GROWTH PROBLEMS FOR SUBHARMONIC FUNCTIONS OF FINITE ORDER IN SPACE

BY

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ABSTRACT. For a function u(x) subharmonic (or C^2) in R^m, we compare the "harmonics" (defined in §1) of u with those of a related subharmonic function whose total Riesz mass in |x| < r is the same as that of u, but whose L^2 norm on |x| = r is maximal, for all 0 < r < \infty. We deduce estimates on the growth of the Riesz mass of u in |x| < r, as r \to \infty.

Introduction. Following Hayman [7], [8], we study the growth and distribution of the Riesz mass of subharmonic functions in R^m (m \geq 2) from the point of view of classical value distribution theory. Thus, if u(x) is subharmonic we define the characteristic

\begin{equation}
T(r, u) = \sigma_m^{-1} \int_{|\omega|=1} u(\omega)^* \, d\omega
\end{equation}

of u(x) and its order

\begin{equation}
\lambda = \limsup_{r \to \infty} \frac{\log T(r, u)}{\log r};
\end{equation}

d\omega denotes (m - 1)-dimensional surface area on \Sigma = \Sigma_m = \{|x| = 1\} and \sigma_m = \int_\Sigma d\omega. We always suppose u^+ is unbounded: T(r, u) \to \infty when r \to \infty, and u is harmonic near 0 with u(0) = 0. We compare the growth of T(r, u) with that of

\begin{equation}
N(r) = N_u(r) = \sigma_m^{-1} \int_\Sigma u(\omega) \, d\omega
\end{equation}

which, by Jensen’s theorem [1, p. 133], is a weighted average of the Riesz mass of u in the ball |x| \leq r:

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(4) \[ n(r) = (\sigma_m d_m)^{-1} \int_{|x| \leq r} d(\Delta u(x)), \quad N(r) = d_m \int_0^r n(t)^{1-m} \, dt. \]

Here \( \Delta \) denotes the Laplacian, \( \Delta u \) exists as a distribution and \( \mu = (\sigma_m d_m)^{-1} \Delta u \) is a positive measure when \( u \) is subharmonic [1, p. 127]; and \( d_m = m - 2 \) for \( m > 2, d_2 = 1 \). (For definitions and a discussion of basic results, see §1.)

When \( f(z) \) is an entire function of one complex variable and \( u(x,y) = \log|f(x + iy)| \), \( n(r) \) counts the number of zeros of \( f(z) \) in \( |z| < r \), and it is a classical problem to find good lower bounds for

\[ k(u) = \limsup_{r \to \infty} \frac{N(r)}{T(r,u)} \]

in terms of \( \lambda \). For example, it is known in this case that

\[ k(u) \geq \begin{cases} 1 & (0 \leq \lambda \leq \frac{1}{2}), \\ \sin \pi \lambda & (\frac{1}{2} < \lambda \leq 1) \end{cases} \]

(Edrei and Fuchs [3]), where equality holds for \( f(z) = \) polynomial (\( \lambda = 0 \)), \( = e^z \) (\( \lambda = 1 \)) and

\[ f(z) = \prod_{n=1}^{\infty} \left(1 - z/n^{1/\lambda}\right) \quad (0 < \lambda < 1). \]

Hayman has extended (6) to arbitrary subharmonic \( u \) in the plane and found the sharp analogue for functions of orders \( \lambda < 1 \) in \( \mathbb{R}^m, m \geq 3 \) ([7], [8]).

For \( \lambda > 1 \), precise results are not in general available even for entire functions. A recent result in this direction is

\[ k(u) \geq (0.9) \left| \frac{\sin \pi \lambda}{\lambda + 1} \right| \quad (1 < \lambda < \infty) \]

(Miles and Shea [10]), and well-known examples [2] show that (8) would fail for large \( \lambda \) if the 0.9 factor were replaced by any constant greater than unity. Inequality (8) is an easy corollary of the main result of [10].

**Theorem A.** Let \( f(z) \) be an entire function of finite order \( \lambda \) in the plane, and put \( u(z) = \log|f(z)| \),

\[ m_2(r,u) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |u(re^{i\theta})|^2 \, d\theta \right\}^{1/2}. \]

Then

\[ \limsup_{r \to \infty} \frac{N(r)}{m_2(r,u)} \geq \frac{|\sin \pi \lambda|}{\pi \lambda} \left\{ \frac{2}{1 + (\sin 2\pi \lambda)/2\pi \lambda} \right\}^{1/2}. \]
Equality is possible in (10) for each \( \lambda \geq 0 \).

Our first purpose in this note is to find the appropriate extension of Theorem A to subharmonic functions. The proof in [10] rests on some simple properties of the Fourier coefficients

\[
c_k(r;f) = \frac{1}{2\pi} \int_0^{2\pi} \log|f(re^{i\theta})|e^{-ik\theta} d\theta \quad (k = 0, \pm 1, \pm 2, \ldots),
\]

in particular on the inequality

\[
|c_k(r;f)| \leq |c_k(r;f^*)| \quad (r > 0, k = 0, \pm 1, \pm 2, \ldots)
\]

where \( f^* \) is a suitable entire function whose zeros have the same moduli as those of \( f \) but are projected onto the positive real axis. Thus, if \( u^* = \log|f^*| \), then \( N_u(r) = N_{u^*}(r) \) and

\[
m_2(r,u) \leq m_2(r,u^*) \quad (0 < r < \infty)
\]

by Parseval's theorem, and to prove (10) it suffices to consider just the \( f^* \).

In §2, we study the spherical harmonics of subharmonic functions in \( \mathbb{R}^m \) and prove an analogue of (11) for all \( m \geq 2 \) (Theorem 2.1). From this we deduce

**Theorem 1.** Let \( u(x) \) be subharmonic and of finite order \( \lambda \) in \( \mathbb{R}^m \), and put

\[
m_2(r,u) = \left\{ a_m^{-1} \int_{\sum} |u(\omega)|^2 d\omega \right\}^{1/2}
\]

Then

\[
\limsup_{r \to \infty} \frac{N(r)}{m_2(r,u)} \geq C(\lambda, m) \quad (0 \leq \lambda < \infty, m \geq 2),
\]

where

\[
C(\lambda, m) = \left\{ 1 + \frac{\lambda^2(\lambda + m - 2)^2}{(m - 2)!} \sum_{k=1}^{\infty} \frac{(k + m - 3)! (2k + m - 2)}{k! (k - \lambda)^2(k + \lambda + m - 2)^2} \right\}^{-1/2}
\]

When \( m = 2 \), the bound in (13) is the same as that in (10), and when \( m = 3 \) or \( 4 \) inequality (13) remains sharp for all \( \lambda \), with
When \( m \geq 5 \) the series in (14) diverges and \( C(\lambda, m) = 0 \), which just reflects the fact that for these \( m \) the extremal functions for this problem (studied in §4) fail to be square-integrable on spheres \( |x| = r, 0 < r < \infty \).

By Schwarz's inequality and Jensen's theorem, \( m_2(r, u) \geq 2T(r, u) - N(r) \), and we deduce easily a bound for \( k(u) \) defined in (5):

**Corollary 1.** If \( u(x) \) is subharmonic

\[
k(u) \geq \frac{\sin \pi \lambda}{\pi \lambda (\lambda + 1)^{1/m - 1}} \quad (0 < \lambda < \infty; m = 2, 3, 4).
\]

In §4 we consider a class of examples which, we conjecture, minimize \( k(u) \) for any given order \( \lambda \) and dimension \( m \); in particular we show that there exist subharmonic functions \( u_{\lambda, m}(x) \) of order \( \lambda \) in \( \mathbb{R}^m \) with

\[
k(u_{\lambda, m}) \leq C_m \frac{\sin \pi \lambda}{(\lambda + 1)^{1/m}} \quad (1 < \lambda < \infty).
\]

Thus the bound in (15) has the right order of magnitude for large \( \lambda \).

