THE GENERALIZED GREEN'S FUNCTION FOR
AN nTH ORDER LINEAR DIFFERENTIAL OPERATOR

BY

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ABSTRACT. The generalized Green's function $K(t,s)$ for an nth order linear
differential operator $L$ is characterized in terms of the 2nth order differential
operators $L^*L$ and $LL^*$. The development is operator oriented and takes
place in the Hilbert space $L^2[a,b]$. Two features of the characterization are a
determination of the jumps occurring in the derivatives of orders $n$, $n+1$,
\ldots, $2n-1$ at $t=s$ and a determination of the boundary conditions satis-
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jumps occurring in the derivatives of orders 0, 1, ..., \(n - 1\). These jump conditions take a particularly simple form when specified in terms of the quasi-derivatives as is seen in §4. For each \(c\) with \(a < c < b\) we show in §5 that \(K(c, \cdot)\) satisfies the \(2n\) boundary conditions which determine \(LL^*\), included among these being the boundary conditions determining \(L^*\).

In §6 we establish a representation theorem for \(K(c, \cdot)\) in the case \(a < c < b\) in terms of functions \(\phi_1, \phi_2, \ldots, \phi_{2n}\) which form a basis for the solution space of \(\tau^*\phi = 0\). The representation is uniquely determined by the jump conditions, the boundary conditions determining \(LL^*\), and the property that \(K(c, \cdot)\) is orthogonal to the null space of \(L^*\). The representation theorem is extended to the cases \(c = a\) and \(c = b\) in §7, and the boundary conditions satisfied by \(K(a, \cdot)\) and \(K(b, \cdot)\) are determined. It should be emphasized that these two functions do not satisfy the boundary conditions determining \(L^*\).

In §8 we introduce the generalized Green’s function \(K_1(t, s)\) for the differential operator \(L^*\), showing that \(K(t, s) = K_1(s, t)\) for \(i \neq s\). Letting \(\psi_1, \psi_2, \ldots, \psi_{2n}\) be a basis for the solution space of \(\tau^*\psi = 0\), in §9 we represent \(K(t, s)\) in terms of the functions \(\psi_i(t)\) and \(\phi_j(s)\). We relate \(K(t, s)\) to the orthogonal projection operators on the null spaces of \(L\) and \(L^*\) in §10, and we conclude our discussion by giving several examples of generalized Green’s functions in §11.

2. Basic definitions. For a closed interval \([a, b]\) let \(S\) be the real Hilbert space \(L^2[a, b]\) with the standard inner product \((f, g)\) and norm \(\|f\|\). We denote the domain, range, and null space of any operator \(L\) in \(S\) by \(\mathcal{D}(L), \mathcal{R}(L),\) and \(\mathcal{N}(L)\), respectively. For each positive integer \(n\) let \(H^n[a, b]\) denote the subspace of \(S\) consisting of all functions \(f\) in \(C^{n-1}[a, b]\) with \(f^{(n-1)}\) absolutely continuous on \([a, b]\) and \(f^{(n)}\) in \(S\). Also, let \(H^0_0[a, b]\) denote the subspace consisting of all functions \(f \in H^n[a, b]\) which are identically zero in neighborhoods of \(a\) and \(b\) (the neighborhoods can vary with the \(f\)). The space \(H^n[a, b]\) becomes a Banach space under the norm

\[
\|f\|_n = \sum_{i=0}^{n-1} \max_{a < i < b} |f^{(i)}(t)| + \|f^{(n)}\|, \quad f \in H^n[a, b].
\]

We will refer to this structure as the \(H^n\)-structure for \(H^n[a, b]\).

Given an \(n\)th order formal differential operator

\[
\tau = \sum_{i=0}^{n} a_i(t) \left( \frac{d}{dt} \right)^i,
\]

where the coefficients \(a_i(t)\) belong to \(C^\infty[a, b]\) and \(a_n(t) \neq 0\) on \([a, b]\), and given \(k\) \((0 < k \leq 2n)\) linearly independent boundary values

\[
B_j(f) = \sum_{j=0}^{n-1} \alpha_j f^{(j)}(a) + \sum_{j=0}^{n-1} \beta_j f^{(j)}(b), \quad i = 1, \ldots, k,
\]

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we define a linear differential operator \( L \) in \( S \) by

\[
\mathcal{D}(L) = \{ f \in H^n[a,b] | B_i(f) = 0, i = 1, \ldots, k \}, \quad Lf = \tau f.
\]

Let

\[
\tau^* = \sum_{i=0}^{n} b_i(t) \left( \frac{d}{dt} \right)^i
\]

be the formal adjoint of \( \tau \) where the coefficients are given by

\[
b_i(t) = \sum_{j=i}^{n} (-1)^j \binom{j}{i} \left( \frac{d}{dt} \right)^{j-i} a_j(t), \quad i = 0, 1, \ldots, n,
\]

with \( b_n(t) = (-1)^n a_n(t) \neq 0 \) on \([a,b]\), and let

\[
B_i^*(f) = \sum_{j=0}^{n-1} \alpha_{ij}^* f^{(j)}(a) + \sum_{j=0}^{n-1} \beta_{ij}^* f^{(j)}(b), \quad i = 1, \ldots, 2n - k,
\]

be a set of \( 2n - k \) linearly independent adjoint boundary values. We know that the adjoint operator \( L^* \) is the differential operator in \( S \) given by

\[
\mathcal{D}(L^*) = \{ f \in H^n[a,b] | B_i^*(f) = 0, i = 1, \ldots, 2n - k \}, \quad L^* f = \tau^* f.
\]

The restriction of \( L \) to the subspace \( \mathcal{D}(L) \cap \mathcal{R}(L) \) is a 1-1 closed linear operator, and its inverse \( H = [L|\mathcal{D}(L) \cap \mathcal{R}(L) ]^{-1} \) is a 1-1 linear operator with domain \( \mathcal{R}(L) \) and range \( \mathcal{D}(L) \cap \mathcal{R}(L) \). By the open mapping theorem \( H \) is continuous from \( \mathcal{R}(L) \) under the \( L^2 \)-topology to \( \mathcal{D}(L) \cap \mathcal{R}(L) \) under the \( H^n \)-topology.

Let \( P \) and \( Q \) denote the orthogonal projections from \( S \) onto \( \mathcal{R}(L) \) and \( \mathcal{R}(L^*) \), respectively. Note that \( I - P \) and \( I - Q \) are the orthogonal projections from \( S \) onto the closed subspaces \( \mathcal{R}(L^*) \) and \( \mathcal{R}(L) \), respectively. Also,

\[
L^H f = f \quad \text{for all } f \in \mathcal{R}(L),
\]

and

\[
H L f = f - Pf \quad \text{for all } f \in \mathcal{D}(L).
\]

Let \( L^\dagger : S \to S \) be the linear operator defined by

\[
L^\dagger f = H(I - Q)f, \quad f \in S.
\]

Clearly \( L^\dagger \) is continuous from \( S \) under the \( L^2 \)-topology to \( \mathcal{D}(L) \cap \mathcal{R}(L) \).
under the $H^n$-topology, and $L^\dagger|_{\mathbb{R}(L)} = H$. Also, it can be verified that $L^\dagger$ has the following properties:

(i) $LL^\dagger f = Lf$ for all $f \in \mathbb{D}(L)$,
(ii) $L^\dagger LL^\dagger f = L^\dagger f$ for all $f \in S$,
(iii) $LL^\dagger f = f - Qf$ for all $f \in S$,
(iv) $L^\dagger Lf = f - Pf$ for all $f \in \mathbb{D}(L)$.

The operator $L^\dagger$ is the generalized inverse of $L$, and it has been studied in [8].

Fix a point $t \in [a, b]$, and let $F: S \to \mathbb{R}$ be the linear functional defined by

$$F(f) = L^\dagger f(t) = H(I - Q)f(t) \quad \text{for } f \in S.$$  

Since evaluation at a point is a continuous linear functional on $H^n[a, b]$ under its $H^n$-structure, it follows that $F$ is a continuous linear functional on $S$. Therefore, by the Riesz representation theorem there exists a unique function $K(t, \cdot) \in S$ such that

$$(1) \quad L^\dagger f(t) = F(f) = \int_a^b K(t, s)f(s)ds \quad \text{for all } f \in S,$$

and equation (1) is valid for all $t \in [a, b]$. The function $K(t, s)$ is called the generalized Green's function for $L$. We are going to establish its properties which are analogous to the well-known properties which characterize the standard Green's function when $L$ is invertible, i.e., when $P = Q = 0$ and $L^\dagger = H = L^{-1}$.

