INEQUALITIES FOR POLYNOMIALS ON THE UNIT INTERVAL

BY

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Abstract. Let \( p_n(z) = \sum_{k=0}^{n} a_k z^k \) be a polynomial of degree at most \( n \) with real coefficients. Generalizing certain results of I. Schur related to the well-known inequalities of Chebyshev and Markov we prove that if \( p_n(z) \) has at most \( n - 1 \) distinct zeros in \((-1, 1)\), then

\[
|a_n| < 2^{n-1} \left( \cos \frac{\pi}{4n} \right)^{2n} \max_{-1 < x < 1} |p_n(x)|,
\]

\[
\max_{-1 < x < 1} |p'_n(x)| < \left( n \cos \frac{\pi}{4n} \right)^2 \max_{-1 < x < 1} |p_n(x)|.
\]

1. Introduction. Let \( p_n(z) = \sum_{k=0}^{n} a_k z^k \) be a polynomial of degree at most \( n \). According to a well-known result of A. Markov [4],

\[
\max_{-1 < x < 1} |p'_n(x)| \leq n^2 \max_{-1 < x < 1} |p_n(x)|.
\]

In (1) equality holds if and only if \( p_n(z) \) is a constant multiple of \( T_n(z) \) where

\[
T_n(z) = 2^{n-1} \prod_{\nu=1}^{n} \left( z - \cos \left( \frac{\nu - \frac{1}{2}}{n} \pi \right) \right)
\]

is the so-called Chebyshev polynomial of the first kind of degree \( n \).

The influence of the location of the zeros of \( p_n(z) \) on the bound in Markov’s inequality (1) has been studied by Schur [7], Erdős [1], Eröd [2], Rahman [5], Scheick [6] and others. It was shown by Erdős [1] that if all the zeros of \( p_n(z) \) are real but lie outside \((-1, 1)\), then (1) can be replaced by

\[
\max_{-1 < x < 1} |p'_n(x)| < \frac{1}{2} en \max_{-1 < x < 1} |p_n(x)|.
\]

Scheick [6] obtained the same estimate under the weaker assumption that \( p_n(z) \) is real for real \( z \) and does not vanish in \( |z| < 1 \). Schur [7] prescribed one of the zeros of \( p_n(z) \) to lie at one of the end points of the interval \([-1, +1]\) and showed that then

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An analogous problem concerning Bernstein’s inequality for polynomials on the unit disk was recently studied by Giroux and Rahman [3, Theorems 1, 2].

With respect to the problem considered by Schur it is natural to ask what can be said about

\[
\left( \frac{\max_{-1 < x < 1} |p''(x)|}{\max_{-1 < x < 1} |p_n(x)|} \right) / \left( \frac{\max_{-1 < x < 1} |p_n(x)|}{\max_{-1 < x < 1} |p_n(x)|} \right)
\]

if we simply assume that \( p_n(z) \) is a real polynomial of degree \( n \) having at most \( n - 1 \) distinct zeros in \((-1, 1)\). This question is answered in Theorem 1.

Improving upon the well-known estimate of Chebyshev

\[(4) \quad |a_n| \leq 2^{n-1} \max_{-1 < x < 1} |p_n(x)| \]

for the leading coefficient of a polynomial \( p_n(z) \) of degree \( n \) in terms of \( \max_{-1 < x < 1} |p_n(x)| \), Schur [7, Theorem III*] proved that if \( p_n(z) = \sum_{k=0}^{n} a_k z^k \) is a polynomial of degree \( n \) vanishing at +1 or −1, then

\[(5) \quad |a_n| \leq 2^{n-1} \left( \cos \frac{\pi}{4n} \right)^{2n} \max_{-1 < x < 1} |p_n(x)|.\]

We show that the same estimate holds (see Theorem 2 below) for all real polynomials having at most \( n - 1 \) distinct zeros in \((-1, 1)\).

2. Statement of results.

**Notation.** We shall denote by \( \mathcal{P}_n \) the class of all polynomials \( p_n(z) = \sum_{k=0}^{n} a_k z^k \) of degree \( n \) with real coefficients.

