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LIFTING IDEMPOTENTS AND EXCHANGE RINGS([1])

BY

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ABSTRACT. Idempotents can be lifted modulo a one-sided ideal L of a ring R if, given $x \in R$ with $x - x^2 \in L$, there exists an idempotent $e \in R$ such that $e - x \in L$. Rings in which idempotents can be lifted modulo every left (equivalently right) ideal are studied and are shown to coincide with the exchange rings of Warfield. Some results of Warfield are deduced and it is shown that a projective module P has the finite exchange property if and only if, whenever $P = N + M$ where N and M are submodules, there is a decomposition $P = A \oplus B$ with $A \subseteq N$ and $B \subseteq M$.

In 1972 Warfield showed that if M is a module over an associative ring R then M has the finite exchange property if and only if end M has the exchange property as a module over itself. He called these latter rings exchange rings and showed (using a deep theorem of Crawley and Jónsson) that every projective module over an exchange ring is a direct sum of cyclic submodules. Let $J(R)$ denote the Jacobson radical of R. Warfield showed that, if $R/J(R)$ is (von Neumann) regular and idempotents can be lifted modulo $J(R)$, then R is an exchange ring and so generalized theorems of Kaplansky and Müller.

The main purpose of this paper is to prove the following theorem: *A ring R is an exchange ring if and only if idempotents can be lifted modulo every left (respectively right) ideal.* The properties of these rings are examined in the first section and the theorem is proved in the second section. The theorems of Warfield are then easily deduced and a new condition that a projective module have the finite exchange property is given.

1. **Suitable rings.**

In this section, the rings of interest are defined, some of their properties are deduced, and several examples are given. All rings are assumed to be associative with identity and $J(R)$ denotes the Jacobson radical of a ring R.

1.1. PROPOSITION. *If R is a ring, the following conditions are equivalent for an element x of R.*

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(1) *There exists* $e^2 = e \in R$ *with* $e - x \in R(x - x^2)$.

(2) *There exists* $e^2 = e \in Rx$ *and* $c \in R$ *such that* $(1 - e) - c(1 - x) \in J(R)$.

(3) *There exists* $e^2 = e \in Rx$ *such that* $R = Re + R(1 - x)$.

(4) *There exists* $e^2 = e \in Rx$ *such that* $1 - e \in R(1 - x)$.

PROOF. (1) $\Rightarrow$ (2). If $e - x = r(x - x^2)$ then $1 - e = (1 - rx)(1 - x)$.

(2) $\Rightarrow$ (3) Given (2) it is clear that $e + c(1 - x)$ is a unit.

(3) $\Rightarrow$ (4) Given (3) write $1 = te + s(1 - x)$ and define $f = e + (1 - e)te$. Then $f^2 = f \in Rx$ and $1 - f = (1 - e)s(1 - x)$.

(4) $\Rightarrow$ (1) Given (4), $e - x = e(1 - x) - (1 - e)x \in R(x - x^2)$. □

1.2. DEFINITION. A ring is called *suitable* if each element satisfies the conditions in Proposition 1.1.

It would appear that these rings should be called *left* suitable as it is not clear that the definition is left-right symmetric. However, it will be proved below (Theorem 2.1) that every "left suitable" ring is "right suitable", and conversely. Thus the left-right analogs of all the results below are true.

If L is an additive subgroup of a ring R, we say idempotents can be *lifted* modulo L if, given $x \in R$ with $x - x^2 \in L$, there exists $e^2 = e \in R$ such that $e - x \in L$. Jacobson [3, p. 53] defines a condition on the radical of a ring he calls "suitable for building idempotents". Hence the next result explains our use of the term suitable.