In studying the generalized Green's function we will utilize the $2n$th order selfadjoint differential operators $LL^*$ and $L^*L$, which have been considered in [8]. They are given by

$$\mathbb{D}(LL^*) = \{ f \in H^{2n}[a, b]| B_i^*(f) = 0, i = 1, \ldots, 2n - k; \quad B_j^*(\tau f) = 0, j = 1, \ldots, k \}, \quad LL^* f = \tau^* f,$$

and

$$\mathbb{D}(L^*L) = \{ f \in H^{2n}[a, b]| B_i(f) = 0, i = 1, \ldots, k; \quad B_j^*(\tau f) = 0, j = 1, \ldots, 2n - k \}, \quad L^*L f = \tau^* f,$$

where

$$\tau^* = \sum_{i=0}^{2n} c_i(t) \left( \frac{d}{dt} \right)^i = \sum_{i=0}^{n} \sum_{j=0}^{i} \binom{i}{j} a_i(t)b_j^{(i-j)}(t) \left( \frac{d}{dt} \right)^{j+i},$$

from which we see that

$$c_{2n}(t) = a_n(t)b_n(t) = (-1)^n[b_n(t)]^2 \neq 0 \quad \text{on } [a, b].$$
If we set $B_i^+(f) = B_i(r^* f)$ for $f \in H^{2n}[a, b]$, then using Leibniz's rule we can express the boundary value $B_i^+$ as

$$B_i^+(f) = \sum_{j=0}^{n-1} \alpha_j \sum_{l=0}^{n} \sum_{p=1}^{l+j} \binom{j}{p-1} b_l^{(j-p+1)}(a) f^{(p)}(a)$$

$$+ \sum_{j=0}^{n-1} \beta_j \sum_{l=0}^{n} \sum_{p=1}^{l+j} \binom{j}{p-1} b_l^{(j-p+1)}(b) f^{(p)}(b)$$

for all $f \in H^{2n}[a, b]$, $i = 1, \ldots, k$. Another expression for this higher order boundary value is given in [8, p. 178].

A tool that we will frequently use is Green's formula [4, p. 1288]:

$$\int_a^b f(t) g(t) \, dt = \int_a^b f(t) r^* g(t) \, dt + F_{\tau}(f, g) - F_{\bar{\tau}}(f, g),$$

for all $f, g \in H^n[a, b]$, where

$$F_{\tau}(f, g) = \sum_{j=0}^{n-1} F_{\tau}^j(f, g) = \sum_{j=0}^{n-1} F_{\bar{\tau}}^j(f, g)$$

is the boundary form for $\tau$ at the point $t$ with

$$F_{\tau}^j(\tau) = \sum_{i=j}^{n-1} (-1)^i \binom{i}{j} \left( \frac{d}{dt} \right)^{i-j} a_{l+i+1}(t), \quad j \leq n - 1,$n

$$F_{\tau}^j(\tau) = 0, \quad j + l > n - 1.$$

For $j + l = n - 1$ we have

$$F_{\tau}^{n-1}(\tau) = (-1)^{n-1} a_n(t),$$

and hence, the $n \times n$ matrix $[F_{\tau}^j(\tau)]$ is nonsingular for each $t \in [a, b]$. We have similar results for the boundary forms $F_{\tau}(r^*), F_{\bar{\tau}}(r^*),$ and $F_{\tau}^j(r^*).$

3. Continuity properties and jump conditions. We begin our characterization of the generalized Green's function by establishing an orthogonality property.

**Theorem 1.** $K(t, \cdot) \in \mathcal{M}(L^*) = \mathcal{R}(L)$ for all $t \in [a, b]$.

**Proof.** Fix a point $t \in [a, b]$. Then for any function $f(s)$ in $\mathcal{M}(L^*)$ we have $(I - Q)f = 0$, and hence, from (1)

$$\int_a^b K(t, s)f(s) \, ds = H(I - Q)f(t) = 0,$$

so $K(t, \cdot) \in \mathcal{M}(L^*)$. Q.E.D.
In deriving the continuity and differentiability properties of \( K(t,s) \), the following lemma is very useful.

**Lemma 1.** \( \int_a^b K(t,s) \tau^* f(s) \, ds = \tau^* f(t) \) for all \( t \in [a,b] \) and for all \( f \in \mathcal{D}(LL^*) \).

**Proof.** Fix \( t \in [a,b] \) and take any \( f \in \mathcal{D}(LL^*) \). Setting \( g = L^* f = \tau^* f \), we have \( g \in \mathcal{D}(L) \cap \mathcal{D}(L)^\perp \) and

\[
\int_a^b K(t,s) \tau^* f(s) \, ds = H(I - Q)Lg(t) = g(t). \quad \text{Q.E.D.}
\]

**Theorem 2.** For each \( c \) with \( a < c < b \), \( K(c, \cdot) \in C^\infty[a,c] \) with \( \tau^* K(c,s) = 0 \) for \( a \leq s \leq c \), and \( K(c, \cdot) \in C^\infty[c,b] \) with \( \tau^* K(c,s) = 0 \) for \( c \leq s \leq b \) (the left-hand and the right-hand derivatives are not required to be equal at \( s = c \)). Moreover, \( K(a, \cdot) \in C^\infty[a,b] \) and \( K(b, \cdot) \in C^\infty[a,b] \) with \( \tau^* K(a,s) = 0 \) and \( \tau^* K(b,s) = 0 \) for all \( s \in [a,b] \).

**Proof.** For a fixed \( c \) with \( a < c < b \), consider a function \( f(t) \) in \( H^0_{2n}[a,c] \). We can extend \( f(t) \) to all of \( [a,b] \) by taking \( f(t) = 0 \) on \( [b,c] \), in which case \( f(t) \in \mathcal{D}(LL^*) \). By Lemma 1 we have

\[
\int_a^c K(c,s) \tau^* f(s) \, ds = \int_a^b K(c,s) \tau^* f(s) \, ds = \tau^* f(c) = 0.
\]

By Lemma 9 in [4, p. 1291] we conclude that \( K(c, \cdot) \in C^\infty[a,c] \) and \( (\tau^*)^* K(c,s) = \tau^* K(c,s) = 0 \) for all \( s \in [a,c] \).

The same proof establishes the other parts of the theorem. Q.E.D.

We want to study \( K(c,s) \) as a function of \( c \) at the point \( s = c \). It will be shown that \( K(c,s) \) is continuous in its derivatives of orders 0, 1, \ldots, \( n - 2 \) at \( s = c \), while its derivatives of orders \( n - 1, n, \ldots, 2n - 1 \) have prescribed jump discontinuities at \( s = c \).

Fix a point \( c \) with \( a < c < b \), and let \( \eta_-(s) = K(c,s) \) for \( a \leq s \leq c \) and \( \eta_+(s) = K(c,s) \) for \( c \leq s \leq b \). Take any \( f \in H^0_{2n}[a,b] \). Clearly \( f \in \mathcal{D}(LL^*) \), and from Lemma 1 and Green's formula (3) we have

\[
\tau^* f(c) = \int_a^b K(c,s) \tau^* f(s) \, ds \\
= F_c(\tau^*)(f,\eta^-) - E_a(\tau^*)(f,\eta^-) + E_b(\tau^*)(f,\eta^+) - F_c(\tau^*)(f,\eta^+) \\
= F_c(\tau^*)(f,\eta^-) - F_c(\tau^*)(f,\eta^+)
\]

or
generalized green's function

\[ \sum_{j=0}^{n} b_j(c) f^{(j)}(c) = \sum_{l=0}^{2n-1} \sum_{j=0}^{2n-1} F_{c,j}(\tau \tau^*) [\eta_j^{(l)}(c) - \eta_{j+1}^{(l)}(c)] f^{(l)}(c) \]

for all \( f \in H_0^{2n}[a, b] \). By carefully choosing the function \( f \in H_0^{2n}[a, b] \) it is possible to assign any value we wish to the quantities \( f(c), f'(c), \ldots, f^{(2n-1)}(c) \). In particular, we can make one of them 1 and all the others 0, and hence, from (4) we conclude that

\[ \sum_{j=0}^{2n-1} F_{c,j}(\tau \tau^*) [\eta_j^{(l)}(c) - \eta_{j+1}^{(l)}(c)] = b_j(c), \quad l = 0, 1, \ldots, n, \]

\[ \sum_{j=0}^{2n-1} F_{c,j}(\tau \tau^*) [\eta_j^{(l)}(c) - \eta_{j+1}^{(l)}(c)] = 0, \quad l = n + 1, \ldots, 2n - 1. \]

Since the \( 2n \times 2n \) matrix \( [F_{c,j}(\tau \tau^*)] \) is nonsingular, equations (5) and (6) are uniquely solvable for the quantities \( \eta_j^{(l)}(c) - \eta_{j+1}^{(l)}(c), j = 0, 1, \ldots, 2n - 1, \) in terms of the quantities \( F_{c,j}(\tau \tau^*) \) and \( b_j(c) \).