**Theorem 1.** Inequality (3) holds for all polynomials \( p_n(z) \) in \( \mathcal{P}_n \) which have at most \( n - 1 \) distinct zeros in \((-1, 1)\). Equality is attained if and only if \( p_n(z) \) is a constant multiple of

\[ T_n \left( \pm \left( \cos \frac{\pi}{4n} \right)^2 z + \left( \sin \frac{\pi}{4n} \right)^2 \right). \]

In particular (3) holds for all polynomials \( p_n(z) \) in \( \mathcal{P}_n \) which vanish at +1 or −1. Here the restriction that \( p_n(z) \) has real coefficients can be easily dropped. In fact, if \( p_n(z) = \sum_{k=0}^{n} a_k z^k \) is an arbitrary polynomial of degree \( n \) vanishing at +1 or −1 and the maximum of \( |p_n'(x)| \) in \([-1, 1]\) is attained at \( x_0 \in [-1, 1] \) where \( p_n'(x_0) = |p_n(x_0)| e^{i\gamma} \), then \( A_n(z) = \sum_{k=0}^{n} \Re(a_k e^{-i\gamma}) z^k \) is a polynomial in \( \mathcal{P}_n \) vanishing at +1 or −1 with

\[ \max_{-1 < x < 1} |A_n'(x)| = \max_{-1 < x < 1} |p_n'(x)|, \quad \max_{-1 < x < 1} |A_n(x)| \leq \max_{-1 < x < 1} |p_n(x)|. \]

Since by Theorem 1,

\[ \max_{-1 < x < 1} |A_n'(x)| \leq \left( n \cos \frac{\pi}{4n} \right)^2 \max_{-1 < x < 1} |A_n(x)|, \]
we see that (3) holds for all polynomials $p_n(z)$ of degree $n$ vanishing at $+1$ or $-1$. We thus get an alternative proof of Schur’s result in its full generality.

Note that if in Theorem 1, $p_n(z)$ is allowed to have complex coefficients, then nothing better than Markov’s result can hold.

**Theorem 2.** Inequality (5) holds for all polynomials $p_n(z)$ in $\mathcal{P}_n$ which have at most $n - 1$ distinct zeros in $(-1, 1)$. Equality is attained if and only if $p_n(z)$ is a constant multiple of $T_n(\pm (\cos(\pi/4n))^2 z + (\sin(\pi/4n))^2)$.

Here again the coefficients of $p_n(z)$ cannot be allowed to be complex. Nevertheless, Schur’s result that (5) holds for all polynomials $p_n(z) = \sum_{k=0}^{n} a_k z^k$ vanishing at $+1$ or $-1$ can be easily deduced.

As an immediate consequence of Theorem 2, we obtain

**Corollary.** All the zeros of a monic polynomial $p_n(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ in $\mathcal{P}_n$ with

$$\max_{-1 < x < 1} |p_n(x)| < 2^{1-n} \left(\cos \frac{\pi}{4n}\right)^{-2n}$$

are distinct and lie in $(-1, 1)$.

3. Lemmas.

**Notation.** We shall denote by $\mathcal{T}_n$ the class of all real trigonometric polynomials

$$t(\theta) = a_0 + \sum_{\nu=1}^{n} (a_{\nu}\cos \nu \theta + b_{\nu}\sin \nu \theta)$$

with $a_n^2 + b_n^2 = 4^{1-n}$, and having a double zero at $\theta = 0$, i.e.

$$\sum_{\nu=0}^{n} a_{\nu} = 0 = \sum_{\nu=1}^{n} b_{\nu}.$$

Theorem 2 will be deduced from the following two lemmas.

**Lemma 1.** Let $t(\theta)$ be a trigonometric polynomial in the class $\mathcal{T}_n$ with $\max_{-\pi < \theta < \pi} |t(\theta)| = M$. If $|t(\theta)|$ is equal to $M$ at $2n - 1$ different points in $[-\pi, \pi)$, then

$$t(\theta) = \pm 2^{1-n} \left(\cos \frac{\pi}{4n}\right)^{-2n} T_n \left( - \left(\cos \frac{\pi}{4n}\right)^2 \cos \theta + \left(\sin \frac{\pi}{4n}\right)^2 \right),$$

where, as usual, $T_n(z)$ is the Chebyshev polynomial of the first kind of degree $n$.

