be shown.

Let $n > 1$. Let F, f, G and γ satisfy the conditions of the theorem. Then, by Proposition 4.5, there is a function P such that $P_{(n)} = F$. Moreover $(G')^{(n-2)}$ is the indefinite integral of a function in $BV[a,b]$ so, by Lemma 5.5, there is a function S such that $S_{(n)} = P_{(n)}G' = FG'$. Then, by Proposition 3.4, FG' is Z_{n-1}-integrable. Thus, since F, f, G, γ satisfy Proposition 5.6, we see that

$$\begin{aligned}[FG]_a^b &= (Z_n)\int_a^b (FG' + fG)\\ &= (Z_n)\int_a^b fG + (Z_n)\int_a^b FG'\\ &= (Z_n)\int_a^b fG + (Z_{n-1})\int_a^b FG'. \quad \square\end{aligned}$$

To end this section we define

$$G(x) = \begin{cases} x^{n^2+3n-2}\cos 1/x^{n+3}, & x \neq 0,\\ 0, & x = 0;\end{cases}$$

$$f(x) = \begin{cases} x^{1-3n-n^2}\cos 1/x^{n+3}, & x \neq 0,\\ 0, & x = 0.\end{cases}$$

It can be shown that G is n times differentiable in a neighborhood of zero, and that f is an nth exact Peano derivative and yet fG is not Z_m-integrable for any m in any neighborhood of zero. The function f is of course Z_{n-1}-integrable. Thus it is essential to assume more than the n-fold differentiability of G to obtain an integration by parts theorem for the Z_{n-1}-integral.

6. Relation between the Z_n-integral and Burkill's C_nP-integral. Burkill ([1] and [2]) defined a notion of differentiation, continuity and integration inductively. The reader is referred to those definitions. We recall here only that $C_n(f,x,h)$ denotes the nth Cesàro mean of a $C_{n-1}P$-integrable function.

PROPOSITION 6.1. *The Z_n-integral is identical to the C_nP-integral* ($n = 0, 1, 2, \ldots$).

The proof is by induction. For $n = 0$ it is certainly true as both the Z_0-integral and the C_0P-integral are precisely the Perron integral. We assume that, for $0 \leqslant k \leqslant n-1$, the C_kP-integral is identical to the Z_k-integral. We need some lemmas (which will also be useful later).

LEMMA 6.2. *If $P_{(n)}$ is finite in $[a,b]$ then for every $x \in [a,b]$ and for every h such that $x+h \in [a,b]$ we have $\Theta_n(P,x,h) = C_n(P_{(n)},x,h)$.*

PROOF. According to the definition and the induction assumption,

$$\frac{h^n}{n!}C_n(P_{(n)},x,h) = \frac{1}{(n-1)!}(Z_{n-1})\int_x^{x+h}(x+h-\xi)^{n-1}P_{(n)}(\xi)\,d\xi.$$

Applying Propositions 5.1 and 3.4 we see that this is

$$\begin{aligned}
&\frac{1}{(n-1)!}[P_{(n-1)}(\xi)(x+h-\xi)^{n-1}]_x^{x+h} \\
&\qquad + \frac{1}{(n-2)!}(Z_{n-2})\int_x^{x+h}(x+h-\xi)^{n-2}P_{(n-1)}(\xi)\,d\xi \\
&= -\frac{h^{n-1}}{(n-1)!}P_{(n-1)}(x) + \frac{1}{(n-2)!}(Z_{n-2})\int_x^{x+h}(x+h-\xi)^{n-2}P_{(n-1)}(\xi)\,d\xi \\
&= \cdots = \text{(repeating the first step)} \\
&= -\frac{h^{n-1}}{(n-1)!}P_{(n-1)}(x) - \frac{h^{n-2}}{(n-2)!}P_{(n-2)}(x) - \cdots \\
&\quad + \frac{1}{0!}(Z_0)\int_x^{x+h}P_{(1)}(\xi)\,d\xi \\
&= -\sum_{k=1}^{n-1}\frac{h^k}{k!}P_{(k)}(x) + P(x+h) - P(x) \quad \text{as was to be shown.} \quad \square
\end{aligned}$$

LEMMA 6.3. *Suppose $P_{(n)}$ is finite in $[a,b]$. Then*
(a) $C_n D_* P_{(n)} = \delta_{n+1}P$,
(b) $C_n D^* P_{(n)} = \Delta_{n+1}P$, *and*
(c) $C_n D P_{(n)} = P_{(n+1)}$ *provided one side exists.*