For each \( t \in [a, b] \) consider the \( 2n \times 2n \) linear system

\[ \sum_{j=0}^{2n-1} F_{t,j}(\tau \tau^*) \eta_j(t) = b_j(t), \quad l = 0, 1, \ldots, n, \]

\[ \sum_{j=0}^{2n-1} F_{t,j}(\tau \tau^*) \eta_j(t) = 0, \quad l = n + 1, \ldots, 2n - 1, \]

for the \( 2n \) unknowns \( \eta_0(t), \eta_1(t), \ldots, \eta_{2n-1}(t) \). From Cramer's rule it follows that (7)–(8) is uniquely solvable for the quantities \( \eta_0(t), \eta_1(t), \ldots, \eta_{2n-1}(t) \) for each \( t \in [a, b] \), and as functions of \( t \) these quantities each belong to \( C^\infty[a, b] \).

In terms of these functions we can rewrite (4) as

\[ \tau^* f(c) = \sum_{j,l=0}^{2n-1} F_{c,j}(\tau \tau^*) f^{(l)}(c) \eta_j(c) \]

for all \( f \in H_0^{2n}[a, b] \) and for all \( c \) with \( a < c < b \). If we take any function \( f \in H_0^{2n}[a, b] \), then we can find a function \( \tilde{f} \in H_0^{2n}[a, b] \) with \( \tilde{f}^{(j)}(c) = f^{(j)}(c) \) for \( j = 0, 1, \ldots, 2n - 1, \) and hence, we see that (9) is valid for all \( f \in H_0^{2n}[a, b] \) and for all \( a < c < b \). Finally, letting \( c \to a \) and \( c \to b \), we conclude that

\[ \tau^* f(t) = \sum_{j,l=0}^{2n-1} F_{t,j}(\tau \tau^*) f^{(l)}(t) \eta_j(t) \]

for all \( f \in H_0^{2n}[a, b] \) and for all \( t \in [a, b] \). Let us summarize these results as a theorem.
Theorem 3. For each \( t \in [a, b] \) the \( 2n \times 2n \) linear system (7)-(8) has a unique solution \( \eta_0(t), \eta_1(t), \ldots, \eta_{2n-1}(t) \). Moreover, these functions belong to \( C^\infty[a, b] \), and the following properties hold:

(a) \( \tau^* \) for all \( f \in H^2[a, b] \) and for all \( t \in [a, b] \).

(b) For each point \( c \) with \( a < c < b \),

\[
\lim_{s \to c^-} \frac{\partial^j K(c, s)}{\partial s^j} - \lim_{s \to c^+} \frac{\partial^j K(c, s)}{\partial s^j} = \eta_j(c),
\]

for \( j = 0, 1, \ldots, 2n - 1 \).

Fix a point \( c \) with \( a < c < b \), and let us consider the quantities

\[
\eta_j(c) = \eta_j(c) - \eta_j(c), \quad j = 0, 1, \ldots, 2n - 1,
\]

which are determined by equations (7)-(8) with \( t = c \). We know that

\[
F(t) = 0 \quad \text{for} \quad j + l > 2n - 1,
\]

and

\[
F_{c}^{l2n-l-1}(t) = (-1)^{2n-l-1} c_{2n}(c) = (-1)^{n-l-1} b_n(c)^2
\]

for \( l = 0, 1, \ldots, 2n - 1 \), and hence, (7)-(8) is a triangular system. From equation (8) we obtain

\[
\eta_0(c) = \eta_1(c) = \cdots = \eta_{n-2}(c) = 0,
\]

and substituting this into the last equation in (7) yields

\[
\eta_{n-1}(c) = -b_n(c)^{-1} = (-1)^{n-1} a_n(c)^{-1}.
\]

In equations (11) and (12) we have obtained the well-known jump conditions satisfied by the standard Green's function.

At this point it is possible to determine the quantities \( \eta_n(c), \eta_{n+1}(c), \ldots, \eta_{2n-1}(c) \) from the first \( n \) equations in (7) and from (11) and (12). Proceeding in this way leads to an \( n \times n \) triangular system which can be solved for these quantities in terms of the quantities \( F(t)^j(t) \) and \( b_j(c) \). By expressing the boundary form \( F(t)^j(t) \) in terms of \( F(t)^j(t) \) and \( F(t)^j(t) \), it is possible to derive formulas for \( \eta_n(c), \eta_{n+1}(c), \ldots, \eta_{2n-1}(c) \) in terms of \( b_0(c), b_1(c), \ldots, b_n(c) \) and their derivatives. Instead of following this approach, we are going to reexamine the above discussion and derive a new set of equations for the quantities \( \eta_n(c), \eta_{n+1}(c), \ldots, \eta_{2n-1}(c) \). This new method is much simpler, and the final system of equations is easily solved.

Lemma 2. Let \( c \) be a point with \( a < c < b \), and let \( \eta_-(s) = K(c, s) \) for \( a \leq s \leq c \) and \( \eta_+(s) = K(c, s) \) for \( c \leq s \leq b \). Then
GENERALIZED GREEN’S FUNCTION

\[ F_c(\tau)(f, \eta_-) - F_c(\tau)(f, \eta_+) = f(\tau) \]

and

\[ F_c(\tau^*)(\eta_-, f) - F_c(\tau^*)(\eta_+, f) = -f(\tau) \]

for all \( f \in H^n[a, b] \).

**Proof.** Take any \( f \in H^n[a, b] \). Using (11) and (12) we have

\[ W(\tau, T^-_\alpha) - W(\tau, T^+\alpha) = 2 \sum_{j,l=0}^{n-1} F_c^{(l)}\tau f(\eta_j(c)) = 2 F_c^{-x}(T_\alpha f(\eta_j(c)) - x(c) \eta_j(c) \eta_{n-1}(c)) \]

\[ = (-1)^{n-1} a_n(c) f(c) (-1)^{n-1} a_n(c)^{-1} = f(c). \]

The second equation is established similarly. Q.E.D.

Now proceeding as above, for each \( f \in H_0^{2n}[a, b] \) we have

\[ \tau^* f(c) = \int_c^a \eta_-(s) \tau^* f(s) ds + \int_c^b \eta_+(s) \tau^* f(s) ds \]

\[ = \int_c^a \tau^* \eta_-(s) \tau f(s) ds + F_c(\tau)(\tau^* f, \eta_-) \bigg|_a^c \]

\[ + \int_c^b \tau^* \eta_+(s) \tau f(s) ds + F_c(\tau)(\tau^* f, \eta_+) \bigg|_c^b \]

\[ = F_c(\tau)(\tau^* f, \eta_-) - F_c(\tau)(\tau^* f, \eta_+) \]

\[ + F_c(\tau)(f, \tau^* \eta_-) - F_c(\tau)(f, \tau^* \eta_+). \]

But by Lemma 2

\[ \tau^* f(c) = F_c(\tau)(\tau^* f, \eta_-) - F_c(\tau)(\tau^* f, \eta_+), \]

and hence, comparing these last two equations we conclude that

(13) \[ F_c(\tau^*)(f, \tau^* \eta_-) - F_c(\tau^*)(f, \tau^* \eta_+) = 0 \]

for all \( f \in H_0^{2n}[a, b] \).

We can rewrite (13) as

\[ \sum_{j,l=0}^{n-1} F_c^{(l)}(\tau^*) f(\eta_j(c)) \left[ \left( \frac{d}{ds} \right)^j [\tau^* \eta_-(s)]_{s=c} - \left( \frac{d}{ds} \right)^j [\tau^* \eta_+(s)]_{s=c} \right] = 0 \]

for all \( f \in H_0^{2n}[a, b] \), which implies that
for \( l = 0, 1, \ldots, n - 1 \). From this \( n \times n \) triangular system we see that

\[
\left( \frac{d}{ds} \right)^j [\tau^* \eta_-(s)]_{s=c} - \left( \frac{d}{ds} \right)^j [\tau^* \eta_+(s)]_{s=c} = 0, \quad j = 0, 1, \ldots, n - 1.
\]

**Remark 1.** Equation (14) implies that the function \( \tau^* \mathcal{K}(c, \cdot) \) has a removable singularity at \( s = c \) and that by assigning the right value we can consider \( \tau^* \mathcal{K}(c, \cdot) \) as a function in \( H^n[a, b] \). We will examine this function in §10, relating it to the projection operator \( P \).