1.3. COROLLARY. *A ring is suitable if and only if idempotents can be lifted modulo every left ideal.*

PROOF. Clear by (1) of Proposition 1.1. □

If L is a left ideal of a ring R the *idealizer* of L is $I(L) = \{a \in R \mid La \subseteq L\}$. It is well known that L is an ideal of $I(L)$ and $L/I(L) \cong \operatorname{end}_R (R/L)$. In these terms, the condition in Corollary 1.3 can be rephrased as follows: A ring R is suitable if and only if, for every ideal of R, idempotents can be lifted in $I(L)$ modulo L. (If the condition is satisfied use the fact that $x \in I(R(x - x^2))$ for each x.)

The next result is an immediate consequence of Proposition 1.1.

1.4. PROPOSITION. *Every homomorphic image of a suitable ring is suitable.*

1.5. PROPOSITION. *A ring R is suitable if and only if $R/J(R)$ is suitable and idempotents can be lifted modulo $J(R)$.*

PROOF. Assume the conditions and let $x \in R$. Write $\bar{x} = x + J(R)$ and $\bar{R} = R/J(R)$. There exists $\bar{a}^2 = \bar{a} \in \bar{R}\bar{x}$ and $\bar{c} \in \bar{R}$ such that

$$\bar{1} - \bar{a} = \bar{c}(\bar{1} - \bar{x}).$$

We may assume $a \in Rx$. Choose $f^2 = f$ such that $\bar{f} = \bar{a}$. Then $u = 1 - f + a$

is a unit in R and so $e = u^{-1}fu = u^{-1}fa$ satisfies $e^2 = e \in Rx$. Since $\bar{e} = \bar{f} = \bar{a}$ it follows that $(1 - e) - c(1 - x) \in J(R)$ so R is suitable by Proposition 1.1. The converse is immediate. $\square$

It should be noted that Proposition 1.5 remains true if $J(R)$ is replaced by any ideal $A \subseteq J(R)$. This result enables us to show that the class of suitable rings is quite large and, in fact, contains all semiperfect and all (von Neumann) regular rings. Call a ring R *semiregular* if $R/J(R)$ is regular and idempotents can be lifted modulo $J(R)$.

1.6. PROPOSITION. *Every semiregular ring is suitable.*

PROOF. We may assume R is regular by Proposition 1.5. If $x \in R$ choose $y \in R$ such that $xyx = x$ and write $f = yx$. If $e = f + (1 - f)xf$ then $e^2 = e \in Rx$ and $1 - e = (1 - f)(1 - x)$. $\square$

1.7. EXAMPLE. Let D denote a division ring and let S be a subring of D containing 1. Define

$$R(D, S) = \{(x_1, x_2, \ldots, x_n, s, s, s, \ldots) | n \geqslant 1, x_i \in D, s \in S\}.$$

Then R is a ring (with componentwise operations) and R is suitable if and only if the same is true of S. In fact, S is a homomorphic image of R while, if S is suitable, the same is true of R by a componentwise calculation. Furthermore, every nonzero left (or right) ideal of R contains a nonzero idempotent so $J(R) = 0$. $\square$

A ring R will be called *local* if $R/J(R)$ is a division ring. Such rings are suitable and the ring L of all rational fractions with odd denominators is local. Hence, if Q denotes the rational numbers, $R(Q, L)$ is a commutative suitable ring with zero Jacobson radical which is not regular (L is a homomorphic image). This means $R(Q, L)$ is suitable but not semiregular.

The next result provides another class of suitable rings and gives a characterization of suitable rings among rings with central idempotents. Call a ring *clean* if every element is the sum of a unit and an idempotent.

1.8. PROPOSITION. (1) *Every clean ring is suitable.*

(2) *A ring with central idempotents is clean if and only if it is suitable.*

PROOF. (1) If $x = e + u$ where $e^2 = e$ and u is a unit then

$$u[x - u^{-1}(1 - e)u] = ue + u^2 - u + eu = x^2 - x$$

and the result follows from condition (1) of Proposition 1.1.