Using other methods, we obtain

**Theorem 2.** If \( u(x) \) is subharmonic and of order \( \lambda \) in \( \mathbb{R}^m \), then

\[
k(u) \geq A_m \frac{\sin \pi \lambda}{(\lambda + 1)^{1/(m+1)}} \quad (0 < \lambda < \infty; m \geq 5)
\]

where \( A_m \) depends only on \( m \).

Hayman [8] has obtained \( k(u) \geq (q + 1 - \lambda)(\lambda - q)/(\lambda(q + 1)^{4m+q} \), with \( q = \lfloor \lambda \rfloor \), as a consequence of an inequality between \( N(r) \) and \( M(r, u) = \sup_{|x| = r} u(x) \). Using the Poisson formula to estimate \( M(r, u) \) in terms of \( T(\sigma r, u), \sigma > 1 \), we can easily adapt the proof of (17) to find that

\[
\lim_{r \to \infty} \sup_{r} \frac{N(r)}{M(r, u)} \geq B_m \frac{\sin \pi \lambda}{(\lambda + 1)^{1/m}} \quad (0 < \lambda < \infty, m \geq 2).
\]

The conjectured extremal functions \( u_{\lambda, m} \) mentioned above are harmonic in \( \mathbb{R}^m \) except on the positive \( x_1 \)-axis, along which the Riesz mass is distributed...
regularly: \( N_{u,m}(r) = r^\lambda \), and \( u_{\lambda,m}(x) = |x|^\lambda I(\cos \theta; \lambda, m) \) where \( \theta \) denotes the angle between the vector \( x \) and the positive \( x_1 \)-axis, and \( I \) is defined in §4. If we put

\[
K(\lambda, m) \overset{\text{def}}{=} k(u_{\lambda,m}) = T(1, u_{\lambda,m})^{-1} \quad (m \geq 2, 0 \leq \lambda < \infty)
\]

then Hayman’s sharp result noted earlier, for \( \lambda < 1 \) and \( m \geq 2 \), is: \( k(u) \geq K(\lambda, m) \), and our approximations (15) and (17) for \( \lambda > 1 \) have been compared with \( K(\lambda, m) \) via (16). Complementary to these lower bounds for \( k(u) \), when \( u \) is an arbitrary subharmonic function, is

**Theorem 3.** Let \( u(x) \) be subharmonic in \( \mathbb{R}^m \) of finite nonintegral order \( \lambda \) with all its Riesz mass distributed along a ray through 0. Then

\[
\liminf_{r \to \infty} \frac{N(r)}{T(r,u)} \leq K(\lambda, m)
\]

where besides (16) \( K(\lambda, m) \) satisfies

\[
K(\lambda, m) < 1 \quad (m \geq 3, 0 < \lambda < \infty)
\]

and

\[
K(\lambda, 2) = \frac{|\sin \pi \lambda|}{q + |\sin \pi \lambda|} \quad (q \leq \lambda < q + \frac{1}{2})
\]

\[
K(\lambda, 2) = \frac{|\sin \pi \lambda|}{q + 1} \quad (q + \frac{1}{2} \leq \lambda < 1)
\]

for \( q = 0, 1, 2, \ldots \).

Inequality (18) remains valid for integral orders \( \lambda \), but then requires different methods; cf. [15].

For entire functions in the plane this is due to Ostrovskiï [12]. There exist other related studies of this type, e.g. by Edrei and Fuchs [2], also [4], [5], [9].

All the results mentioned above for entire functions have extensions to meromorphic functions, provided the definitions of \( N(r) \) and \( T(r,u) \) are generalized in a natural way. If \( f \) is meromorphic in the plane and \( u(z) = \log |f(z)| = \log |g(z)| - \log |h(z)| \) where \( g, h \) are entire functions having no common zeros, we define \( \mu = \Delta u = \Delta \log |g| - \Delta \log |h| = \mu^+ - \mu^- \) where \( \mu^+ \) and \( \mu^- \) are positive measures,

\[
n(r,u) = \frac{1}{2\pi} \int_{|z|<r} d\mu^-(z), \quad n(r,-u) = \frac{1}{2\pi} \int_{|z|<r} d\mu^+(z),
\]

\[
N(r,u) = \int_{0}^{r} n(t,u)t^{-1} dt, \quad N(r) = N_u(r) = N(r,u) + N(r,-u),
\]

with
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\[ T(r, u) = \frac{1}{2\pi} \int_{0}^{2\pi} u(re^{i\theta})^+ d\theta + N(r, u), \quad k(u) = \limsup_{r \to \infty} \frac{N(r)}{T(r, u)}. \]

Thus \( k(u) \) gives a measure of the "total deviation from harmonicity" of \( u = \log|f| \). The Edrei-Fuchs inequality (6) remains valid in this more general setting [3], as does Theorem A [10].

We prove Theorems 1 and 2 for \( u(x) \) in the class \( \mathcal{D}_m \) of functions "delta-subharmonic" in \( \mathbb{R}^m \).

**Definition 1.** A function \( u \) defined (a.e.) in \( \mathbb{R}^m \) is in \( \mathcal{D}_m \) if there exist subharmonic functions \( u_1, u_2 \) in \( \mathbb{R}^m \) with \( u = u_1 - u_2 \).

A more intrinsic definition is: \( u \in \mathcal{D}_m \) if for every compact set \( F \), \( u \in \mathcal{D}(F) \) and

\[ \left| \int u(x) \Delta \varphi(x) \, dx \right| \leq C(F) \| \varphi \|_\infty \]

for some constant \( C(F) \) and every \( \varphi \in C^\infty(\mathbb{R}^m) \) vanishing outside of \( F \).

It is immediate from the second definition that any \( u \in C^2(\mathbb{R}^m) \) is delta-subharmonic. The equivalence of the two definitions and other basic facts needed here are discussed further in §1.

If \( f : \mathbb{C}^M \to \mathbb{C} \) is an entire function of order \( \lambda \), then Theorem 2 applies to \( u = \log|f| \) and yields

\[ \limsup_{r \to \infty} \frac{N(r, 0; f)}{T(r, f)} \geq A(M) \frac{\sin \pi\lambda}{\pi} \quad (0 < \lambda < \infty) \]

with \( c = M + \frac{1}{2} \) and \( N(r, 0; f) = N_u(r) \). Our examples \( u_{\lambda, 2M}(x) \) show that \( c > M \) for subharmonic functions in \( \mathbb{R}^{2M} \) generally, but it remains an interesting question whether (22) with \( c \approx M \) is a good estimate for entire functions when \( M \geq 2 \).

1. *Definitions and auxiliary results.* A function \( u : \mathbb{R}^m \to [-\infty, \infty) \) is subharmonic, \( u \in \mathcal{S}_m \), if \( u \) is upper semicontinuous, \( \neq -\infty \) and

\[ u(x) \leq \sigma_{m}^{-1} \int_{\Sigma} u(x + \delta \omega) \, d\omega \]

for all \( x \in \mathbb{R}^m \) and \( \delta > 0 \). It is well known [1, p. 128], [8], [14] that

(1.1) \[ u \in L^1(F) \quad \text{for every compact } F, \]

(1.2) \[ \Delta u \text{ exists as a distribution and } \mu = (\sigma_{m} d_{m})^{-1} \Delta u \]

is a positive Borel measure, finite for compact sets.