**Proof.** We show first that under the assumptions of the lemma, $t(\theta)$ is a cosine polynomial. Since $t(0) = t'(0) = 0$, we see that $t(\theta)$ has exactly $2n$ critical points in $[-\pi, \pi]$, which we may list as

$$-\pi < \varphi_1 < \varphi_2 < \cdots < \varphi_{2n} < \pi.$$
where for some \( k \) (\( 1 < k < 2n \)) \( \varphi_k = 0 \). Further, in each of the subintervals \([-\pi, 0)\) and \((0, \pi)\) the signs of \( t(\theta) \) at consecutive critical points are alternating (provided the subinterval in question contains at least two critical points). If \( \varphi_j \) and \( \varphi_{j+1} \) (\( j \neq k \neq j + 1 \)) are two consecutive critical points of \( t(\theta) \) such that

\[
\text{sgn} \ t(\varphi_j) = -\text{sgn} \ t(\varphi_{j+1}),
\]

and

\[
|t(\varphi_j)| = |t(\varphi_{j+1})| = \max_{-\pi < \theta < \pi} |t(\theta)|,
\]

then for every \( \epsilon \) (\( 0 < \epsilon < 1 \)) the graph of \((1 - \epsilon)t(-\theta)\) crosses the graph of \( t(\theta) \) in \((\varphi_j, \varphi_{j+1})\). Hence, whatever \( k \) (\( 1 < k < 2n \)) may be, \( s(\epsilon, \theta) = t(\theta) - (1 - \epsilon)t(-\theta) \) has at least \( 2n - 3 \) zeros in

\[
\mathcal{E} = \{ \theta : \varphi_1 < \theta < \varphi_{2n} \} \cap \{ \theta : |\theta| > \delta \}
\]

where \( \delta \) is a suitably small positive number not depending on \( \epsilon \). As \( E \) is a closed set the number of zeros of \( s(\epsilon, \theta) \) in \( E \) cannot decrease when \( \epsilon \to 0 \). Hence \( s(\theta) = t(\theta) - t(-\theta) \) has at least \( 2n - 3 \) zeros in \( E \). If \( \varphi_1 = -\pi \), then taking the periodicity of \( t(\theta) \) into account we see that one of the zeros of \( s(\epsilon, \theta) \) lying in \( E \) tends to \(-\pi\) as \( \epsilon \to 0 \), where it becomes a zero of multiplicity at least two. If \( \varphi_1 > -\pi \), then \( s(\theta) \) has at least a simple zero at \(-\pi\), since \( t(-\pi) = t(\pi) \). Besides, in any case \( s(\theta) \) has a zero of multiplicity at least three at \( \theta = 0 \). Hence \( s(\theta) \) has at least \( 2n + 1 \) zeros in \([-\pi, \pi]\) if a multiple zero is counted as many times as its multiplicity. Since \( s(\theta) \) is of degree at most \( n \) this is possible only if \( s(\theta) \equiv 0 \), i.e. \( t(\theta) \) is a cosine polynomial.

A similar discussion shows that taking into account the multiplicity of the zero at \( \theta = 0 \) each of the two polynomials

\[
t(\theta) \pm 2^{1-n} \left( \cos \frac{\pi}{4n} \right)^{2n} T_n \left( -\left( \cos \frac{\pi}{4n} \right)^2 \cos \theta + \left( \sin \frac{\pi}{4n} \right)^2 \right)
\]

has at least \( 2n - 1 \) zeros in \([-\pi, \pi]\). But clearly, one of these two polynomials is of degree at most \( n - 1 \), and hence must be identically zero. This completes the proof of Lemma 1.

**Lemma 2.** Let \( t(\theta) \) be a trigonometric polynomial in the class \( \mathcal{T}_n \) with \( \max_{-\pi < \theta < \pi} |t(\theta)| = M \). If \( |t(\theta)| \) is equal to \( M \) at less than \( 2n - 1 \) different points in \([-\pi, \pi]\), then \( t(\theta) \) cannot be of smallest supremum norm in \( \mathcal{T}_n \).