(2) If R is suitable and $x \in R$ choose $e^2 = e \in Rx$ with $1 - e \in R(1 - x)$. If $e = ax$ we may assume $ea = a$ so that $axa = a$. If idempotents are central then $xa = x(ax)a = xa(ax) = (xa)ax = a(xa)x = ax$. Similarly write $1 - e = b(1 - x)$ where $(1 - e)b = b$ and $b(1 - x) = (1 - x)b$. Then an easy calculation shows that $a - b$ is the inverse of $x - (1 - e)$. $\square$

We note in passing that the proof of (2) shows that every central element in a suitable ring is the sum of a unit and an idempotent.

A ring is said to be *reduced* if it has no (nonzero) nilpotent elements. These rings have central idempotents. Hence if a ring is either commutative or reduced, it is suitable if and only if it is clean. In particular every *strongly regular* ring (regular and reduced) is clean. However, the ring $R(Q, L)$ described following Example 1.7 is a commutative, clean, reduced ring with zero Jacobson radical which is not strongly regular.

It is easily verified that every local ring is clean so there exist clean rings with nonzero nilpotent ideals. Also, the ring of all $n \times n$ matrices over an algebraically closed field is clean since every matrix is similar to a Jordan matrix. Moreover it is not difficult to show that a ring R is clean if and only if $R/J(R)$ is clean and idempotents can be lifted modulo $J(R)$. Hence the class of clean rings is quite large.

Call a ring R *potent* if idempotents can be lifted modulo $J(R)$ and every left (equivalently right) ideal not contained in $J(R)$ contains a nonzero idempotent. Potent rings for which $J(R)$ is a nil ideal have been variously called I-rings and Zorn rings.

1.9. PROPOSITION. *Every suitable ring is potent.*

PROOF. It suffices to show that there is a nonzero idempotent in Rx for each $x \notin J(R)$. Suppose $x \in R$ is such that $e^2 = e \in Rx$ implies $e = 0$. Given $a \in R$ choose $e^2 = e \in Rax$ such that $1 - e \in R(1 - ax)$. Then $e = 0$ and so $1 \in R(1 - ax)$. This means $x \in J(R)$. $\square$

If Z denotes the rational integers then, in the notation of Example 1.7, the ring $R(Q, Z)$ is a commutative potent ring with $J(R) = 0$ which is not suitable.

The next result is easily proved and will be referred to below.

1.10. PROPOSITION. *If R is suitable and $e^2 = e \in R$ the ring eRe is suitable.*

PROOF. If $x \in eRe$ choose $f^2 = f \in Rx$ such that $1 - f \in R(1 - x)$. Then $fe = f$ so $(ef)^2 = ef \in (eRe)$ and $e - ef = e(1 - f)e \in eRe(e - x)$. $\square$

Using this result, a natural extension of property (4) in Proposition 1.1 can be proved which leads to a result about lifting orthogonal idempotents modulo a left ideal.

1.11. PROPOSITION. *Let R be suitable and suppose $x_1 + x_2 + \cdots + x_n = 1$ in R. Then there exist orthogonal idempotents $e_1, \ldots, e_n$ such that $e_i \in Rx_i$ for each i and $e_1 + \cdots + e_n = 1$.*

PROOF. Assume $n \geqslant 2$ and proceed by induction. Given $x_1 + \cdots + x_{n+1} = 1$ choose an idempotent $f \in R(x_1 + \cdots + x_n)$ with $1 - f \in Rx_{n+1}$. Write

$f = r(x_1 + \cdots + x_n)$. Since fRf is suitable, choose (by induction) orthogonal idempotents $f_1, f_2, \ldots, f_n$ in fRf such that $f_i \in fRfrx_if$ for each i and $f_1 + \cdots + f_n = f$. For each i write $f_i = fr_ifrx_if$ and define $e_i = f_ir_ifrx_i$. Then

$$e_ie_j = (f_ir_ifrx_i)(f_jr_jfrx_j) = f_if_j(r_jfrx_j)$$

holds for each i and j so these e_i are orthogonal idempotents. Write $e = e_1 + \cdots + e_n$. Since $e_i \in Rx_i$, $1 \leqslant i \leqslant n$, it remains to prove $1 - e \in Rx_{n+1}$. But $e_if = f_i$ for each i and so $ef = f$. Consequently $1 - e = (1 - e)(1 - f) \in Rx_{n+1}$. $\square$

The lifting theorem for orthogonal idempotents is now easy.