Further, Riesz's theorem holds: If
\[(1.3) \quad K(x) = \log|x| \quad (m = 2), \quad = -|x|^{2-m} \quad (m \geq 3),\]

then for any compact \(F\),

\[(1.4) \quad u(x) = \int_F K(x - y) \, d\mu(y) + h(x)\]

where \(\mu\) is the measure in (1.2) and \(h\) is harmonic in the interior of \(F\). Conversely, given a positive locally finite measure \(\mu\) on \(\mathbb{R}^m\), any \(u\) having the representation (1.4) for compact \(F\) and \(h\) harmonic in the interior of \(F\) is subharmonic in \(\mathbb{R}^m\) with \(\Delta u = \sigma_m d_m \mu\).

The measure \(\mu\) in (1.2) is termed the Riesz measure of \(u\).

Let \(u \in \mathfrak{D}_m\), so that \(u = u_1 - u_2\) where \(u_j \in \mathcal{S}_m\). Then it is clear that (21) holds with \(C(F) = \mu_1(F) + \mu_2(F)\) if \(\mu_j = \Delta u_j\) for \(j = 1, 2\). Conversely, suppose \(u \in L^1_{\text{loc}}\) satisfies (21). Then \(\Delta u\) is a (locally finite, signed) Borel measure \(\sigma_m d_m \mu\) [1, p. 93]. Let \(|\mu|\) be the total variation of \(\mu\), and let \(\mu^+ = \frac{1}{2}(|\mu| + \mu), \quad \mu^- = \frac{1}{2}(|\mu| - \mu)\). Then as in Weierstrass's classical theorem we can construct [8, Chapter 4] functions \(u^+, u^- \in \mathcal{S}_m\) with \(\Delta u^\pm = \sigma_m d_m u^\pm\) and \(u = u^+ - u^- + h\) where \(h\) is harmonic in \(\mathbb{R}^m\); thus \(u \in \mathfrak{D}_m\) according to Definition 1.

For convenience, we shall continue to refer to the measure defined in (1.2) as the Riesz measure of \(u\), for any \(u \in \mathfrak{D}_m\), and to the mass of the total variation measure \(|\mu| = \mu^+ + \mu^-\) as the Riesz mass of \(u\).

If \(u \in \mathfrak{D}_m\),

\[(1.5) \quad \mu = (\sigma_m d_m)^{-1} \Delta u = \mu^+ - \mu^-\]

and we assume throughout §§1–3 that \(\mu^+, \mu^-\) have no mass in a neighborhood of 0, that

\[(1.6) \quad u(0) = 0,\]

and that

\[(1.7) \quad T(r, u) \to \infty \quad (r \to \infty).\]

This involves no restriction for the kind of asymptotic problems studied here.

Generalizing definitions (1), (4) we put

\[
\begin{align*}
n(r, u) &= \mu^-([|x| \leq r]), \quad n(r, -u) = \mu^+([|x| \leq r]), \\
N(r, u) &= d_m \int_0^r n(t, u) t^{1-m} \, dt, \\
N(r) &= \int_0^r u^+(t) \, dt, \\
N(r, u) &= m^{-1} \int_\Sigma u^+(r \omega) \, d\omega + N(r, u), \\
T(r, u) &= \sigma_m^{-1} \int_\Sigma u^+(r \omega) \, d\omega + N(r, u),
\end{align*}
\]

and (2), (5) remain unchanged.
Applying Green’s formula to \( u_2 \), we have

\[
(1.9) \quad u_2(0) = c_1^{-1} \int_{\Sigma} u_2(r\omega) d\omega + \int_{|y|<r} [K(y) - K(re)| d\mu^\perp(y)
\]

where \( e = (1,0,\ldots,0) \), and integration by parts converts the last integral in (1.9) to \( N(r,u) \). Thus

\[
T(r,u) = c_1^{-1} \int_{\Sigma} [(u_1 - u_2)^+ + u_2](r\omega) d\omega - u_2(0)
\]

where \( v = \max(u_1, u_2) \in \mathbb{S}_m \), so that by (3) and (4), \( T(r,u) \) is a continuous, increasing function convex in \( \log r \) \( (m = 2), r^{2m} \) \( (m \geq 3) \).

Applying (1.9) to \( u \), we obtain the analogue for \( u \in \mathbb{S}_m \) of Nevanlinna’s first fundamental theorem,

\[
(1.10) \quad T(r,u) = T(r,-u) \quad (0 < r < \infty).
\]

If \( x, y \in \mathbb{R}^m \) we write

\[
x \vee y = x \cdot y / |x| |y| = \cos \theta
\]

where \( \theta \) is the angle between \( \overrightarrow{0x} \) and \( \overrightarrow{0y} \). Then

\[
K(x - y) = - \sum_{k=0}^{\infty} P_k(x \vee y) \frac{|x|^k}{|y|^{k+m-2}} \quad (|x| < |y|)
\]

(1.11)

\[
= - \sum_{k=0}^{\infty} P_k(x \vee y) \frac{|y|^k}{|x|^{k+m-2}} \quad (|y| < |x|)
\]

where the \( P_k \) are the Gegenbauer polynomials [16, pp. 302, 329]. On the other hand, for fixed \( y \), \( K(x - y) \) is real-analytic in \( x \) and thus \( P_k(x \vee y) \) \( \cdot |x|^k / |y|^{k+m-2} \) is the sum of terms of degree \( k \) in the Taylor expansion of \( K(x - y) \) in a neighborhood of the origin. Thus \( P_k(x \vee y)|x|^k \) is a homogeneous harmonic polynomial of degree \( k \) in \( x \) (except when \( m = 2, k = 0 \)), and [1, p. 169]

\[
(1.12) \quad \int_{\Sigma} P_j(r\omega \vee y) P_k(r\omega \vee z) d\omega = 0 \quad (j \neq k)
\]

for all \( r = |x| > 0 \) and \( y, z \in \mathbb{R}^m \setminus \{0\} \).

For any integer \( q \geq 0 \), we define

\[
(1.13) \quad K_q(x,y) = K(x - y) + \sum_{k=0}^{q} P_k(x \vee y) \frac{|x|^k}{|y|^{k+m-2}} \quad (x,y \in \mathbb{R}^m).
\]

Thus
\begin{equation}
K_q(x,y) = - \sum_{k=q+1}^{\infty} P_k(x \vee y) \frac{|x|^k}{|y|^{k+m-2}} \quad (|x| < |y|).
\end{equation}

Assume that \( u \in \mathcal{O}_m \) is of finite order \( \lambda \), so that by (1.8) and (1.10):
\[
\limsup_{r \to \infty} \frac{\log N(r)}{\log r} \leq \lambda,
\]
and let \( \mu \) be the associated Riesz measure. Then for \( \alpha > \lambda \),
\begin{equation}
\int_0^\infty \frac{N(r)}{r^{\alpha+1}} \, dr = \frac{d_m}{\alpha} \int_0^\infty \frac{n(t)}{t^{\alpha+m-1}} \, dt
= \frac{d_m}{\alpha(\alpha + m - 2)} \int_{\mathbb{R}^m} \frac{d|\mu|(x)}{|x|^{\alpha+m-2}}
\end{equation}
converges. Let \( q = q(\mu) \) denote the least integer \( \geq 0 \) for which
\begin{equation}
\int \frac{d|\mu|(x)}{|x|^{q+m-1}} < \infty,
\end{equation}
and put
\begin{equation}
u_\mu(x) = \int_{\mathbb{R}^m} K_q(x,y) \, d\mu(y).
\end{equation}

By (1.16), (1.13), (1.11) and (1.4), \( u_\mu \in \mathcal{O}_m \) and
\begin{equation}
u_\mu(\omega) \in L^1(\Sigma, d\omega) \quad (0 < \omega < \infty).
\end{equation}

For some purposes it is convenient to have explicit estimates of \( K_q \), and we state

**Lemma 1.1.** There exists a constant \( C = C(m,q) \) such that, if \( |x| = r \),
\[
|K_q(x,y)| \leq Cr^{q+1}/|y|^{q+m-1} \quad (r \leq \frac{1}{2} |y|),
\]
\[
K_q(x,y) \leq Cr^{q+1}/|y|^{q+m-2}(r + |y|) \quad (x,y \in \mathbb{R}^m),
\]
the latter except when \( m = 2 \) and \( q = 0 \), in which case
\[
K_0(x,y) = \log |1 - x/y| \leq \log(1 + r/|y|).
\]

When \( m = 2 \), Lemma 1.1 is well known [6, p. 26]; analogous estimates yield the result for \( m > 3 \), e.g. see [8].