**Proof.** We may assume that \( |t(\theta)| \) attains its maximum at exactly \( 2n - 2 \) points in \([-\pi, \pi]\) with alternating signs in the subintervals \([-\pi, 0)\) and \((0, \pi)\), for otherwise we can add a trigonometric polynomial of degree less than \( n \) such that the resulting trigonometric polynomial still belongs to \( \mathcal{T}_n \), but has smaller supremum norm.
Since \( t'(\theta) \) is a real trigonometric polynomial it has an even number of zeros in \([-\pi, \pi)\). Hence either \( \xi = 0 \) is a zero of \( t(\theta) \) of multiplicity three, or else there is one (and only one) critical point \( \eta \) of \( t(\theta) \) other than 0 with \( |t(\eta)| < M \). It is easily seen that \( \xi \) and \( \eta \) must be consecutive critical points if \( t(\theta) \) is to be a trigonometric polynomial of smallest supremum norm in \( \mathcal{T}_n \). In any case, we may assume without loss of generality, that we have two consecutive critical points \( \xi = 0 \) and \( \eta \) with \( \xi < \eta \) and \( 0 = |t(\xi)| < |t(\eta)| < M \). If a multiple zero is counted as many times as its multiplicity then we see that \( t(\theta) \) has a total number of \( 2n \) zeros \( \theta_v \) \((1 \leq v \leq 2n)\) in \([-\pi, \pi)\), which may be arranged as

\[-\pi < \theta_1 < \theta_2 < \cdots < \theta_{2n} < \pi.\]

Putting \( \theta_0 = \theta_{2n} - 2\pi \) and \( \theta_{2n+1} = \theta_1 + 2\pi \), we have for some \( k \) \((2 < k < 2n)\),

\[\theta_{k-2} < \theta_{k-1} = \theta_k = \xi = 0 < \eta < \theta_{k+1}.\]

As

\[|t(\theta)| = 2^n \prod_{v=1}^{2n} \left| \sin \frac{\theta - \theta_v}{2} \right| = 2^{-n} \prod_{v=1}^{2n} |e^{i(\theta - \theta_v)/2} - e^{-i(\theta - \theta_v)/2}| \]

(6)

\[= 2^{-n} \prod_{v=1}^{2n} |e^{i\theta} - e^{i\theta_v}|,\]

it is sufficient to show that we can decrease the maximum modulus of \( F(z) = \prod_{v=1}^{2n} (z - e^{i\theta_v}) \) on the unit circle by moving some of the \( \theta_v \)'s on the real axis keeping \( \theta_{k-1} = \theta_k \). For this purpose we consider

\[F(\alpha, z) = \frac{D(\alpha, z)}{D(0, z)} F(z),\]

where

\[D(\alpha, z) = (z - e^{-i\alpha})^2 (z - e^{i(\theta_{k+1} + 2\alpha)}).\]

On discussing the behaviour of

\[|D(\alpha, e^{i\theta})| = 8 \left| \sin \left( \frac{\theta + \alpha}{2} \right) \right|^2 \sin \left( \frac{\theta - \theta_{k+1} - 2\alpha}{2} \right)\]

we see that, indeed, for small positive \( \alpha \),

\[\max_{|\alpha| = 1} |F(\alpha, z)| < \max_{|\alpha| = 1} |F(z)|.\]

Through the relationship (6) there corresponds to \( F(\alpha, z) \) a trigonometric polynomial \( t(\alpha, \theta) \) which is simply a translation of an element in \( \mathcal{T}_n \) and has smaller supremum norm than \( t(\theta) \).
4. Proofs of the theorems. We will prove Theorem 2 first since we shall need it for the proof of Theorem 1.