1.12. Proposition. *Let R be suitable, let L be a left ideal of R, and let $x_1, \ldots, x_n$ be orthogonal idempotents modulo L; that is, $x_i \equiv x_i^2 \pmod L$ for each i and $x_ix_j \equiv 0 \pmod L$ for all $i \neq j$. Then there exist orthogonal idempotents $e_1, \ldots, e_n$ such that $e_i \in Rx_i$ and $e_i \equiv x_i \pmod L$ for each i.*

Proof. Write $x = x_1 + \cdots + x_n$ and, by Proposition 1.11 choose orthogonal idempotents $e_1, \ldots, e_n$ such that $e_i \in Rx_i$ for each i and

$$1 - e \in R(1 - x),$$

where we write $e = e_1 + \cdots + e_n$. For each i, the hypotheses imply that $xx_i \equiv x_i \pmod L$ and, since $e_i \in Rx_i$, that $ex_i \equiv e_i$. If $1 - e = r(1 - x)$ this gives

$$x_i - e_i \equiv (1 - e)x_i \equiv r(1 - x)x_i \equiv 0 \pmod L$$

for each i. This completes the proof. $\square$

It should be noted that, in the notation of Proposition 1.12, if $L \subseteq J(R)$ and $x_1 + \cdots + x_n \equiv 1 \pmod L$ then necessarily $e_1 + \cdots + e_n = 1$.

2. **Exchange rings.** A left R-module M is said to have the *exchange property* (see Crawley and Jónsson [2]) if for any module X and decompositions

$$X = M' \oplus Y = \bigoplus_{i \in I} N_i$$

where $M' \simeq M$, there exists submodules $N_i' \subseteq N_i$ for each i such that

$$X = M' \oplus (\oplus N_i').$$

If this condition holds for finite sets I (equivalently for $|I| = 2$) the module M is said to have the *finite exchange property*. This is equivalent to the full exchange property for finitely generated modules. Crawley and Jónsson [2, Theorem 7.1], proved that, if the modules N_i are all countably generated and have the exchange property, then any two direct sum decompositions of $\oplus_{i \in I} N_i$ have isomorphic refinements.

In 1969 Warfield [9] showed that an indecomposable module has the exchange property if and only if it has a local ring of endomorphisms. In 1972 he called a ring R an *exchange ring* if ${}_R R$ has the (finite) exchange property. He verified that the definition was left-right symmetric and that a module has the finite exchange property if and only if its endomorphism ring is an exchange ring [10, Theorem 2]. In addition, he proved that every semiregular ring is an exchange ring. In particular, this implies (using the Crawley-Jónsson theorem) that every projective module over a semiregular ring is a direct sum of cyclic submodules (generalizing results of Kaplansky [4] for local and regular rings and Müller [6] semiperfect rings). Later Monk [5] gave a ring-theoretic description of these exchange rings and showed that there exist exchange rings which are not semiregular.

For the purpose of the next theorem only, call a ring R *left suitable* if each element has the properties in Proposition 1.1 and define *right suitable* rings analogously. The theorem implies that these conditions are both equivalent to R being an exchange ring and will be used to deduce results of Warfield and Monk. If M is a left R-module its R-endomorphisms will be written on the right of their arguments.