Using Lemma 1.1 we find that if
then \( u_\mu \) has order \( \lambda_0 \), \( q \leq \lambda_0 \leq q + 1 \). Further, arguments like those used for the classical Hadamard representation theorem (worked out in Hayman's book \([8, \text{Chapter IV}]\)), give

**Lemma 1.2.** Let \( u \in \mathcal{Q}_m \) have finite order \( \lambda \), let \( q(u) \) be determined as in (1.16) and put \( g = \max(q, [\lambda]) \).

Then

\[
(1.20) \quad u(x) = u_\mu(x) + h(x)
\]

where \( h \) is a harmonic polynomial of degree at most \( g \).

Observe that \( g = q(\mu) \) when \( \lambda \) is not a positive integer.

Finally, we collect some facts about spherical harmonics needed for Theorem 1; for proofs see \([1, \text{pp. 168-170}]\) and \([11, \text{pp. 43, 44}]\). Let \( \mathcal{X}_k \) denote the space of all homogeneous harmonic polynomials of degree \( k \). The restrictions of these to \( \Sigma \) are the spherical harmonics of order \( k \), and they form a finite-dimensional subspace \( \mathcal{C}_k \) of \( L^2(\Sigma, d\omega) \). For each \( k \geq 0 \), let \( \{\varphi_{k,j}\}_{j=0}^{n(k)} \) be an orthonormal basis of \( \mathcal{C}_k \); then the set \( \Phi = \{\varphi_{k,j} : k \geq 0, 0 \leq j \leq n(k)\} \) is complete in \( L^2(d\omega) \). If \( \varphi, \psi \in \Phi \) are of different degrees then

\[
\int_{\Sigma} \varphi(\omega) \psi(\omega) d\omega = 0;
\]

this fact generalizes (1.12).

Let \( f \in L^1(\omega) \), and define the \( k \)th harmonic of \( f \) to be

\[
(1.21) \quad f_k = \sum_{j=0}^{n(k)} \left\{ \int_{\Sigma} f(\omega) \varphi_{k,j}(\omega) d\omega \right\} \varphi_{k,j}.
\]

We note that

\[
\|f_k\|_\infty \leq C(k) \|f\|_1 \quad (k \geq 0),
\]

that \( f = \sum f_k \) holds for all \( f \) in the linear span \( \Phi^* \) of \( \Phi \), and that if \( f_k = 0 \) for each \( k \geq 0 \) then \( f = 0 \), since \( \Phi^* \) is dense in \( C(\Sigma) \). Further, \( f_k \) is the orthogonal projection of \( f \) onto \( \mathcal{C}_k \) for all \( f \in L^2(\omega) \), and thus \( f_k \) does not depend on the basis chosen.

Finally, we write

\[
(1.22) \quad c_k = c_k(f) = \left\{ \int_{\Sigma} f_k^2(\omega) d\omega \right\}^{1/2} = \|f_k\|_2
\]

and observe that, if \( f \in L^2(\omega) \),

\[
(1.23) \quad \|f\|_2 = \left\{ \sum_{k=0}^{\infty} c_k^2 \right\}^{1/2}
\]

since \( \Phi \) is complete.
In the next section we study the harmonics of \( u \) defined in (1.17), and for this we must compute the harmonics of \( K_q \). For a given \( r > 0 \) and \( y \in \mathbb{R}^m \), let \( \{\varphi_{k,j}\}_{j=0}^{m(k)} \) be as described above with \( \varphi_{k,0}(\omega) = \alpha_k P_k(\omega \vee y) \), where the positive number \( \alpha_k \) is determined by \( \|\varphi_{k,0}\|_2 = 1 \). Then it is obvious from (1.12)–(1.14) that the \( k \)th harmonic of \( f(\omega) = K_q(r\omega, y) \) is

\[
 f_k(\omega) = Q_k P_k(\omega \vee y) \quad (\omega \in \Sigma)
\]

for a suitable factor \( Q_k(r, |y|) \). When \( |y| > r \) we compute \( Q_k \) using (1.14),

\[
 Q_k(r, |y|) = -r^k/|y|^{k+m-2} \quad (k > q), \quad = 0 \quad (k \leq q).
\]

When \( |y| < r \) we use (1.13) in a similar way and, for \( |y| = r \), \( Q_k \) is defined by continuity since \( K_q(r\omega, y) \to K_q(r\omega, y) \) in \( L^1(d\omega) \) when \( \sigma \to 1 \). Then the values of \( Q_k \) can be tabulated as follows:

| \( k \) | \( Q_k(r, |y|) \) |
|------|------------------|
| \( k > q \) | \(-r^k/|y|^{k+m-2}\) |
| \( 1 \leq k \leq q \) | \( r^k/|y|^{k+m-2} - r^k/r^{k+m-2} \) |
| \( k = 0 \) | \( K(r) - K(t) \) |

Finally, we observe from (1.17) and Fubini’s theorem that the \( k \)th harmonic of \( u^*_\mu(r\omega) \) is

\[
(1.25) \quad \int_{\mathbb{R}^m} Q_k(r, |y|) P_k(\omega \vee y) d\mu(y).
\]

2. An extremal property of spherical symmetrizations of potentials; proof of Theorem 1. Let \( u \in \mathcal{D}_m \) have finite nonintegral order \( \lambda \), and let \( q = [\lambda] \). Let \( \mu \) be the Riesz measure of \( u \), and denote by \( \tilde{\mu} \) the measure obtained by projecting the mass of \( \mu \) onto the positive \( x_1 \)-axis according to

\[
 \tilde{\mu}([a, b]) = \mu([a \leq |x| \leq b]) \quad (0 < a < b < \infty)
\]

where \([a, b]\) denotes the interval on the \( x_1 \)-axis with endpoints \((a,0,\ldots,0)\), \((b,0,\ldots,0)\). We also introduce the total variation measure \( \mu^* = |\tilde{\mu}| \) and the associated subharmonic function

\[
(2.1) \quad u^*_\mu(x) = \int_0^\infty K_q(x, te) d\mu^*(t).
\]

We shall compare the harmonics of \( u^*_\mu \) and \( u^*_\mu \). Recalling (1.22) and (1.25) we define

\[
 C_k(r, u^*_\mu) = c_k(u^*_\mu(r\omega)) = \left\| \int_{\mathbb{R}^m} Q_k(r, |y|) P_k(\omega \vee y) d\mu(y) \right\|_2.
\]
If \( u_\mu(r, \omega) \in L^2(d\omega) \), \( m^2(r, u_\mu) \) defined in Theorem 1 satisfies
\[
(2.2) \quad m^2(r, u_\mu) = \sigma_m^{-1/2}\|u_\mu\|_2 = \left\{ \sum_{k=0}^{\infty} C_k(r, u_\mu)^2 \right\}^{1/2},
\]
see (1.23). In any case, we have