**Proof of Theorem 2.** We will prove the equivalent fact that if \( p_n(z) \) is a monic polynomial in \( \mathbb{P}_n \) having at most \( n - 1 \) zeros in \((-1, 1)\) and \( M = \max_{-1 < x < 1} |p_n(x)| \), then

\[
M > 2^{1-n} \left( \cos \frac{\pi}{4n} \right)^{-2n}
\]

where equality is possible if and only if \( p_n(z) = 2^{1-n} \left( \cos \frac{\pi}{4n} \right)^{-2n} P_\ast(z) \) or \( p_n(z) = (-1)^n 2^{1-n} \left( \cos \frac{\pi}{4n} \right)^{-2n} P_\ast(-z) \), where

\[
P_\ast(z) = T_n \left( \left( \cos \frac{\pi}{4n} \right) z + \left( \sin \frac{\pi}{4n} \right)^2 \right).
\]

Since \( 2 > (\cos(\pi/4n))^{-2n} \) \((n > 1)\), Chebyshev's inequality (4) shows that (7) holds for all monic polynomials of degree less than \( n \). If \( p_n(z) \) has a real zero outside \([-1, 1]\) or pairs of complex conjugate zeros, \( \max_{-1 < x < 1} |p_n(x)| \) can be decreased by moving these zeros appropriately and keeping them outside the unit interval. So, we may suppose that \( p_n(z) \) is a polynomial of degree \( n \) vanishing at one of the end points of the unit interval, or having a double zero in \((-1, 1)\). Then the trigonometric polynomial \( P_n(\cos \theta) \) is also of degree \( n \) and has at least one double zero in \([-\pi, \pi)\). For a suitable choice of \( \alpha \) the trigonometric polynomial \( t(\theta) = p_n(\cos(\theta - \alpha)) \) belongs to \( \mathbb{F}_n \). Lemmas 1 and 2 show that

\[
\pm 2^{1-n} \left( \cos \frac{\pi}{4n} \right)^{-2n} P_\ast(-\cos \theta)
\]

are the only elements of smallest supremum norm in \( \mathbb{F}_n \). Hence (7) holds, with equality if and only if

\[
p_n(z) = 2^{1-n} \left( \cos \frac{\pi}{4n} \right)^{-2n} P_\ast(z) \quad \text{or} \quad p_n(z) = (-1)^n 2^{1-n} \left( \cos \frac{\pi}{4n} \right)^{-2n} P_\ast(-z).
\]

With this Theorem 2 is proved.

**Proof of Theorem 1.** Without loss of generality we may restrict ourselves to polynomials whose absolute value does not exceed 1 on the unit interval. Now let \( \mathcal{C} \) denote the (sub-) class consisting of all polynomials \( p_n(z) \) in \( \mathbb{P}_n \) which have at most \( n - 1 \) distinct zeros in \((-1, 1)\) and which satisfy \( |p_n(x)| < 1 \) for \(-1 < x < 1\). Then

\[
P_\ast(\pm z) = T_n \left( \pm \left( \cos \frac{\pi}{4n} \right)^2 z + \left( \sin \frac{\pi}{4n} \right)^2 \right) \in \mathcal{C}.
\]

A straightforward calculation shows that
\[ \max_{-1 < x < 1} |P_n'(x)| = (n \cos(\pi/4n))^2 = P_n'(1). \]

In view of this and the fact that for a polynomial \( p_n(z) \) in \( \mathbb{P}_n \) for which \( \max_{-1 < x < 1} |P_n'(x)| < 1 \) we have [7, p. 275]

\[ \max_{-1 < x < 1} |p_n'(x)| < \frac{n^2}{2} < P_n'(1) \quad (n > 2), \]

whenever \( \max_{-1 < x < 1} |p_n'(x)| \) is attained in \((-1, 1)\), it is enough to show (in order to establish Theorem 1) that \( |p_n'(1)| < (n \cos(\pi/4n))^2 = P_n'(1) \) for all \( p_n(z) \in \mathbb{C} \) with equality if and only if \( p_n(z) = \pm P_n^*(z) \).