2.1. THEOREM. *If R is a ring, the following conditions are equivalent for a left R-module M*:

(1) end *M is right suitable*,

(2) *M has the finite exchange property*,

(3) end *M is left suitable.*

PROOF. (1) $\Rightarrow$ (2). Suppose $X = M \oplus Y = N_1 \oplus N_2$ are R-modules. Write $E = \operatorname{end} X$ and choose $\pi^2 = \pi \in E$ with $M = X\pi$. Let τ_1 and τ_2 be orthogonal idempotents in E such that $\tau_1 + \tau_2 = 1$, $N_i = X\tau_i$, $N_1 = \ker \tau_2$ and $N_2 = \ker \tau_1$. Then $\pi = \pi\tau_1\pi + \pi\tau_2\pi$ so, since $\pi E \pi \cong \operatorname{end} M$ is right suitable, choose orthogonal idempotents $\nu_i \in \pi\tau_i\pi E\pi$ such that $\nu_1 + \nu_2 = \pi$. Write $\nu_i = \pi\tau_i\alpha_i$ where $\alpha_i \in \pi E \pi$ and where we assume $\alpha_i\nu_i = \alpha_i$. Now define $\eta_i = \tau_i\alpha_i\tau_i$ $(i = 1,2)$. These are orthogonal idempotents and $X\eta_i \subseteq N_i$ for $i = 1, 2$. Hence $N_i = X\eta_i \oplus N_i'$ where $N_i' = N_i \cap \ker \eta_i$ and so $X = (X\eta_1 \oplus X\eta_2) \oplus (N_1' \oplus N_2')$. We show $X = M \oplus N_1' \oplus N_2'$.

Note first that $\pi\eta_i = \nu_i\tau_i$ so that $\pi\eta_i\alpha_i = \nu_i\tau_i\alpha_i = \nu_i^2 = \nu_i$. Suppose now that $x \in M \cap (N_1' \oplus N_2')$. Then $x\eta_1 = 0 = x\eta_2$ and so $x = x\pi = \Sigma x\nu_i = \Sigma x\pi\eta_i\alpha_i = \Sigma x\eta_i\alpha_i = 0$. Next choose $x \in X$ and write $x = x_1 + x_2 + \omega$ where $x_i \in X\eta_i$ *and* $\omega \in N_1' \oplus N_2'$. We have

$$\eta_i\alpha_i\eta_j = (\eta_i\tau_i)(\alpha_i\pi)\eta_j = (\eta_i\tau_i)(\alpha_i\nu_i)(\nu_j\tau_j) = \delta_{ij}\eta_i$$

where $\delta_{ij} = 0$ or 1 according as $i \neq j$ or $i = j$. It follows that $x_i - x_i\alpha_i \in N_i'$ for each i and so

$$x - x_1\alpha_1 - x_2\alpha_2 = (x_1 - x_1\alpha_1) + (x_2 - x_2\alpha_2) + \omega \in N_1' \oplus N_2'.$$

Since $x_i\alpha_i \in M$ for each i this shows $X = M \oplus N_1' \oplus N_2'$.

(2) $\Rightarrow$ (3). Given (2) write $X = M \oplus M$ and write $N_1 = \{(x,0) | x \in M\}, N_2 = \{(0,x)| x \in M\}$ and $D = \{(x,x)| x \in M\}$. Suppose $\alpha, \beta \in \operatorname{end} M$ are such that $\alpha + \beta = 1$. If $M' = \{(x\alpha, -x\beta)| x \in M\}$ then $M' \cong M$ and $X = M' \oplus D = N_1 \oplus N_2$. Hence there exist $N_i' \subseteq N_i$ such that $X = M' \oplus N_1' \oplus N_2'$. If $x \in M$ this gives a unique decomposition

$$(*) \quad (x,x) = (y\alpha, -y\beta) + (x_1, 0) + (0, x_2)$$

where $(x_1, 0) \in N_1'$ and $(0, x_2) \in N_2'$. Define $\alpha', \beta' \in \operatorname{end} M$ by $x\alpha' = x_2$ and $x\beta' = x_1$. Then the decompositions