Now by Leibniz’s rule

\[
\left( \frac{d}{ds} \right)^j [\tau^* \eta_-(s)] = \sum_{i=0}^{n} \sum_{j=0}^{i} \binom{j}{i} b^i_{(j-i)}(s) \eta_{i+1}(s) = \sum_{i=0}^{n} \sum_{p=i}^{i+j} \binom{j}{j-p+i} b_{(j-p+i)}(s) \eta_{i+p}(s),
\]

and hence, equation (14) can be rewritten as

\[
\sum_{i=0}^{n} \sum_{p=i}^{i+j} \binom{j}{j-p+i} b_{(j-p+i)}(s) \eta_{i+p}(c) = 0, \quad j = 0, 1, \ldots, n - 1.
\]

In this last double sum the terms corresponding to \( p = 0, 1, \ldots, n - 2 \) are all zero, so (15) reduces to

\[
\sum_{i=0}^{n} \binom{j}{n-i-1} b_{(j-n+i+1)}(c) \eta_{n-i}(c) + \sum_{p=n}^{n+j} \left[ \sum_{i=p-j}^{n} \binom{j}{p-i} b_{(j-p+i)}(c) \eta_{i+p}(c) \right] = 0
\]

for \( j = 0, 1, \ldots, n - 1 \). Finally, using (12) we obtain the equation

\[
\eta_{n+j}(c) = \sum_{i=n-j-1}^{n-1} \binom{j}{n-i-1} b_{(j-n+i+1)}(c) b_n(c)^{-2}
\]

for \( j = 0, 1, \ldots, n - 1 \) (in case \( j = 0 \) the second sum does not occur). In this equation the quantity \( \eta_{n+j}(c) \) is given in terms of the preceding ones
\( \eta_n(c), \eta_{n+1}(c), \ldots, \eta_{n+j-1}(c) \) and the quantities \( b_0(c), b_1(c), \ldots, b_n(c) \) and their derivatives. These equations are easily solved to yield the first few quantities, e.g.,

\[(17) \quad \eta_n(c) = b_{n-1}(c)b_n(c)^{-2} \]

and

\[(18) \quad \eta_{n+1}(c) = b_{n-2}(c)b_n(c)^{-2} + b'_{n-1}(c)b_n(c)^{-2} - b_{n-1}(c)^2b_n(c)^{-3} - b_{n-1}(c)b'_n(c)b_n(c)^{-3}. \]

**Remark 2.** In equations (11), (12), and (16) we have formulas for the quantities \( \eta_0(c), \eta_1(c), \ldots, \eta_{2n-1}(c) \) which are valid for each \( c \) with \( a < c < b \). By the continuity of these functions on \([a,b]\), it follows that these formulas are valid for each \( c \in [a,b] \). Similarly, equation (15) holds for each \( c \in [a,b] \).

4. **Quasi-derivatives.** The jump conditions can also be interpreted in terms of quasi-derivatives (see [11, Part II, pp. 48–50]). Since \( \tau \tau^* \) is formally selfadjoint, it can be written in the form

\[ \tau \tau^* = \sum_{i=0}^{n} (-1)^i \left( \frac{d}{dt} \right)^i A_i(t) \left( \frac{d}{dt} \right)^i A_i(t) \]

where \( A_n(t) = (-1)^n c_{2n}(t) = b_n(t)^2 \). For any function \( f \in H^{2n}[a,b] \), the quasi-derivatives of \( f \) relative to \( \tau \tau^* \) are defined by

\[ f^{[i]}(t) = f^{(i)}(t), \quad i = 0, 1, \ldots, n-1, \]

\[ f^{[n]}(t) = A_n(t)f^{(n)}(t), \]

and

\[ f^{[n+1]}(t) = A_{n-1}(t)f^{(n-1)}(t) - \frac{d}{dt}(f^{[n+1]}(t)), \quad i = 1, \ldots, n. \]

With these definitions \( \tau \tau^* f = f^{[2n]} \), and Green's formula becomes

\[ \int_a^b \tau \tau^* f(s)g(t)\,dt = \int_a^b f(t)\tau \tau^* g(t)\,dt + \sum_{i=1}^{n} \left\{ f^{[i+1]}(t)g^{[2n-i]}(t) - f^{[2n-i]}(t)g^{[i+1]}(t) \right\}_{t=a}^{t=b} \]

for all \( f, g \in H^{2n}[a,b] \).

Fix a point \( c \) with \( a < c < b \). We already know that


\[ qM(c) - rW(c) = 0 \quad \text{for } i = 0, 1, \ldots, n - 2. \]

Proceeding as in the last section, for each \( f \in H^2[a, b] \) we have

\[
\sum_{i=0}^{n} b_i(c) f^{(i)}(c) = \int_a^c \eta_-(s) f^*(s) \, ds + \int_c^b \eta_+(s) f^*(s) \, ds
\]

\[
= \sum_{i=1}^{n} \{ f^{[i-1]}(s) \eta_-^{[2n-i]}(s) - f^{[2n-i]}(s) \eta_-^{[i-1]}(s) \}_{s=a}^{s=c}
\]

\[
+ \sum_{i=1}^{n} \{ f^{[i-1]}(s) \eta_+^{[2n-i]}(s) - f^{[2n-i]}(s) \eta_+^{[i-1]}(s) \}_{s=c}^{s=b}
\]

\[
= \sum_{i=1}^{n} f^{(i-1)}(c) \{ \eta_-^{[2n-i]}(c) - \eta_+^{[2n-i]}(c) \}
\]

\[
- b_n(c)^2 f^{(n)}(c) \{ \eta_-^{[n-1]}(c) - \eta_+^{[n-1]}(c) \},
\]

which implies that

\[ \eta_-^{[n-1]}(c) - \eta_+^{[n-1]}(c) = -b_n(c)^{-1} \]

and

\[ \eta_-^{[n+j]}(c) - \eta_+^{[n+j]}(c) = b_{n-j-1}(c) \quad \text{for } j = 0, 1, \ldots, n - 1. \]

In equations (19), (20), and (21) we have formulas for the jumps in the quasi-derivatives at \( s = c \). They are much simpler than the analogous formulas for the jumps in the derivatives.

5. Boundary conditions. Recall that the selfadjoint differential operator \( LL^* \) is determined by the boundary conditions \( B_i^*(f) = 0, i = 1, \ldots, 2n - k \), and \( B_j^+(f) = 0, j = 1, \ldots, k \) (see equation (2)).

**Theorem 4.** For each \( c \) with \( a < c < b \) the function \( K(c, \cdot) \) satisfies the boundary conditions \( B_i^*(f) = 0, i = 1, \ldots, 2n - k \), and \( B_j^+(f) = B_j(f^*) = 0, j = 1, \ldots, k \).

**Proof.** Fix \( c \) with \( a < c < b \), and choose a function \( g \) in \( H^2[a, b] \) which coincides with \( K(c, \cdot) \) in neighborhoods of both \( a \) and \( b \). Then for any \( f \in \mathcal{D}(LL^*) \) it follows by Green’s formula that

\[
\int_a^b \tau^* f(s)[g(s) - K(c,s)] \, ds = \int_a^b f(s)\tau f(s) \, ds - F_i(\tau^*)(f, \eta_- - \eta_+)
\]

\[
= \int_a^b f(s)\tau f(s) \, ds - \tau f(c) \quad \text{by Theorem 3(a)}
\]

\[
= \int_a^b f(s)\tau f(s) \, ds - \int_a^b K(c,s)\tau f(s) \, ds \quad \text{by Lemma 1},
\]

and hence,
The representation theory. Let $\phi_1, \ldots, \phi_{2n}$ and $\psi_1, \ldots, \psi_{2n}$ be functions in $C^\infty[a, b]$ which form bases for the solution spaces of $\tau^*\phi = 0$ and $\tau^*\psi = 0$, respectively. Setting $p = \dim \mathcal{H}(L)$ and $q = \dim \mathcal{H}(L^*)$, it is well known [11, Part I, p. 12] that $n - k = p - q$. Also, with no loss of generality we can assume that $\mathcal{H}(L^*) = \langle \phi_1, \ldots, \phi_q \rangle$ and that $\psi_1, \ldots, \psi_n$ form a basis for the solution space of $\tau\psi = 0$.

We are going to represent the generalized Green's function $K(t, s)$ in terms of the functions $\psi_i(t), \phi_j(s)$. Before doing this we need to take a closer look at the boundary conditions $\tau^*(f) = 0, j = 1, \ldots, k$.

Let $r$ be the rank of the $k \times n$ matrix having entries $B_j(\psi_i), i = 1, \ldots, k; j = 1, \ldots, n$. Clearly $\psi = \sum_{j=1}^n x_j \psi_j$ belongs to $\mathcal{H}(L)$ iff

$$\sum_{j=1}^n x_j B_j(\psi_j) = 0, \quad i = 1, \ldots, k.$$  

and hence, the mapping $(x_1, \ldots, x_n) \rightarrow \sum_{j=1}^n x_j \psi_j$ is an isomorphism between the solution space of (22) and $\mathcal{H}(L)$, so $p = n - r$.