**Theorem 2.1.** Let \( \mu \) be a measure satisfying (1.16). Then
\[
(2.3) \quad C_k(r, u_\mu) \leq C_k(r, u_\mu^*) \quad (0 < r < \infty; k \geq 0).
\]
Thus
\[
(2.4) \quad m_2(r, u_\mu) \leq m_2(r, u_\mu^*)
\]
for all \( r \) such that \( m_2(r, u_\mu^*) < \infty \). [This holds everywhere when \( m = 2 \), and a.e. when \( m = 3, 4 \); for, by (2.1) it is sufficient to show, a.e.,
\[
(2.5) \quad \psi_\tau(\omega) = \int_{|\tau|/2}^{2\tau} |\tau\omega - \tau e|^{2-m} \mu^*(t) \in L^2(d\omega).
\]
Fix any \( r \) such that \( \varphi(t) = u_\mu^*([0, t]) \) has a finite derivative at \( r \), and let \( \delta \) and \( K \) satisfy \( |\varphi(t) - \varphi(r)| \leq K|t - r| \) when \( |t - r| \leq 2\delta \). Then
\[
\psi_\tau(\cos \theta) = \int_{r/2}^{2r} \left\{ r^2 + r^2 - 2r \cos \theta \right\}^{-\nu} d\varphi(t) \quad (\nu = \frac{1}{2}(m - 2))
\]
\[
\leq C \int_{r/2}^{2r} \left\{ |r - \tau| + \theta \right\}^{-2\nu} d\varphi(t).
\]
\[
\leq C \left\{ \int_{|r - \tau| \leq \theta} \theta^{-2\nu} d\varphi(t) + \sum_{j=0}^{k} \int_{2^{j+1}\theta < |r - \tau| \leq 2^j\theta} |r - \tau|^{-2\nu} d\varphi(t) \right. \]
\[
+ \left. \int_{\delta < |r - \tau| \leq r} |r - \tau|^{-2\nu} d\varphi(t) \right\}
\]
where \( C \) depends only on \( r \) and \( k = \lfloor \log(\delta/\theta)/\log 2 \rfloor \). It follows that \( \psi_\tau(\cos \theta) \in L^2([0, \pi]; \sin^{m-2}\theta d\theta) \) when \( m = 3, 4 \).

**Proof of Theorem 2.1.** For each \( k > 0 \) we have by Schwarz's inequality and the fact that, by (1.24), \( Q_k \) is of one sign only,
\[
C_k(r, u_\mu)^2 = \int_\Sigma \left\{ \int_{R^m} Q_k(r, |y|) P_k(\omega \vee y) d\mu(y) \right\}^2 d\omega
\]
\[
\leq \int_\Sigma \left\{ \int_{R^m} |Q_k(r, |y|)| P_k(\omega \vee y) d|\mu|(y) \int_{R^m} |Q_k(r, |y|)| d|\mu|(y) \right\} d\omega
\]
\[
= \left\{ \int_\Sigma P_k^2(\omega \vee e) d\omega \right\} \left\{ \int_0^\infty Q_k(r, t) d\mu^*(t) \right\}^2
\]
\[
= \int_\Sigma \left\{ \int_0^\infty Q_k(r, t) P_k(\omega \vee e) d\mu^*(t) \right\}^2 d\omega = C_k(r, u_\mu^*),
\]
as claimed. When \( k = 0 \) we have as in (1.9) that

\[
\sigma_m^{-1/2} C_0(r, u_\mu) = \left| \int_{|y| < r} [K(re) - K(y)] d\mu(y) \right|
\]

\[
= |N(r, -u) - N(r, u)|
\]

\[
\leq N(r, -u) + N(r, u) = N(r, -u^*_\mu) = \sigma_m^{-1/2} C_0(r, u^*_\mu).
\]

To prove Theorem 1(2), let \( u \in \mathcal{O}_m \) have order \( \lambda \neq 0 \) positive integer and put \( q = [\lambda] \). Then (1.20) holds with \( h \) an harmonic polynomial of degree \( \leq q \) and \( u_\mu \in \mathcal{O}_m \) of order \( \lambda \). Further, \( N(r) = N_\mu(r) = N(r, -u^*_\mu) \) has order \( \lambda \) by (1.19); thus there exists a strong proximate order \( \lambda(t) \) in the sense of [19, p. 41], that is, \( \lambda(t) \in C^2(0, \infty) \) and

\[
\lambda(t) \to \lambda, \quad \lambda'(t) t \log t \to 0, \quad \lambda''(t) t^2 \log t \to 0 \quad (t \to \infty),
\]

and, if

\[
N_1(t) = t^{\lambda(t)}, \quad n_1(t) = d_m^{-1} t^{m-1} N_1(t),
\]

then also

\[
(2.6) \quad N(t) \leq N_1(t) \quad (0 < t < \infty), \quad N(r_n) = N_1(r_n) \quad (n \geq 1)
\]

where \( r_n \) increases to \(+\infty\), and

\[
(2.7) \quad n_1'(t) = \{\lambda(\lambda + m - 2)/d_m + o(1)\} t^{m-3} N_1(t) \quad (t \to \infty).
\]

For proof, see pp. 35 and 39 in [19].

In particular, \( n_1(t) \) is eventually increasing, say for \( t \geq r_1 \). Define \( \hat{n}, \hat{N} \) by

\[
\hat{N}(r) = d_m \int_0^r \hat{n}(t) t^{1-m} dt.
\]

Clearly, \( \hat{n} \) increases on \((0, \infty)\) and thus

\[
\hat{u}(x) = \int_0^\infty K_q(x, te) d\hat{n}(t) \in \mathcal{S}_m.
\]

Further,

(2) We thank Dr. F. Abi-Khuzam for pointing out an error in a previous version of the proof of Theorem 1.
\[
\lim_{n \to \infty} \inf \frac{N(r_n)}{m_2(r_n, u)} \geq \lim_{n \to \infty} \inf \frac{N(r_n)}{m_2(r_n, u_\mu) + m_2(r_n, h)} \geq \lim_{n \to \infty} \inf \frac{N(r_n)}{m_2(r_n, u_\mu^*)}
\]

where we have used (2.4) and \(m_2(r_n, h) = O(r_n^q) = o(N(r_n))\), by (2.6).

We proceed to estimate \(m_2(r_n, u_\mu^*)\). For each \(k \geq 1\), we have from the proof of Theorem 2.1 that

\[
\sigma_m^{-1} C_k(r, u_\mu^*)^2 = I_k^2 \left\{ \int_0^\infty Q_k(r, t) \, d\mu^*(t) \right\}^2
\]

where [11, pp. 15, 33, 4]

\[
I_k^2 = \sigma_m^{-1} \int_{\Sigma} P_k^2(\omega \vee e) \, d\omega
\]

\[
(2.10) \quad I_k^2 = \frac{(m - 2) \Gamma(k + m - 2)}{\Gamma(m - 2) \Gamma(k + 1) (2k + m - 2)} \quad (k \geq 1, m \geq 3),
\]

\[
I_k^2 = \frac{1}{2\pi} \int_0^{2\pi} \frac{\cos^2 k \theta}{k^2} \, d\theta = \frac{1}{2k^2} \quad (k \geq 1, m = 2).
\]

By (1.24), (1.8) and two integrations by parts,

\[
\left| \int_0^\infty Q_k(r, t) \, d\mu^*(t) \right| = \int_0^\infty |Q_k(r, t)| \, d\mu^*(t)
\]

\[
= \beta_k N(r) + \frac{k(k + m - 2)}{d_m} \int_0^\infty N(t) \left| Q_k \left( \frac{r}{t}, 1 \right) \right| \frac{dt}{t}
\]

where

\[
d_m \beta_k = 2k + m - 2 \quad (1 \leq k \leq q), \quad = -(2k + m - 2) \quad (k > q).
\]

Thus at the \(r_n\), by (2.6) and (2.7),

\[
C_k(r_n, u_\mu^*) \leq C_k(r_n, u) \quad (k \geq 0).
\]