$$(x\alpha'\alpha, x\alpha'\alpha) = (x\alpha'\alpha, -x\alpha'\beta) + (0,0) + (0, x\alpha'),$$
$$(x\beta'\beta, x\beta'\beta) = (-x\beta'\alpha, x\beta'\beta) + (x\beta', 0) + (0,0)$$

show that $\alpha'\alpha\alpha' = \alpha'$ and $\beta'\beta\beta' = \beta'$ and so $\alpha'\alpha$ and $\beta'\beta$ are idempotents. Furthermore, equating components in $(*)$ yields $y = x(\alpha' - \beta')$ and so $x = y\alpha + x\beta' = x(\alpha'\alpha + \beta'\beta)$. Hence $\alpha'\alpha + \beta'\beta = 1_M$ and it follows that $\operatorname{end} M$ is left suitable.

(3) $\Rightarrow$ (1). Applying (1) $\Rightarrow$ (3) to $_RR$ shows right suitability implies left suitability for any ring. The converse for any ring is analogous and, in particular, it holds for $\operatorname{end} M$. $\square$

If R is any ring it is clear that $R \cong \operatorname{end}_R R$. Hence exchange rings and suitable rings are the same. The following is thus immediate:

2.2. Corollary (Warfield [10, Theorem 2]). *A module M has the finite exchange property if and only if* end *M is an exchange ring.*

Proposition 1.6 now gives an easy proof of another result of Warfield:

2.3. Corollary (Warfield [10, Theorem 3]). *Every semiregular ring has the exchange property.*

The next result was proved for commutative rings by Shutters [8] and is immediate from Proposition 1.5.

2.4. Corollary. *A ring R is an exchange ring if and only if $R/J(R)$ is an exchange ring and idempotents left modulo $J/(R)$.*

The original proof of the next result was different and more involved than the present one.

2.5. Corollary (Monk [5, Theorem 1]). *A module M has the exchange property if and only if, given $\alpha \in$* end *M there exists γ and σ in* end *M such that $\gamma\alpha\gamma = \gamma$ and $\sigma(1-\alpha)(1-\gamma\alpha) = 1 - \gamma\alpha$.*

Proof. Write $E = \operatorname{end} M$ and let $\alpha \in E$. If such γ, σ exist let $\tau = \gamma\alpha$.

Then $\tau^2 = \tau \in E\alpha$ and $(1-\tau)\sigma(1-\alpha)(1-\tau) = 1-\tau$. If

$$\pi = 1 - (1-\tau)\sigma(1-\alpha)$$

then $\pi^2 = \pi = \pi\tau \in E\alpha$ and $1 - \pi \in E(1-\alpha)$. Hence E is suitable. Conversely, let $\pi^2 = \pi \in E\alpha$ and $1-\pi \in E(1-\alpha)$. If $\pi = \beta\alpha$ and $1 - \pi = \sigma(1-\alpha)$ we are done with $\gamma = \beta\alpha\beta$. $\square$

If R is a suitable ring and $e^2 = e \in R$ it was shown in Proposition 1.10 that eRe is also suitable. In the light of Theorem 2.1 this shows immediately that if a module M has the finite exchange property the same is true of any direct summand of M [2, Lemma 3.10]). It is also shown in [2] that the direct sum of two modules with the finite exchange property has the same property. Hence:

2.6. Corollary. *If R is a ring and $e^2 = e \in R$ then R is suitable if and only if eRe and $(1-e)R(1-e)$ are both suitable.*

It would be of interest to see a direct ring-theoretic proof of this and of the fact that left and right suitability are equivalent.

The next result gives a new condition that a projective module has the finite exchange property. Two preliminary results in projective modules will be needed.