Let

$$\mathfrak{B} = \{ B \in \langle B_1, \ldots, B_k \rangle | B(\psi_i) = 0 \text{ for } i = 1, \ldots, n \}.$$  

We assert that $\dim \mathfrak{B} = q$. Note that $B = \sum_{i=1}^k y_i B_i \in \mathfrak{B}$ iff

$$\sum_{i=1}^k y_i B_i(\psi_j) = 0, \quad j = 1, \ldots, n,$$  

and it follows that the mapping $(y_1, \ldots, y_k) \rightarrow \sum_{i=1}^k y_i B_i$ is an isomorphism between the solution space of (23) and $\mathfrak{B}$, and hence,

$$\dim \mathfrak{B} = k - r = k - (n - p) = q.$$  

Consequently, with no loss of generality we can assume that $\mathfrak{B} = \langle B_1, \ldots, B_q \rangle$, and this implies that

$$B_i(f) = 0 \quad \text{for all } f \in \langle \psi_1, \ldots, \psi_n \rangle, i = 1, \ldots, q.$$  

Fix a point $c$ with $a < c < b$, and let us examine the function $K(c, \cdot)$. We know it can be represented in the form
\[ K(c, s) = \sum_{i=1}^{2n} \alpha_i(c)\phi_i(s), \quad a \leq s < c, \]

(25)

\[ K(c, s) = \sum_{i=1}^{2n} \beta_i(c)\phi_i(s), \quad c < s \leq b. \]

We want to show that the \(4n\) constants \(\alpha_i(c), \beta_i(c)\) can be calculated from (i) the \(2n\) jump conditions (see Theorem 3), (ii) the \(2n\) boundary conditions (see Theorem 4), and (iii) the \(q\) orthogonality conditions (see Theorem 1). This makes a total of \(4n + q\) conditions for the \(4n\) constants. At first glance it seems reasonable to discard the \(q\) orthogonality conditions. However, these conditions turn out to be necessary, and we will actually show that \(q\) of the boundary conditions \(B_j^+(f) = 0, j = 1, \ldots, k\), can be deleted.

Suppose \(\alpha_1(c), \ldots, \alpha_{2n}(c), \beta_1(c), \ldots, \beta_{2n}(c)\) are constants, and let

\[ \eta_-(s) = \sum_{i=1}^{2n} \alpha_i(c)\phi_i(s), \quad a \leq s \leq c, \]

\[ \eta_+(s) = \sum_{i=1}^{2n} \beta_i(c)\phi_i(s), \quad c \leq s \leq b. \]

We want to determine conditions on these constants that will make \(\eta_-\) and \(\eta_+\) coincide with \(K(c, \cdot)\) on the intervals \([a, c)\) and \((c, b]\), respectively. A first necessary condition is that the jump conditions be satisfied, i.e.,

(26) \[ \sum_{i=1}^{2n} \alpha_i(c)\phi_i(c) - \sum_{i=1}^{2n} \beta_i(c)\phi_i(c) = \eta_j(c), \quad j = 0, 1, \ldots, 2n - 1. \]

Assume the constants \(\alpha_i(c), \beta_i(c)\) satisfy (26). Then

\[ \eta_-^{(j)}(c) - \eta_+^{(j)}(c) = \eta_j(c) \quad \text{for } j = 0, 1, \ldots, 2n - 1, \]

and hence, using Leibniz’s rule and (15) we obtain

\[
\left( \frac{d}{ds} \right)^j [\tau^* \eta_-](s)_{s=c} - \left( \frac{d}{ds} \right)^j [\tau^* \eta_+](s)_{s=c} = \sum_{i=0}^{n} \sum_{l=0}^{i+j} \binom{j}{l-i} b^{(j-l+i)}(c) \eta_l(c) = 0
\]

for \( j = 0, 1, \ldots, n - 1 \). Setting \( \phi(s) = \eta_-(s) \) for \( a \leq s < c \) and \( \phi(s) = \eta_+(s) \) for \( c < s \leq b \), the above discussion shows that the function \(\tau^* \phi\) has a removable singularity at \(s = c\). By assigning the right value at \(s = c\) we get \(\tau^* \phi \in \langle \psi_1, \ldots, \psi_n \rangle\), and hence, from (24) we conclude that

\[ B_j^+(\phi) = B_j(\tau^* \phi) = 0 \quad \text{for } j = 1, \ldots, q. \]
Let us summarize these results as a lemma.

**Lemma 3.** Let $c$ be a point with $a < c < b$, let $\alpha_1(c), \ldots, \alpha_{2n}(c), \beta_1(c), \ldots, \beta_{2n}(c)$ be constants satisfying (26), and let

$$
\phi(s) = \sum_{i=1}^{2n} \alpha_i(c) \phi_i(s), \quad a < s < c,
$$

$$
\phi(s) = \sum_{i=1}^{2n} \beta_i(c) \phi_i(s), \quad c < s < b.
$$

Then $\tau^* \phi$ has a removable singularity at $s = c$, $\tau^* \phi \in \langle \psi_1, \ldots, \psi_n \rangle$, and

$$
B_j^{+}(\phi) = B_j(\tau^* \phi) = 0 \quad \text{for} \quad j = 1, \ldots, q.
$$

A second necessary condition is that the boundary conditions in Theorem 4 be satisfied. Let

$$
B_i^{*} = C_i^{*} + D_i^{*} \quad \text{for} \quad i = 1, \ldots, 2n - k,
$$

and let

$$
B_i^{+} = C_i^{+} + D_i^{+} \quad \text{for} \quad i = 1, \ldots, k,
$$

where $C_i^{*}$ and $C_i^{+}$ are boundary values at $a$ and $D_i^{*}$ and $D_i^{+}$ are boundary values at $b$. Then the boundary conditions become

$$
\sum_{i=1}^{2n} \alpha_i(c) C_j^{*}(\phi_i) + \sum_{i=1}^{2n} \beta_i(c) D_j^{*}(\phi_i) = 0, \quad j = 1, \ldots, 2n - k,
$$

and

$$
\sum_{i=1}^{2n} \alpha_i(c) C_j^{+}(\phi_i) + \sum_{i=1}^{2n} \beta_i(c) D_j^{+}(\phi_i) = 0, \quad j = q + 1, \ldots, k.
$$

Notice that in (28) we have omitted the boundary conditions $B_j^{+}(f) = 0, j = 1, \ldots, q$, because of Lemma 3.

Finally, a third necessary condition is the orthogonality condition of Theorem 1:

$$
\sum_{i=1}^{2n} \alpha_i(c) \int_a^c \phi_i(s) \phi_j(s) \, ds + \sum_{i=1}^{2n} \beta_i(c) \int_c^b \phi_i(s) \phi_j(s) \, ds = 0,
$$

\[ j = 1, \ldots, q, \]

where we are assuming that $\mathcal{R}(L^*) = \langle \phi_1, \ldots, \phi_q \rangle$. In equations (26)–(29) we have obtained a $4n \times 4n$ linear system for the $4n$ unknowns $\alpha_i(c), \beta_i(c)$. The
The existence of the generalized Green’s function $K(c, \cdot)$ guarantees this system has a solution.

**Theorem 5.** For each point $c$ with $a < c < b$ the $4n \times 4n$ linear system (26)-(29) has a unique solution $\alpha_1(c), \ldots, \alpha_{2n}(c), \beta_1(c), \ldots, \beta_{2n}(c)$. Moreover, in terms of these quantities the generalized Green’s function $K(c, \cdot)$ is given by

$$K(c, s) = \frac{2n}{i=1} \alpha_i(c) \phi_i(s), \quad a < s < c,$$

$$K(c, s) = \frac{2n}{i=1} \beta_i(c) \phi_i(s), \quad c < s < b.$$

**Proof.** Suppose $\alpha^0_1, \beta^0_1$ is a solution to the homogeneous system associated with (26)-(29). Let

$$\phi(s) = \frac{2n}{i=1} \alpha^0_i \phi_i(s), \quad a < s < c,$$

$$\phi(s) = \frac{2n}{i=1} \beta^0_i \phi_i(s), \quad c < s < b.$$

From (26) we see that $\phi \in H^{2n}[a, b]$, and clearly $\tau^* \phi(s) = 0$ for all $s \in [a, b]$. Also, by (24), (27), and (28) we have $\phi \in \mathfrak{U}(LL^*)$, which implies that $\phi \in \mathfrak{U}(LL^*) = \mathfrak{U}(L^*)$. On the other hand from (29) we obtain $\phi \in \mathfrak{U}(L^*)^\perp$. We conclude that $\phi(s) = 0$ on $[a, b]$, so $\alpha^0_i = \beta^0_i = 0$ for $i = 1, \ldots, 2n$. Q.E.D.

**7. The end point theory.** Up to this point we have been restricting our attention to the case $a < c < b$. Note that the linear system (26)-(29) makes perfectly good sense in case $c = a$ or $c = b$, and the coefficients in this system are infinitely differentiable functions of $c$ on the closed interval $[a, b]$.

**Theorem 6.** For each $c \in [a, b]$ the $4n \times 4n$ linear system (26)-(29) has a unique solution $\alpha_1(c), \ldots, \alpha_{2n}(c), \beta_1(c), \ldots, \beta_{2n}(c)$. Moreover, these quantities are infinitely differentiable functions of $c$ on the interval $[a, b]$.