PROOF. According to Lemma 6.2,

$$\begin{aligned}
\theta_{n+1}(P,x,h) &= \frac{P(x+h) - \sum_{k=0}^{n} h^k P_{(k)}(x)/k!}{h^{n+1}/(n+1)!} \\
&= \frac{h^n C_n(P_{(n)},x,h)/n! - h^n P_{(n)}(x)/n!}{h^{n+1}/(n+1)!} \\
&= \frac{C_n(P_{(n)},x,h) - P_{(n)}(x)}{h/(n+1)}.
\end{aligned}$$

Now simply take the lower limit of both sides to see (a). The proof of (b) is similar and (c) follows from (a) and (b). □

LEMMA 6.4. *Let M be defined on $[a,b]$. Then M is C_n-continuous on $[a,b]$ if and only if there is a function G on $[a,b]$ such that $G_{(n)} = M$.*

PROOF. Suppose $G_{(n)} = M$. Then according to Lemma 6.2,

$$\lim_{h\to 0} C_n(M,x,h) = \lim_{h\to 0} \frac{G(x+h) - \sum_{k=0}^{n-1} h^k G_{(k)}(x)/k!}{h^k/k!}$$

$$= G_{(n)}(x) = M(x) \quad \text{so that } M \text{ is } C_n\text{-continuous.}$$

Conversely suppose M is C_n-continuous on $[a,b]$. Then it is $C_{n-1}P$-integrable and thus Z_{n-1}-integrable by the induction hypothesis. Let $g_{n-1}(x) = (Z_{n-1})\int_a^x M$. By Proposition 4.5, g_{n-1} is Z_{n-2}-integrable and so we may define successively

$$g_k(x) = (Z_k)\int_a^x g_{k+1} \qquad (0 \leqslant k \leqslant n-2).$$

Put $G = g_0$. Then applying Proposition 5.1 $n-2$ times we have

$$\frac{1}{(n-1)!}(Z_n)\int_x^{x+h} (x+h-\xi)^{n-1} M(\xi)\,d\xi$$

$$= G(x+h) - G(x) - \sum_{k=1}^{n-1} \frac{h^k}{k!} g_k(x).$$

Then, as M is C_n-continuous, we see that $G_{(n)}(x) = M(x)$ for each x. (Also $G_{(k)}(x) = g_k(x)$ $(k = 0,\ldots,n-1)$.) □

Thus from Lemmas 6.3 and 6.4 we see that the class of $(n+1)$th e.P.d.'s is exactly the same as the class of exact C_n-derivatives.

We now finish the proof of Proposition 6.1 by showing that the $C_n P$-integral is identical to the Z_n-integral.

Let f be any function defined on $[a,b]$. Let M be a C_n-majorant of f on $[a,b]$. Then M is C_n-continuous so, by Lemma 6.4, there is a function P such that $P_{(n)} = M$ and, by Lemma 6.3,

$$\delta_{n+1} P(x) = C_n D_* P_{(n)}(x) = C_n D_* M(x)$$

so that $-\infty < \delta_{n+1} P(x) \geqslant f(x)$. Thus M is an n-majorant as well and so

$$(Z_n^*)\int_a^b f \leqslant [M]_a^b$$

for each such M. It follows that

$$(Z_n^*)\int_a^b f \leqslant (C_n P^*)\int_a^b f.$$

If we consider C_n-minorants we see that

$$(Z_{n*})\int_a^b f \geqslant (C_n P_*)\int_a^b f$$

as well, so that the Z_n-integral extends the $C_n P$-integral.

On the other hand, if M is an n-majorant of f on $[a,b]$ then there is a function P such that $P_{(n)} = M$ so that M is C_n-continuous. Moreover, $C_n D_* M(x) = \delta_{n+1} P(x)$ by Lemma 6.3 and so M is a C_n-majorant of f. But then $(C_n P^*)\int_a^b f \leqslant [M]_a^b$ for each such M and so

$$(C_n P^*)\int_a^b f \leqslant (Z_n^*)\int_a^b f.$$

Similarly considering n-minorants we see that

$$(C_n P_*)\int_a^b f \geqslant (Z_{n*})\int_a^b f.$$

Thus we see that if f is integrable in the sense of either Z_n or $C_n P$ then it is also integrable in the other sense and

$$(C_n P)\int_a^b f = (Z_n)\int_a^b f. \quad \square$$

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