2.7. Lemma. *If P is a projective module and $P = M_1 + \cdots + M_n$ where the M_i are submodules, there exist $\alpha_i \in \operatorname{end} P$ such that $P\alpha_i \subseteq M_i$ for each i and $\alpha_1 + \cdots + \alpha_n = 1_P$.*

Proof. Let $M_i = \sum_{j \in J_i} Rx_{ij}$ for each i and let F be a free module on basis $\{y_{ij} \mid 1 \leqslant i \leqslant n, j \in J_i\}$. Define $\varphi\colon F \to P$ by setting $y_{ij}\varphi = x_{ij}$ for all i,j. Then φ is onto and so there exists $\psi\colon P \to F$ such that $\psi\varphi = 1_P$. Let $\pi_{ij}\colon F \to R$ denote the projections and write $\varphi_{ij} = \psi\pi_{ij}$ for all i, j. Then $\{\varphi_{ij}, x_{ij}\}$ is a dual basis for P and we are finished with $\alpha_i = \sum_{j \in J_i} \varphi_{ij}x_{ij}$ for each i. $\square$

2.8. Lemma. *Let P be a projective module and suppose $P = P_1 + N$ where P_1 is a direct summand of P and N is a submodule. Then there exists $P_2 \subseteq N$ such that $P = P_1 \oplus P_2$.*

Proof. Choose $\gamma^2 = \gamma \in \operatorname{end} P$ with $P\gamma = P_1$. If $\varphi\colon P \to P/N$ is the natural map let $\alpha \in \operatorname{end} P$ satisfy $\alpha\gamma\varphi = \varphi$. Define $\delta = \gamma + (1-\gamma)\alpha\gamma$. Then $\delta^2 = \delta$, $P\delta = P\gamma = P_1$ and $\ker\delta = M(1-\gamma)(1-\alpha\gamma) \subseteq N$. Take $P_2 = \ker\delta$. $\square$

$$\begin{tikzcd}
 & & P \arrow[dll, "\alpha"'] \arrow[d, "\varphi"] & \\
P \arrow[r, "\gamma"] & P_1 \arrow[r, "\varphi|_{P_1}"] & P/N \arrow[r] & 0
\end{tikzcd}$$

2.9. PROPOSITION. *The following conditions are equivalent for a projective module P.*

(1) *P has the finite exchange property.*

(2) *If $P = M_1 + \cdots + M_n$ where the M_i are submodules there is a decomposition $P = P_1 \oplus \cdots \oplus P_n$ with $P_i \subseteq M_i$ for each i.*

(3) *If $P = M + N$ where M and N are submodules there exists a summand P_1 of P such that $P_1 \subseteq M$ and $P = P_1 + N$.*

PROOF. (1) $\Rightarrow$ (2). If $P = M_1 + \cdots + M_n$ use Lemma 2.7 to find $\alpha_i \in \operatorname{end} P$ such that $P\alpha_i \subseteq M_i$ for each i and $\alpha_1 + \cdots + \alpha_n = 1_P$. By Proposition 1.11 there exist orthogonal idempotents $\pi_i \in (\operatorname{end} P)\alpha_i$ such that $\pi_1 + \cdots + \pi_n = 1_P$. Then (2) follows with $P_i = P\pi_i$.

(2) $\Rightarrow$ (3). Obvious.

(3) $\Rightarrow$ (1). Let $\alpha_1, \alpha_2 \in \operatorname{end} P$ be such that $\alpha_1 + \alpha_2 = 1$. Then $P = P\alpha_1 + P\alpha_2$ so, by (3) and Lemma 2.8, let $P = P_1 \oplus P_2$ where $P_i \subseteq P\alpha_i$ for each i. Let π_1, π_2 be idempotents in end P with $\pi_1 + \pi_2 = 1$ and $P\pi_i = P_i$. There exist $\beta_i \in \operatorname{end} P$ such that $\beta_i\alpha_i = \pi_i$. Hence end P is suitable and (1) follows by Theorem 2.1. □

A submodule K in a module M is said to be *small* in M if $K + N = M$ where N is a submodule implies $N = M$. An epimorphism $P \to M \to 0$ is called a *projective cover* of M if P is projective and the kernel is small in P. A projective module is said to be *semiperfect* if every homomorphic image has a projective cover.