**Proof.** Consider the system (26)-(29) in the special case $c = a$. Suppose $\alpha^0_1, \beta^0_1$ is a solution of the associated homogeneous system, and set

$$\phi(s) = \frac{2n}{i=1} \beta^0_i \phi_i(s), \quad a < s < b.$$

Clearly $\phi \in C^\infty[a, b]$ and $\tau^* \phi(s) = 0$ on $[a, b]$. Now
\[ B_j^*(\phi) = C_j^*(\phi) + D_j^*(\phi) \]
\[ = \sum_{i=0}^{n-1} \alpha_i^j \left[ \sum_{i=1}^{2n} \beta_i^0 \phi_i^{(i)}(a) \right] + \sum_{i=1}^{2n} \beta_i^0 D_j^*(\phi_i) \]
\[ = \sum_{i=1}^{2n} \alpha_i^0 C_j^*(\phi_i) + \sum_{i=1}^{2n} \beta_i^0 D_j^*(\phi_i) \text{ by (26)} \]
\[ = 0 \text{ by (27) for } j = 1, \ldots, 2n - k. \]

By (24) we have \( B_j^*(\phi) = 0 \) for \( j = 1, \ldots, q \), while for \( j = q + 1, \ldots, k \) the above argument yields
\[ B_j^+(\phi) = C_j^+(\phi) + D_j^+(\phi) = 0 \text{ by (26) and (28).} \]

Therefore, \( \phi \in \mathcal{R}(L^*) = \mathcal{R}(L^*)^\perp \). Since \( \phi \in \mathcal{R}(L^*)^\perp \text{ by (29), we conclude that } \phi(s) = 0 \text{ on } [a, b]. \) This certainly implies that \( \beta_i^0 = 0 \) for \( i = 1, \ldots, 2n. \) Note that (26) becomes
\[ \sum_{i=1}^{2n} \alpha_i^0 \phi_i^{(j)}(a) = 0, \quad j = 0, 1, \ldots, 2n - 1. \]

In this \( 2n \times 2n \) linear system the determinant of the coefficient matrix is the Wronskian \( W(\phi_1, \ldots, \phi_{2n}) \) evaluated at \( a \), and hence, \( \alpha_i^0 = 0 \) for \( i = 1, \ldots, 2n. \)

We have shown that the linear system (26)-(29) is uniquely solvable for \( c = a \), and similarly, we have unique solvability for \( c = b \). By Cramer’s rule the \( \alpha_i(c) \) and \( \beta_i(c) \) are infinitely differentiable functions of \( c \) on \( [a, b] \). Q.E.D.

We are now in a position to be able to characterize the functions \( K(a, \cdot) \) and \( K(b, \cdot) \).

**Theorem 7.** The function \( K(a, \cdot) \) satisfies the boundary conditions \( B_i^*(f) = \alpha_{i,n-1} b_a(a)^{-1}, i = 1, \ldots, 2n - k, \) and \( B_j^+(f) = 0, j = 1, \ldots, k, \) and
\[ K(a,s) = \sum_{i=1}^{2n} \beta_i(a)\phi_i(s) \text{ for } a \leq s \leq b. \]

The function \( K(b, \cdot) \) satisfies the boundary conditions \( B_i^*(f) = -\beta_i^{*n-1} b_b(b)^{-1}, i = 1, \ldots, 2n - k, \) and \( B_j^+(f) = 0, j = 1, \ldots, k, \) and
\[ K(b,s) = \sum_{i=1}^{2n} \alpha_i(b)\phi_i(s) \text{ for } a \leq s \leq b. \]

**Proof.** Let \( \phi(s) = \sum_{i=1}^{2n} \beta_i(a)\phi_i(s), a \leq s \leq b. \) Clearly \( \phi \in C^\infty[a, b] \) and \( \tau \tau^*\phi(s) = 0 \) on \( [a, b]. \) We want to show that \( \phi(s) = K(a, s) \) for all \( s \in [a, b] \) and check the boundary conditions. For \( j = 1, \ldots, 2n - k \) we have
\[ B_j^+(\phi) = \sum_{i=0}^{n-1} \alpha_j^+ \left[ \sum_{i=1}^{2n} \alpha_i(a)\phi_i^{(l)}(a) - \eta_i(a) \right] + \sum_{i=1}^{2n} \beta_i(a)D_j^*(\phi_i) \quad \text{by (26)} \]

\[ = \sum_{i=1}^{2n} \alpha_i(a)C_j^*(\phi_i) + \sum_{i=1}^{2n} \beta_i(a)D_j^*(\phi_i) + \alpha_{j,n-1} b_n(a)^{-1} \quad \text{by Remark 2} \]

for \( j = 1, \ldots, q \) we have

\[ B_j^+(\phi) = B_j(\tau^* \phi) = 0 \quad \text{by (24)}; \]

and for \( j = q + 1, \ldots, k \) we have

\[ B_j^+(\phi) = \sum_{i=0}^{2n-1} \alpha_j^+ \left[ \sum_{i=1}^{2n} \beta_i(a)\phi_i^{(l)}(a) \right] + \sum_{i=1}^{2n} \beta_i(a)D_j^*(\phi_i) \]

\[ = - \sum_{i=0}^{2n-1} \alpha_j^+ \eta_i(a) \quad \text{by (26) and (28)} \]

\[ = - \sum_{i=0}^{n-1} \alpha_j \sum_{i=0}^{n} \sum_{m=1}^{i} \left( \frac{i}{m-l} \right) b_i^{(l-m+l)}(a)\eta_m(a) \quad \text{by (2)} \]

\[ = 0 \quad \text{by (15) and Remark 2}. \]

Finally, let us show that \( \phi = K(a, \cdot) \). We know that \( \phi \in \mathcal{R}(L) \) by (29) and \( K(a, \cdot) \in \mathcal{R}(L) \) by Theorem 1. Take any \( f \in \mathcal{D}(L^*) \) and set \( g = L^* f = \tau^* f \). Clearly \( f \in \mathcal{D}(L^*) \) and \( g = \tau^* f \in \mathcal{D}(L) \cap \mathcal{R}(L)^\perp \), and by the above \( \tau^* \phi \in \mathcal{R}(L) \). Thus,

\[ \int_a^b \phi(s)\tau^* f(s) \, ds = \int_a^b \tau^* \phi(s)\tau^* f(s) \, ds + F_b(\tau)(g, \phi) \bigg|_a^b \]

\[ = F_b(\tau)(g, \phi) - F_a(\tau)(g, \phi). \]

Choose a function \( \psi \in H^n[a, b] \) such that \( \psi^{(j)}(a) = \psi^{(j)}(b) = 0 \) for \( j = 0, 1, \ldots, n-1 \), except \( \psi^{(n-1)}(a) = b_n(a)^{-1} \). Then

\[ \phi - \psi \in H^n[a, b] \quad \text{and} \quad B_j^*(\phi - \psi) = 0 \quad \text{for} \quad i = 1, \ldots, 2n-k, \]

so \( \phi - \psi \in \mathcal{D}(L^*) \). Thus, we can rewrite the above equation as

\[ \int_a^b \phi(s)\tau^* f(s) ds = F_b(\tau)(g, \phi - \psi) + F_b(\tau)(g, \psi) - F_a(\tau)(g, \phi - \psi) \]

\[ - F_a(\tau)(g, \psi) \]

\[ = - F_a(\tau)(g, \psi) = - \sum_{i=0}^{n-1} F_a^{(l,n-1)}(\tau)g^{(l)}(a)b_n(a)^{-1} \]

\[ = \tau^* f(a) = \int_a^b K(a,s)\tau^* f(s) \, ds \quad \text{by Lemma 1}, \]

or
\[
\int_a^b [\phi(s) - K(a,s)]\tau^* f(s) ds = 0 \quad \text{for all } f \in \mathcal{D}(LL^*).
\]

Since \( \mathcal{R}(LL^*) = \mathcal{R}(L) \), we must have \( \phi(s) = K(a,s) \) for \( a \leq s \leq b \). This same argument can be used at the end point \( b \) to study the function \( K(b, \cdot) \).

Q.E.D.

Combining Theorems 5, 6, and 7 we obtain the following result.

**Theorem 8.** In terms of the functions \( \alpha_i(t), \beta_i(t), i = 1, \ldots, 2n \), which belong to \( C^\infty[a,b] \), the generalized Green's function \( K(t,s) \) has the representation

\[
K(t,s) = \sum_{i=1}^{2n} \alpha_i(t)\phi_i(s), \quad a \leq s < t < b,
\]

\[
K(t,s) = \sum_{i=1}^{2n} \beta_i(t)\phi_i(s), \quad a < t \leq s \leq b.
\]

Moreover, \( K(t,s) \) is infinitely differentiable in both variables for \( t \neq s \).