Let \( Q_*(z) \) be a polynomial in \( \mathbb{C} \) for which

\[ |Q_*(1)| = \sup_{p_n(z) \in \mathbb{C}} |p_n'(1)|. \]

Since \( (n - 1)^2 < (n \cos(\pi/4n))^2 = P_n'(1) \), we see by A. Markov's theorem that \( Q_*(z) \) is of degree \( n \). Suppose

\[ |Q_*(1)| > |P_n'(1)|. \]

Denote by \( \xi_1, \xi_2, \ldots, \xi_k \) the zeros (multiple zeros appearing as many times as their multiplicity) of \( Q_*(z) \) lying in \((-1, 1)\). We distinguish three cases:

**Case (i).** If \( k < n - 2 \), then for suitable choice of the real quantity \( \sigma \)

\[ Q(z) = Q_*(z) + \sigma(z - 1)^2 \prod_{j=1}^{k} (z - \xi_j) \]

is a polynomial of degree \( n \) with \( Q'(1) = Q_*(1) \), and

\[ \mu = \max_{-1 < x < 1} |Q(x)| < 1, \]

so that \( \mu^{-1}Q(z) \) belongs to \( \mathbb{C} \), but \( |\mu^{-1}Q'(1)| > |Q_*(1)| \). This contradicts (8).

**Case (ii).** If \( k = n - 1 \), we denote by \( \xi_n \) the (real) zero of \( Q_*(z) \) lying outside \((-1, 1)\). Note that if \( \xi_n \) were \( < -1 \) then the graph of \( P_*(1)Q_*(x)/Q_*(1) \) would cross that of \( P_*(x) \) at least \( n \) times on \([-1, 1]\). Hence the polynomial

\[ S(x) = P_*(x) - \frac{P'_*(1)}{Q'_*(1)} Q_*(x), \]

which is clearly \( \neq 0 \) would have all its zeros in \([-1, 1]\) which is a contradiction since \( S'(1) = 0 \). On the other hand, the same reasoning can be used to show that in the case \( \xi_n > 1 \) the largest critical point \( \eta \) of \( Q_*(x) \) cannot be larger than 1. But if \( \eta < 1 < \xi_n \), then for sufficiently small \( \varepsilon > 0 \) the polynomial \( Q_*(z + \varepsilon) \) still belongs to \( \mathbb{C} \), and \( |Q'_*(1 + \varepsilon)| > |Q'_*(1)| \), which contradicts (8).

**Case (iii).** If \( k = n \), then all the zeros of \( Q_*(z) \) lie in \((-1, 1)\), and at least one of them is of multiplicity at least two. Denote by \( p \) and \( q \) the coefficients...
of \( z^n \) in \( P_*(z) \) and \( Q_*(z) \) respectively. Without loss of generality we may assume \( q > 0 \). By Theorem 2 we have \( p > q \). Since all the zeros of \( Q_*(z) \) lie in \((-1, 1)\), \( Q'_*(x) \) is monotone increasing for \( x > 1 \). We must have \( Q'_*(1) = 1 \), because otherwise for appropriate \( \epsilon > 0 \) the polynomial \( Q_*(z + \epsilon) \) would contradict the extremal property of \( Q_*(z) \). Consequently, if

\[
U(z) = P_*(z) - Q_*(z),
\]

then

\[
U(1) = 0, \quad U'(1) < 0,
\]

and \( \text{sgn } U(x) = \text{sgn } (p - q) = +1 \) for \( x \to \infty \), so that \( U(x) \) has a zero on \((1, \infty)\). Furthermore, comparing the graphs of \( P_*(x) \) and \( (1 - \epsilon)Q_*(x) \) for \( \epsilon > 0 \) and letting \( \epsilon \) tend to zero, we see that \( U(x) \) has at least (and hence exactly) \( n - 1 \) zeros in \((-1, 1)\). Therefore, \( U(x) \neq 0 \) for \( x < -1 \). It follows that

\[
\text{sgn } U(x) = \text{sgn } (-1)^n(p - q) = (-1)^n \quad \text{for } x < -1,
\]

but

\[
\text{sgn } U(-1) = -\text{sgn } Q_*(-1) = (-1)^{n+1}\text{sgn } q = (-1)^{n+1},
\]

which is a contradiction.

Hence in any case \( |Q'_*(1)| = |P'_*(1)| \). Investigating the above three cases under this hypothesis, we obtain using similar reasonings that \( Q_*(x) = \pm P_*(x) \). This completes the proof of Theorem 1.

REFERENCES

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