It is easy to see from elementary properties of proximate orders that

\[
\lim_{n \to \infty} \frac{N(r_n)}{m_2(r_n, u^*)} = \lim_{r \to \infty} \frac{N(r)}{m_2(r, u^*)} = K_2(\lambda, m),
\]

where

\[
K_2(\lambda, m) = r^\lambda / m_2(r, U_\lambda) \quad (0 < r < \infty)
\]

and
For \( U_\lambda(x) \), clearly

\[
\sigma_m^{-1/2} C_k(1, U_\lambda) = I_k \left\{ \beta_k + \frac{k(k + m - 2)}{d_m} \int_0^\infty \frac{t^{\lambda + m - 3}}{t^m} \right\, dt \]

and a direct calculation using (1.24) and

\[
K_2(\lambda, m)^{-2} = m_2(1, U_\lambda)^2 = 1 + \sigma_m^{-1} \sum_{k=1}^{\infty} C_k(1, U_\lambda)^2
\]

shows that \( K_2(\lambda, m) \) coincides with \( C(\lambda, m) \) defined in (14). In view of (2.9) and (2.11), the proof of Theorem 1 (for general \( u \in \mathcal{O}_m \)) is complete.

The truth of (2.11) can be seen easily from the integral representation for \( \tilde{u}(x) \), together with (2.7), (1.14) and properties of proximate orders; we deduce
we observe, after two differentiations with respect to $\lambda$, that

\begin{equation}
C(\lambda, 2)^{-2} = \frac{1}{2} \left( \frac{\pi \lambda}{\sin \pi \lambda} \right)^2 \left\{ 1 + \frac{\sin 2\pi \lambda}{2\pi \lambda} \right\}.
\end{equation}

A convenient expression for $C(\lambda, 3)$ is given by

\begin{equation}
\frac{2\lambda + 1}{\lambda^2 (\lambda + 1)^2} C(\lambda, 3)^{-2} = \sum_{k=-\infty}^{\infty} \frac{1}{(k - \lambda)^2} - \sum_{k=1}^{\infty} \frac{1}{(k + \lambda)^2} - \sum_{k=0}^{\infty} \frac{1}{(k + \lambda + 1)^2}
= \left( \frac{\pi}{\sin \pi \lambda} \right)^2 - 2 \sum_{k=1}^{\infty} \frac{1}{(k + \lambda)^2},
\end{equation}

\begin{equation}
C(\lambda, 3)^{2} = \left( \frac{\sin \pi \lambda}{\pi \lambda} \right)^2 \frac{2\lambda + 1}{(\lambda + 1)^2} \left\{ 1 - \frac{2}{\pi^2} \left( \frac{\sin^2 \pi \lambda}{\sum_{k=1}^{\infty} \frac{1}{(k + \lambda)^2}} \right) \right\}^{-1}.
\end{equation}

$C(\lambda, 4)$ can be summed explicitly in terms of elementary functions:

\begin{equation}
\frac{4(\lambda + 1)}{\lambda^2 (\lambda + 2)^2} C(\lambda, 4)^{-2} = \frac{1}{\lambda^2} + \sum_{k=1}^{\infty} \left\{ \frac{1}{(k - \lambda)} - \frac{1}{k + \lambda + 1} \right\}
+ (\lambda + 1) \left\{ \sum_{k=1}^{\infty} \frac{1}{(k - \lambda)^2} + \sum_{k=1}^{\infty} \frac{1}{(k + \lambda + 1)^2} \right\}
= (\lambda + 1) \left\{ \frac{\pi}{\sin \pi \lambda} \right\}^2 - \pi \cot \pi \lambda,
\end{equation}

\begin{equation}
C(\lambda, 4)^{2} = \left( \frac{\sin \pi \lambda}{\pi \lambda} \right)^2 \left( \frac{2}{\lambda + 2} \right)^2 \left\{ 1 - \frac{\sin 2\pi \lambda}{2\pi (\lambda + 1)} \right\}^{-1}.
\end{equation}

We deduce easily

**Theorem 2.2.** Let $u \in \mathcal{O}_m$ have finite order $\lambda$. Then

\begin{equation}
\limsup_{r \to \infty} \frac{N(r)}{T(r, u)} \geq \frac{\left| \sin \pi \lambda \right|}{\pi \lambda (\lambda + 1)^{m-1}} \quad (0 \leq \lambda < \infty; m \leq 4).
\end{equation}
For, by Schwarz's inequality and (1.10),

\[
m_2(r,u) \geq \sigma_m^{-1} \int_{\Sigma} u(r\omega)^+ \, d\omega + \sigma_m^{-1} \int_{\Sigma} (-u(r\omega))^+ \, d\omega
\]

(2.16)

\[
= T(r,u) - N(r,u) + T(r,-u) - N(r,-u) = 2T(r,u) - N(r)
\]

and thus

\[
\frac{k(u)}{2 - k(u)} = \limsup_{T \to \infty} \frac{N(r)}{2T(r,u) - N(r)} \geq C(\lambda,m).
\]

Solving this inequality for \( k(u) \) and using simple estimates with (2.12)–(2.14), we obtain (2.15).

3. **Bounds for \( k(u) \) when \( m \geq 5 \).** Theorem 2 is contained in

**Theorem 3.1.** Let \( u \in \Omega_m \) have finite order \( \lambda \). Then

\[
k(u) \geq A_m |\sin \pi \lambda|/(\lambda + 1)^{1/(m+1)} \quad (0 < \lambda < \infty)
\]

where we may take \( A_m = m^{-m} \) (\( m \geq 5 \)).

We assume \( \lambda \) is not a positive integer, and let \( q = [\lambda] \). By Lemma 1.2, (1.20) holds with \( h \) of degree at most \( q \). Then

\[
\sigma_m^{-1} \int_{\Sigma} |u_\mu(r\omega)| \, d\omega \leq \int_{\mathbb{R}_m} \left\{ \sigma_m^{-1} \int_{\Sigma} |K_q(r\omega,\gamma)| \, d\omega \right\} d|m|\gamma
\]

\[
= \int_0^\infty B_q(r/t)t^{2-m} \, dn(t)
\]

where

\[
B_q(r) = \sigma_m^{-1} \int_{\Sigma} |K_q(r\omega,e)| \, d\omega;
\]

here \( e \) denotes the unit vector in the positive \( x_1 \)-direction.

**Lemma 3.1.** When \( 0 < r < \infty \),

(3.1) \[ B_q(r) \leq 2e(m - 2)^{1/(m-2)}(q + 1)^{1/(m-3)}r^{q+1}/(r + 1). \]

Assuming the validity of (3.1), we put

\[
S(r) = r^{q+1}/(r + 1)
\]

and use \( rS'(r) \leq (q + 1)S(r) \) to get
\[
\int_0^\infty S\left(\frac{r}{t}\right) t^{2-m} dt = d_m^{-1} \int_0^\infty \left\{d_m S\left(\frac{r}{t}\right) + S\left(\frac{r}{t}\right)\right\} dN(t)
\leq d_m^{-1} (q + m - 1)(q + 1) \int_0^\infty S\left(\frac{r}{t}\right) N(i) \frac{dt}{t}.
\]

By (2.16), (1.20) and Lemma 3.1,

(3.2) \quad 2T(r, u) \leq N(r) + C_m(q) \int_0^\infty S\left(\frac{r}{t}\right) N(t) \frac{dt}{t} + O(r^q)

where

\[
C_m(q) = 4e(m - 2)^{\frac{1}{2}(m-2)}(q + 1)^{\frac{1}{2}(m+1)}.
\]

For given \(\varepsilon > 0\), there exists [6, p. 101] a sequence \(r_n \to \infty\) with \(N(t) \leq (t/r_n)^{\lambda-\varepsilon} N(r_n) (0 < t < r_n), N(t) \leq (t/r_n)^{\lambda+\varepsilon} N(r_n) (t > r_n)\). Thus

\[
\limsup_{n \to \infty} \frac{T(r_n, u)}{N(r_n)} \leq \frac{1}{2} \left\{1 + C_m(q) \int_0^\infty S(t) t^{-\lambda-1} dt\right\},
\]

and Theorem 3.1 follows.