8. The adjoint relationship. In a series of theorems we have characterized the function \( K(t, \cdot) \), with the differential operators \( L^* \) and \( LL^* \) playing a major role. We now want to study the function \( K(\cdot, s) \), and we expect the differential operators \( L \) and \( L^* L \) to play an analogous role. Indeed, this will turn out to be the situation.

If we set

\[
H_1 = [L^* \mathcal{D}(L^*)]^{-1},
\]

then

\[
L^* H_1 f = f \quad \text{for all } f \in \mathcal{R}(L^*),
\]

\[
H_1 L^* f = f - Qf \quad \text{for all } f \in \mathcal{R}(L^*),
\]

and the generalized inverse of \( L^* \) is the operator \((L^*)^\dagger : S \to S\) given by

\[
(L^*)^\dagger f = H_1 (I - P)f, \quad f \in S.
\]

**Theorem 9.** \((L^*)^\dagger = (L^\dagger)^*\)

**Proof.** Take \( f \in S \) and \( g \in S \), and set \( u = L^\dagger f \) and \( v = (L^*)^\dagger g \). Clearly \( u \in \mathcal{D}(L) \cap \mathcal{R}(L^\dagger), \) \( v \in \mathcal{D}(L^*) \cap \mathcal{R}(L^*)^\perp, \) and \( Lu = f - Qf \) and \( L^* v = g - Pg \) by property (iii) of the generalized inverse. Thus,

\[
(L^\dagger f, g) = (u, L^* v + Pg) = (Lu, v) = (Lu + Qf, v) = (f, (L^*)^\dagger g).
\]

This proves that \((L^*)^\dagger = (L^\dagger)^*\). Q.E.D.
Proceeding as in §2, the generalized Green's function for $L^*$ is the unique function $K_1(t, s)$ determined by the equation

$$ (L^*)^* f(t) = H(t - P) f(t) = \int_a^b K_1(t, s) f(s) ds, $$

which is valid for all $f \in S$ and for all $t \in [a, b]$. Theorems 1–8 can now be applied to establish the basic properties of $K_1(t, s)$. We will discuss these properties after we relate $K_1(t, s)$ to $K(t, s)$.

**Theorem 10.** $K(t, s) = K_1(s, t)$ for all $t \neq s$.

**Proof.** Take functions $f \in S$ and $g \in S$. We know that $K(t, s)$ and $K_1(s, t)$ are very smooth functions for $t \neq s$, and both are bounded on the square $[a, b] \times [a, b]$. Thus, by Fubini's theorem

$$ \int_a^b \int_a^b K(t, s) f(s) g(t) ds dt = \int_a^b f(t) g(t) dt. $$

Or

$$ \int_a^b \int_a^b K_1(t, s) f(s) g(t) ds dt = \int_a^b \int_a^b K_1(s, t) f(t) g(s) ds dt. $$

Since the set of functions $f(s)g(t)$ with $f, g \in S$ is fundamental in $L^2([a, b] \times [a, b])$, we conclude that $K(t, s) = K_1(s, t)$ a.e. on $[a, b] \times [a, b]$. But these functions are continuous for $t \neq s$, so $K(t, s) = K_1(s, t)$ for all $t \neq s$. Q.E.D.

If we fix a point $s \in [a, b]$ and consider $K_1(s, \cdot)$ as a function of the variable $t$, then all our earlier results can be restated for $K_1(s, \cdot)$. In view of Theorem 10 each of these results can be interpreted as a statement about $K(t, s)$ as a function of $t$. Let us summarize our earlier work, stating it in terms of the function $K(t, s)$.

**Theorem 1'.** $K(t, s) \in \mathcal{H}(L) = \mathcal{R}(L^*)$ for all $s \in [a, b]$.

**Theorem 2'.** For each $c$ with $a < c < b$, $K(t, c) \in C^\infty[a, c]$ with $\tau^* \tau K(t, c) = 0$ for $a < t < c$, and $K(t, c) \in C^\infty[c, b]$ with $\tau^* \tau K(t, c) = 0$ for $c < t < b$ (the left-hand and the right-hand derivatives are not required to be equal at $t = c$). Moreover, $K(t, a) \in C^\infty[a, b]$ and $K(t, b) \in C^\infty[a, b]$ with $\tau^* \tau K(t, a) = 0$ and $\tau^* \tau K(t, b) = 0$ for all $t \in [a, b]$.

For each $s \in [a, b]$ we form the $2n \times 2n$ linear system
(31) \[ \sum_{j=0}^{2n-1} F_s^{(j)}(s) \phi_j(s) = a_j(s), \quad l = 0, 1, \ldots, n, \]

(32) \[ \sum_{j=0}^{2n-1} F_s^{(j)}(s) \phi_j(s) = 0, \quad l = n + 1, \ldots, 2n - 1, \]

for the \(2n\) unknowns \(v_0(s), v_1(s), \ldots, v_{2n-1}(s)\).

**Theorem 3'.** For each \(s \in [a,b]\) the \(2n \times 2n\) linear system (31)–(32) has a unique solution \(v_0(s), v_1(s), \ldots, v_{2n-1}(s)\). Moreover, these functions belong to \(C^\infty[a,b]\), and the following properties hold:

(a) \( \tau f(s) = \sum_{j=0}^{2n-1} F_s^{(j)}(s) f^{(j)}(s) \phi_j(s) \) for all \(f \in H^{2n}[a,b]\) and for all \(s \in [a,b]\).

(b) For each point \(c\) with \(a < c < b\),

\[ \lim_{t \to c^-} \frac{\partial^j K}{\partial t^j}(t,c) + \lim_{t \to c^+} \frac{\partial^j K}{\partial t^j}(t,c) = \phi_j(c), \]

for \(j = 0, 1, \ldots, 2n - 1\).

The functions \(v_i(s), i = 0, 1, \ldots, n - 1\), are given by

(33) \[ v_0(s) = v_1(s) = \cdots = v_{n-2}(s) = 0 \]

and

(34) \[ v_{n-1}(s) = -a_n(s)^{-1} = (-1)^{n-1} b_n(s)^{-1}, \]

while the functions \(v_i(s), i = n, n + 1, \ldots, 2n - 1\), are given by

\[ v_{n+j}(s) = \sum_{i=n-j}^{n-1} \left( \begin{array}{c} j \\ n - i - 1 \end{array} \right) a_i(j-n+i+1)(s)a_n(s)^{-2} \]

(35) \[ - \sum_{p=n}^{n+j-1} \left[ \sum_{i=p-j}^{n} \left( \begin{array}{c} j \\ i - p \end{array} \right) a_i(j-p+i)(s)a_n(s)^{-1} \right] \phi_p(s) \]

for \(j = 0, 1, \ldots, n - 1\) (in case \(j = 0\) the second sum does not occur). Explicit formulas can be given for \(v_n(s)\) and \(v_{n+1}(s)\), and the jump conditions for \(K(\cdot,c)\) in case \(a < c < b\) can also be characterized in terms of quasi-derivatives (see equations (19)–(21)).

**Theorem 4'.** For each \(c\) with \(a < c < b\) the function \(K(\cdot,c)\) satisfies the boundary conditions \(B_i(f) = 0, i = 1, \ldots, k\), and \(B_{j}^{*+}(f) = B_{j}^{*}(\tau f) = 0, j = 1, \ldots, 2n - k\).

**Theorem 7'.** The function \(K(\cdot,a)\) satisfies the boundary conditions \(B_i(f) = a_i, a_{n-1}a_n(a)^{-1}, i = 1, \ldots, k, \) and \(B_{j}^{*+}(f) = B_{j}^{*}(\tau f) = 0, j = 1, \ldots, 2n\).
The function $K(\cdot, b)$ satisfies the boundary conditions $B_i(f) = -\beta_{i,n-1} a_n(b)^{-1}$, $i = 1, \ldots, k$, and $B_j^{*}\tau(f) = B_j^{*}(\tau f) = 0, j = 1, \ldots, 2n - k$.

9. A representation theorem. In our next theorem we obtain a representation of $K(t, s)$ in terms of the functions $\phi_1, \ldots, \phi_{2n}$ and $\psi_1, \ldots, \psi_{2n}$, which form bases for the solution spaces of $\tau^*\phi = 0$ and $\tau^*\psi = 0$, respectively.