2.10. COROLLARY. *Every semiperfect module has the finite exchange property.*

PROOF. If P is semiperfect and $P = M + N$, the fact that P/M has a projective cover means P decomposes as $P = P_1 \oplus P_2$ where $P_1 \subseteq M$ and $M \cap P_2$ is small in P [1, Lemma 2.3]. Since $P = P_1 + (M \cap P_2) + N$ it follows that $P = P_1 + N$ and the proof is complete. □

We note in passing that the endomorphism ring of a semiperfect module is, in fact, semiregular (see [7]).

A ring R is called *left perfect* if every left module has a projective cover. Hence, this result shows that, over a left perfect ring, every projective module has the finite exchange property. It appears to be an open question whether the converse is true. The next result collects some information on the subject. An ideal A is *left T-nilpotent* if, given $a_1, a_2, \ldots$ from A, $a_1 a_2 \cdots a_n = 0$ for some n.

2.11. PROPOSITION. *A ring R has the property that every projective left module has the finite exchange property if and only if the same is true of $R/J(R)$ and $J(R)$ is left T-nilpotent.*

PROOF. Assume R has this property and write $J(R) = J$. If F is any free R-module then rad $F = JF$ is small in F by Proposition 2.9 since it contains no nonzero direct summand. Consequently, J is left T-nilpotent. Now it suffices to show any free $R/J(R)$-module has the finite exchange property. Every such module has the form F/JF where F is a free R-module. If $\varphi: F \to F/JF$ is natural, suppose $F\varphi = N_1\varphi + N_2\varphi$ where $F = N_1 + N_2$. If $F = P_1 \oplus P_2$ where $P_i \subseteq N_i$ then $F\varphi = P_1\varphi \oplus P_2\varphi$ and $P_i\varphi \subseteq N_i\varphi$.

For the converse, we use the observation in [8] that every projective R/J-module P has an R-projective cover. Indeed P is a direct sum of cyclic modules [10, Theorem 1] and so, since idempotents lift modulo J, $P \cong \oplus_I(Re_i/Je_i)$ where each $e_i^2 = e_i \in R$. Then $\oplus_I Re_i$ is the required projective cover (since J is left T-nilpotent). Now suppose $F = N_1 + N_2$, where F is R-free and the N_i are submodules. If $\varphi: F \to F/JF$ is natural then $F\varphi = N_1\varphi + N_2\varphi$ so write $F\varphi = A_1 \oplus A_2$ where $A_i \subseteq N_i\varphi$ $(i = 1,2)$. If $Q_i \xrightarrow{\pi_i} A_i \to 0$ are R-projective covers, let $\alpha: Q_1 \oplus Q_2 \to F$ satisfy $\alpha\varphi = \pi_1 \oplus \pi_2$. Since JF is small in F it follows that α is an isomorphism (see [1, Lemma 2.3]) and so $F = Q_1\alpha \oplus Q_2\alpha$ where $Q_i\alpha\varphi = A_i \subseteq N_i\varphi$ for each i. Thus $Q_i\alpha \subseteq N_i + JF$ and so $F = N_1 + JF + Q_2\alpha$. This means $F = N_1 + Q_2\alpha$ (JF is small) and so, by Lemma 2.8, $F = P_1 \oplus Q_2\alpha$ where $P_1 \subseteq N_1$. But then $F = P_1 + N_2 + JF$ so $F = P_1 + N_2$. We are finished by Lemma 2.8. □

We conclude with the observation that the existence of a ring which is left perfect but not right perfect [1, p. 476] shows that the condition in Proposition 2.11 is not left-right symmetric.

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