**Proof of Lemma 3.1.** We first suppose \(0 < r < 1\). Then (1.14) implies

\[
K_q(r, e, \omega) = \sum_{k=q+1}^{\infty} P_k(\omega \vee e) r^k.
\]

Since the \(P_k\) are orthogonal on \(\Sigma\),

\[
B_q(r) = \sum_{k=q+1}^{\infty} I_k^2 r^{2k}
\]

where the \(I_k\) are given in (2.10). By simple estimates,

\[
I_k^2 \leq (m - 2)^2 k^{m-4} \quad (k \geq 1).
\]

Put \(r_0 = \exp(-1/(q + 1))\). Then

\[
B_q(r)^2 \leq \left[e(m - 2)r^{q+1}\right]^2 \sum_{k=q+1}^{\infty} \psi(k) \quad (0 \leq r \leq r_0)
\]

where \(\psi(x) = x^p r_0^{2x} (p = m - 4)\) increases on \(0 < x < x_0 = (q + 1)p/2\) and then decreases, so that
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\[ \sum_{k=q+1}^{\infty} \psi(k) \leq \int_{q+1}^{\infty} \psi(x) \, dx + \psi(x_0) \]

\[ = e^{-2} \left( \frac{q + 1}{2} \right)^{p+1} \left( 2^p + p 2^{p-1} + p(p - 1) 2^{p-2} + \cdots + p! \right) \]

\[ + \left( \frac{p(q + 1)}{2e} \right)^p \]

\[ < p^p (q + 1)^{p+1} \quad (p = m - 4), \]

(3.3) \[ B_q(r) \leq e^{(m - 2)\frac{1}{4} (m-2)} (q + 1)^{\frac{1}{4} (m-3)} r^{q+1} \quad (0 < r \leq r_0). \]

For \( r > r_0 \), (1.13) yields

\[ K_q(r, \omega, e) = K(r, \omega - e) + \sum_{k=0}^{q} P_k(\omega \vee e) r^k, \]

\[ B_q(r) \leq \min\{1, r^{2-m}\} + \sigma^{-1} \int_{\Sigma} \left| \sum_{k=0}^{q} P_k(\omega \vee e) r^k \right| \, d\omega \]

where the second term is dominated by

\[ Q = \left\{ \sigma^{-1} \int_{\Sigma} \sum_{k=0}^{q} (P_k(\omega \vee e) r^k)^2 \, d\omega \right\}^{1/2} = \left\{ \sum_{k=0}^{q} I_k^2 r^{2k} \right\}^{1/2}. \]

Thus

\[ Q^2 \leq \frac{(r/r_0)^{2q}}{2} \sum_{k=0}^{q} I_k^2 \quad (r_0 < r < \infty) \]

and by (2.10)

\[ I_k^2 \leq \frac{(m - 2)^{m-3}}{2\Gamma(m-2)} (q + 1)^{m-4} \quad (1 \leq k \leq q). \]

We deduce

\[ B_q(r) \leq e^{(m - 2)\frac{1}{4} (m-3)} (q + 1)^{\frac{1}{4} (m-3)} r^q \quad (r_0 < r < \infty), \]

and (3.1) follows.

4. Examples. For \( m \geq 3 \), let \( q < \lambda < q + 1 \) for some integer \( q \geq 0 \) and consider

(4.1) \[ U_\lambda(x) = \frac{\lambda(\lambda + m - 2)}{m - 2} \int_{0}^{\infty} K_q(x, te) t^{\lambda + m-3} \, dt, \]

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a subharmonic function whose Riesz mass is distributed along the positive \( x_1 \)-axis with

\[
N(r) = N(r, -U_\lambda) = r^\lambda \quad (0 < r < \infty).
\]

Then

\[
U_\lambda(-x) = \frac{\lambda(\lambda + m - 2)}{m - 2} I_\lambda(\cos \theta)r^\lambda
\]

where \( x = r\omega, \cos \theta = -\omega + e \) and

\[
I_\lambda(\cos \theta) = \int_0^\infty K_q(\tau \omega, -e)\tau^{-\lambda-1} d\tau
\]

\[
= \int_0^\infty \left\{ \sum_{k=0}^g P_k(\omega \vee e)(-1)^k \tau^{-k-m+2} - \frac{1}{(1 + \tau^2 + 2\tau \cos \theta)^\nu} \right\} \tau^\lambda + m - 3 d\tau.
\]

Here and below, \( \nu = (m - 2)/2 \).

We have the representation

\[
I_\lambda(\cos \theta) = \frac{1}{e^{2\pi i \lambda i} - 1} \int_\Gamma \frac{z^{\lambda + m - 3} dz}{(1 + z^2 + 2z \cos \theta)^\nu},
\]

where \( \Gamma \) consists of the circles \(|z| = R\) and \(|z| = e (0 < e < 1 < R)\) respectively oriented positively and negatively, joined by segments along the upper and lower edges of the real axis between \( e \) and \( R \). To see this, use

\[
(1 + z^2 + 2z \cos \theta)^{-\nu} = \sum_{k=0}^\infty (-1)^k P_k(\cos \theta)z^{-k-m+2} \quad (|z| = R)
\]

in (4.4) with Cauchy's theorem and let \( e \to 0, R \to \infty \). Thus we can evaluate \( I_\lambda \) by residues when \( m \) is even. (This procedure is used by Hayman [8, Chapter 4] for orders \( \lambda < 1 \).)

We deduce

\[
I_\lambda(\cos \theta) = \frac{2\pi i}{e^{2\pi i \lambda} - 1} \left\{ \frac{g^{(\nu-1)}(a)}{(\nu - 1)!} + \frac{\bar{g}^{(\nu-1)}(a)}{(\nu - 1)!} \right\}
\]

where \( g(z) = z^{\lambda + m - 3}(z - a)^{-\nu} \), \( \bar{g} \) is the similar expression with \( \bar{a} \) in place of \( a \), and \( a = -e^{i\theta} \). By direct calculation,

\[
I_\lambda(\cos \theta) = \frac{\pi}{\sin \pi \lambda} \frac{\sin(\lambda + 1)\theta}{\sin \theta} \quad (\nu = 1)
\]

and for \( \nu > 1 \),
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\[ I_\lambda(\cos \theta) \]

\[ = \frac{\pi}{\sin \pi \lambda} \left\{ \frac{(\lambda + m - 3) \cdots (\lambda + m - \nu - 1)}{2^{\nu - 1} (\nu - 1)!} \cdot \frac{\cos[(\lambda + \nu) \theta - \pi \nu / 4]}{(\sin \theta)^{\nu}} \right\} + R \]

where

\[ |R| \leq C(\nu)(\lambda + 1)^{\nu - 2}(\sin \theta)^{3 - m} \quad (0 < \theta < \pi) \]

and \( C(\nu) \) does not depend on \( \theta \) or \( \lambda \). This follows easily from (4.5) and

\[ g^{(\nu - 1)}(z) = \sum_{j=0}^{\nu - 1} \binom{\nu - 1}{j} D^{(\nu - j - 1)}(z^{\lambda + m - 3}D^j((z - \alpha)^{-\nu}) \]

where \( D = d/dz \), and the similar expression for \( g^{(\nu - 1)} \).