**Theorem 11.** There exist unique $2n \times 2n$ scalar matrices $\Gamma = [\gamma_{ij}]$ and $\Gamma' = [\gamma'_{ij}]$ such that

$$K(t, s) = \sum_{i,j=1}^{2n} \gamma_{ij} \psi_i(t) \phi_j(s), \quad a < s < t < b,$$

$$K(t, s) = \sum_{i,j=1}^{2n} \gamma'_{ij} \psi_i(t) \phi_j(s), \quad a < t < s < b.$$ 

**Proof.** By Theorem 8 we have

$$K(t, s) = \sum_{j=1}^{2n} \alpha_j(t) \phi_j(s), \quad a < s < t < b,$$

$$K(t, s) = \sum_{j=1}^{2n} \beta_j(t) \phi_j(s), \quad a < t < s < b,$$

where the $\alpha_j(t), \beta_j(t)$ belong to $C^\infty[a, b]$. If we fix $s \in [a, b]$ in these equations and apply $\tau^*\tau$ to the resulting functions of $t$, then

$$0 = \sum_{j=1}^{2n} \tau^* \tau \alpha_j(t) \phi_j(s), \quad a < s < t < b,$$

$$0 = \sum_{j=1}^{2n} \tau^* \tau \beta_j(t) \phi_j(s), \quad a < t < s < b.$$ 

Fixing $t \in (a, b)$ and considering these last equations as functions of $s$ on the intervals $a < s < t$ and $t < s < b$, by the independence of the functions $\phi_1, \ldots, \phi_{2n}$ we get

$$\tau^* \tau \alpha_j(t) = \tau^* \tau \beta_j(t) = 0 \quad \text{for } j = 1, \ldots, 2n.$$ 

From the continuity we conclude that these equations hold for all $t \in [a, b]$, and consequently, there exist unique $2n \times 2n$ scalar matrices $\Gamma = [\gamma_{ij}]$ and $\Gamma' = [\gamma'_{ij}]$ such that
$\alpha_j(t) = \sum_{i=1}^{2n} \gamma_{ij} \psi_i(t), \quad a \leq t \leq b,$

(**)

$\beta_j(t) = \sum_{i=1}^{2n} \gamma_{ij} \psi_i(t), \quad a \leq t \leq b.$

Combine (*) and (**) . Q.E.D.

10. The projections on the null spaces. In this section we relate the generalized Green’s function $K(t,s)$ to the two projection operators $P$ and $Q$. Suppose the functions $\psi_1, \ldots, \psi_p$ and $\phi_1, \ldots, \phi_q$ form orthonormal bases for $\mathcal{H}(L)$ and $\mathcal{H}(L^*)$, respectively. Then

(36) $Pf(t) = \int_a^b \left[ \sum_{i=1}^{p} \psi_i(t)\psi_i(s) \right] f(s)\,ds \quad \text{for all } t \in [a,b], f \in S,$

and

(37) $Qf(s) = \int_a^b \left[ \sum_{i=1}^{q} \phi_i(t)\phi_i(s) \right] f(t)\,dt \quad \text{for all } s \in [a,b], f \in S.$

**Lemma 4.** For each $c \in [a,b]$ the function $\tau^* K(c,s)$ has a removable singularity at $s = c$ and $\tau^* K(c,s) \in \mathcal{H}(L)$.

**Proof.** In case $a < c < b$ the proof is contained in equation (14) and Remark 1, Theorem 4, and Theorem 2. For $c = a$ or $c = b$ we use Theorem 2 and Theorem 7. Q.E.D.

**Theorem 12.** For each $c \in [a,b],$

$\tau^* K(c,s) = - \sum_{i=1}^{p} \psi_i(c)\psi_i(s) \quad \text{for all } s \in [a,b], s \neq c.$

**Proof.** Fix a point $c$ with $a < c < b$, and take any function $f \in S$. Setting $u = (I - P)f \in \mathcal{H}(L)^\perp$ and $v = Pf \in \mathcal{H}(L)$, we have

$\int_a^b \tau^* K(c,s)f(s)\,ds = \int_a^b \tau^* K(c,s)u(s)\,ds + \int_a^b \tau^* K(c,s)v(s)\,ds$

$= \int_a^b \tau^* K(c,s)v(s)\,ds \quad \text{by Lemma 4}$

$= F_c(\tau^*)(\eta_-, v) - F_c(\tau^*)(\eta_+, v) \quad \text{by Theorem 4}$

$= - Pf(c) \quad \text{by Lemma 2}.$

Comparing this to (36) we conclude that
\[ \tau^* K(c,s) = -\sum_{i=1}^{q} \psi_i(c)\psi_i(s) \text{ for all } s \in [a,b], \quad s \neq c. \]

The proof is completed by using the continuity in the \( c = t \) variable. Q.E.D.

We conclude with the analogue of Theorem 12 when the roles of the \( t \) and \( s \) variables are interchanged.

**Theorem 12'** For each \( c \in [a,b] \),

\[ \tau K(t,c) = \sum_{i=1}^{q} \phi_i(t)\phi_i(c) \text{ for all } t \in [a,b], \quad t \neq c. \]

**Proof.** We can use the proof of Theorem 12, or we can combine Theorem 10 and Theorem 12. Q.E.D.

11. **Examples.** We conclude our discussion by presenting two examples of generalized Green’s functions. The actual construction of these functions was carried out using a method developed in [7]. This method requires having bases \( \phi_1, \ldots, \phi_n \) and \( \psi_1, \ldots, \psi_n \) for the solution spaces of \( \tau^* \phi = 0 \) and \( \tau \psi = 0 \), respectively (see Theorem 2 of [7, p. 203] with \( m = q \)).

**Example 1.** In \( S = L^2[0,2\pi] \) consider the differential operator \( L \) given by

\[ \mathcal{D}(L) = \{ f \in H^2[0,2\pi] \mid f(0) - f(2\pi) = 0, f'(0) - f'(2\pi) = 0 \}, \]

\[ Lf = -f''. \]

Clearly \( L \) is determined by the formal differential operator \( \tau \) having coefficients \( a_0(t) = a_1(t) = 0 \) and \( a_2(t) = -1 \) and by the boundary conditions \( B_1(f) = f(0) - f(2\pi) = 0, B_2(f) = f'(0) - f'(2\pi) = 0 \). Also, it is well known that \( L^* = L \), so \( LL^* \) is the differential operator determined by \( \tau \tau^* f = f^{(4)} \) and the four boundary conditions

\[ B_1^+(f) = B_1(f) = 0, \]
\[ B_2^+(f) = B_2(f) = 0, \]
\[ B_1^+(f) = -f''(0) + f''(2\pi) = 0, \]
\[ B_2^+(f) = -f''''(0) + f''''(2\pi) = 0. \]

Proceeding as in [7], the generalized Green’s function for \( L \) is given by

\[ K(t,s) = \frac{1}{4\pi} s^2 + \frac{1}{2} s - \frac{1}{2} st + \frac{1}{4\pi} t^2 + \frac{\pi}{6}, \quad 0 \leq s \leq t \leq 2\pi, \]

\[ K(t,s) = \frac{1}{4\pi} t^2 + \frac{1}{2} t - \frac{1}{2} ts + \frac{1}{4\pi} s^2 + \frac{\pi}{6}, \quad 0 \leq t \leq s \leq 2\pi. \]
Notice that $K(t,s)$ is symmetric, which we expect in view of Theorem 10 and the selfadjointness of $L$. Also, for $0 < c < 2\pi$ the functions $(\partial^2 K/\partial s^2)(c,s)$ and $(\partial^3 K/\partial s^3)(c,s)$ have zero jumps at $s = c$, which agrees with the facts that $\eta_2(c) = 0$ and $\eta_3(c) = 0$ by equations (17) and (18). Finally, the function $K(0,s) = -(1/2)s + (1/4\pi)s^2 + \pi/6$ satisfies the boundary conditions

$$B_1^*(f) = \alpha_{11}^* b_2(0)^{-1} = 0,$$

$$B_2^*(f) = \alpha_{21}^* b_2(0)^{-1} = -1, \quad B_1^+(f) = B_2^+(f) = 0,$$

in agreement with Theorem 7.

**Example 2.** The differential operator $L$ defined by

$$\mathcal{D}(L) = \{ f \in H^2[0, 2\pi] \mid f(0) = 0 \}, \quad Lf = f'' + f,$$

is a nonselfadjoint operator in $S = L^2[0, 2\pi]$. Its adjoint is given by

$$\mathcal{D}(L^*) = \{ f \in H^2[0, 2\pi] \mid f(0) = f(2\pi) = f'(2\pi) = 0 \}, \quad L^*f = f'' + f,$$

and hence, the 4th order differential operator $LL^*$ is given by

$$\mathcal{D}(LL^*) = \{ f \in H^4[0, 2\pi] \mid f(0) = f(2\pi) = f'(2\pi) = f''(0) = f(0) = 0 \}, \quad Lf = f^{(4)} + 2f'' + f.$$

The generalized Green's function for $L$ is

$$K(t,s) = -\cos t\sin s - \frac{1}{2\pi} \sin t\sin s + \frac{s}{2\pi} \sin t\cos s, \quad 0 \leq s \leq t \leq 2\pi,$$

$$K(t,s) = -\sin t\cos s - \frac{1}{2\pi} \sin t\sin s + \frac{s}{2\pi} \sin t\cos s, \quad 0 \leq t \leq s \leq 2\pi.$$

It is easy to verify that $K(t,s)$ satisfies the various properties established in our theorems.

**References**