Since \( I_\lambda(\cos \theta) \) is even in \( \theta \),

\[ r^{-\lambda} T(r, U_\lambda) = \frac{\lambda(\lambda + m - 2)}{m - 2} 2^{\nu - 1} \sigma_m^{-1} \int_0^\pi I_\lambda(\cos \theta)^{+}d\omega(\theta) = K(\lambda, m)^{-1} \]

where

\[ d\omega(\theta) = \sigma_{m-1}(\sin \theta)^{m-2}d\theta. \]

Thus

\[ T(1, U_\lambda) = \frac{\pi \lambda}{\sin \pi \lambda} (\lambda + m - 2) \cdots (\lambda + m - \nu - 1) \left( \frac{\sigma_m^{-1}}{\sigma_m} \right) H_\lambda \]

where

\[ H_\lambda = \int_0^\pi \left\{ (-1)^q \cos \left[ (\lambda + \nu) \theta - \frac{\pi \nu}{4} \right] (\sin \theta)^{\nu} \right\} d\theta + \varepsilon_\lambda, \]

with \( |\varepsilon_\lambda| \leq C_1(\nu)/(\lambda + 1) \) by (4.7). On the other hand, since

\[ \lim_{\beta \to \infty} \int_a^b f(\theta) \cos(\beta \theta + \gamma) d\theta = \lim_{\beta \to \infty} \int_a^b f(\theta)(\cos(\beta \theta + \gamma))^+ \rightarrow d\theta = \frac{1}{\pi} \int_a^b f(\theta) d\theta \]

for any \( f \in L^1(a, b) \) and \( \gamma \) real, we obtain

\[ H_\lambda = \frac{1}{\pi} \int_0^\pi \sin^\nu \theta d\theta + o(1) \]
on letting \( \lambda \to \infty \) so that first \( q = [\lambda] \) is even, then odd.

We deduce that the \( U_\lambda \) satisfy

\[
\frac{N(r)}{T(r, U_\lambda)} = \frac{\sigma_m(m-2)}{2\lambda(\lambda + m - 2)} \left\{ \int_0^\pi I_\lambda(\cos \theta)^+ \, d\omega(\theta) \right\}^{-1}
\]

\[
= \alpha_m |\sin \pi \lambda| \lambda^{-1}\{1 + o(1)\} \quad (0 < r < \infty; \lambda \to \infty)
\]

where \( \alpha_m \) depends only on the dimension; this proves (16) for \( m \) even.

In fact, from (4.4) \( I_\lambda(\cos \theta) \) can be seen to satisfy a differential equation of hypergeometric type [17, p. 178], thus [17, pp. 175, 104]

\[
I_\lambda(\cos \theta) = \beta \, _2F_1\left(\lambda + 2\nu, -\lambda; \nu + \frac{1}{2}; \frac{1 + \cos \theta}{2}\right),
\]

(4.9)

\[
\beta = I_\lambda(1)^2 \Gamma\left(\frac{1}{2} + \nu + \lambda\right) \Gamma\left(\frac{1}{2} - \nu - \lambda\right) / \Gamma\left(\frac{1}{2} + \nu\right) \Gamma\left(\frac{1}{2} - \nu\right)
\]

where the \( _2F_1 \) has a known asymptotic expansion [17, p. 77] for large \( \lambda \) like that in (4.6), but valid for all real \( \nu \). Further, our analysis giving (4.6) from (4.4), when \( \nu \) is integral, remains valid for half-integral \( \nu \) in the case \( \theta = 0 \), and we can asymptotically evaluate the factor \( I_\lambda(1) \) in (4.9). (The \( _2F_1 \) in (4.9) is essentially a Gegenbauer function [17, p. 175].)

We conclude that the functions \( U_\lambda \) satisfy (16) for any \( m \geq 3 \), by known asymptotic results. When \( m = 2 \), (4.1) gives \( U_\lambda(x) = \pi \lambda \csc \pi \lambda (\cos \theta) r^\lambda \) for all \( \lambda \neq \) positive integer, \( |\theta| \leq \pi, r > 0 \).

5. Proof of Theorem 3. We can assume all the Riesz mass of \( u(x) \) is on the negative \( x_1 \)-axis, so that

\[
u(x) = \int_0^\infty K_q(x,-te) \, d\mu(t) + h(x) = u_\mu(x) + h(x)
\]

where \( u \) has order \( \lambda \in (q, q+1) \) and the degree of \( h(x) \) is at most \( q \). For any \( \gamma \in (\lambda, q+1) \),

\[
\int_0^\infty u_\mu(r\omega) r^{-\gamma-1} \, dr = \int_0^\infty d\mu(t) \int_0^\infty K_q(r\omega,-te) r^{-\gamma-1} \, dr
\]

(5.2)

\[
= \int_0^\infty t^{-\gamma-m+2} \, d\mu(t) \int_0^\infty K_q(\tau\omega, -e) r^{-\gamma-1} \, d\tau
\]

\[
= \frac{\gamma(\gamma + m - 2)}{m - 2} I_\gamma(\cos \theta) \int_0^\infty N(i) t^{-\gamma-1} \, dt
\]

where \( I_\gamma \) is defined in (4.3).

Let \( \mathbb{E} \subset \Sigma \cap \{x_m \geq 0\} \) be measurable \( d\omega \), and define \( E \subset [0, \pi] \) by \( E = \{\theta: \omega \vee e = \cos \theta, \omega \in \mathbb{E}\} \), and
Thus \( T(r, u_\mu; \mathcal{E}) \leq T(r, u_\mu) \) and by (5.2)

\[
\int_0^\infty T(r, u_\mu; \mathcal{E}) r^{-\gamma-1} \, dr = \frac{\gamma(m - 2)}{m - 2} \left\{ 2\sigma^{-1}_m \int_E I_\gamma(\cos \theta) \, d\omega(\theta) \right\} \int_0^\infty N(t) t^{-\gamma-1} \, dt
\]

where \( d\omega(\theta) \) was defined in §4.

Using a theorem of Pólya [13] just as in [9, pp. 225–227], we deduce

\[
\liminf_{r \to \infty} \frac{A(\lambda)N(r) + \tau}{T(r, u_\mu)} \leq 1
\]

where \( \tau < \lambda \) is arbitrary and

\[
A(\gamma) = \frac{\gamma(m - 2)}{m - 2} 2\sigma^{-1}_m \int_E I_\gamma(\cos \theta) \, d\omega(\theta).
\]

Since \( N(r) \leq T(r, u_\mu) \), it follows from (5.3) that there exists \( \{r_n\} \to \infty \) with

\[
A(\lambda) \liminf_{n \to \infty} \frac{N(r_n)}{T(r_n, u_\mu)} \leq 1
\]

and

\[
\lim_{n \to \infty} \frac{\log T(r_n, u_\mu)}{\log r_n} = \lambda.
\]

Thus by (5.1)

\[
A(\lambda) \liminf_{r \to \infty} \frac{N(r)}{T(r, u)} \leq 1.
\]

Since \( E \) is an arbitrary subset of \([0, \pi]\) and \( I_\gamma \) is independent of \( r \), we can take

\[
E = \{ \theta : I_\lambda(\cos \theta) \geq 0 \}.
\]

Then by (4.8) and (5.4), (18) follows. Assertion (19) is a simple consequence of

\[
\lim_{\theta \to \pi^-} I_\lambda(\cos \theta) = -\infty \quad (m \geq 3),
\]

clear from (4.3). When \( m \) is even, \( K(\lambda, m) \) can be computed in terms of elementary functions; in particular, (20) follows from the evaluation \( I_\lambda(\cos \theta) = (\pi/\lambda \sin \pi \lambda) \cos \theta \lambda \) when \( m = 2 \).
References


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