STRUCTURALLY STABLE GRASSMANN TRANSFORMATIONS

BY

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ABSTRACT. A Grassmann transformation is a diffeomorphism on a Grassmann manifold which is induced by an $n \times n$ nonsingular matrix. In this paper the structurally stable Grassmann transformations are characterized to be the maps which are induced by matrices whose eigenvalues have distinct moduli. There is exactly one topological conjugacy class of complex structurally stable Grassmann transformations. For the real case the topological classification is determined by the ordering (relative to modulus) of the signs of the eigenvalues of the inducing matrix.

1. **Introduction and preliminaries.** Let $\mathrm{Diff}(M)$ be the space of diffeomorphisms on a compact manifold M. Two diffeomorphisms $f, g \in \mathrm{Diff}(M)$ are *topologically conjugate*, denoted $f \sim g$, if there exists a homeomorphism $h: M \to M$ such that $hf = gh$. This form of topological equivalence implies that the maps have similar qualitative properties. A diffeomorphism f is *structurally stable* provided there exists a neighborhood U of f in $\mathrm{Diff}(M)$ such that each $g \in U$ is topologically conjugate to f. This means that small perturbations of the function do not alter its topological behavior. Given a manifold M two of the fundamental problems in dynamical systems are:

(1) characterization of the structurally stable maps in $\mathrm{Diff}(M)$,

(2) classification up to topological conjugacy of a large subset of $\mathrm{Diff}(M)$.

In this paper we consider these questions for a subset of the set of diffeomorphisms on a certain class of compact manifolds. Before elaborating on this problem, we will give some basic definitions.

For the following definitions M is a compact manifold, $p \in M$, and $f \in \mathrm{Diff}(M)$. If $f(p) = p$ then p is a *fixed point* of f. If there exists a nonzero integer m such that $f^m(p) = p$, then p is said to be a *periodic point* of f. A point $x \in M$ is called a *nonwandering point* of f provided that for any neighborhood U of x there exists a nonzero integer m such that $f^m(U) \cap U \neq \varnothing$. The sets

of fixed points, periodic points, and nonwandering points for a diffeomorphism f are denoted respectively by Fix(f), Per(f), and $\Omega(f)$. Each of these sets is invariant and is preserved by a topological conjugacy (e.g., $h(\Omega(f)) = \Omega(hfh^{-1})$).

An invertible linear map on a Banach space is *hyperbolic* if and only if its spectrum is disjoint from the unit circle. If $p \in$ Fix(f) then p is a *hyperbolic fixed point* of f provided the derivative of f at p, $Df(p)$, is a hyperbolic linear map. Suppose M has metric d and p is a hyperbolic fixed point of f. The *stable manifold* of p, $W^s(p)$, $= \{x \in M \mid d(p, f^m(x)) \to 0$ as $m \to \infty\}$ and the *unstable manifold* of p, $W^u(p)$, $= \{x \in M \mid d(p, f^{-m}(x)) \to 0$ as $m \to \infty\}$. These manifolds are also preserved by conjugacies.

A periodic point p of period m is hyperbolic provided that it is a hyperbolic fixed point of f^m. For a hyperbolic periodic point p the stable and unstable manifolds are defined to be the stable and unstable manifolds of p under f^m.

A diffeomorphism f on a compact manifold M is said to be *Morse-Smale* provided it satisfies the following conditions:

(1) $\Omega(f)$ is finite;
(2) all periodic points are hyperbolic;
(3) for each $p, q \in \Omega(f)$, $W^s(p)$ and $W^u(q)$ have transversal intersection.

The first condition implies $\Omega(f) = $ Per(f). The following theorem is due to Palis and Smale [10].

THEOREM. *$f \in$ Diff(M) is Morse-Smale if and only if $\Omega(f)$ is finite and f is structurally stable.*

A *hyperbolic structure* on $\Omega(f)$ is a continuous splitting of the tangent bundle $T_{\Omega(f)}M$ of M restricted to $\Omega(f)$, $T_{\Omega(f)}M = E^s \oplus E^u$, invariant under the derivative Df, such that Df is contracting on E^s and expanding on E^u. A diffeomorphism f satisfies Smale's *Axiom* A if and only if $\Omega(f)$ has a hyperbolic structure and $\Omega(f)$ is the closure of Per(f).

Suppose f satisfies Axiom A. If $x \in \Omega(f)$ then the definitions of stable and unstable manifold can be generalized to

$$W^s(x) = \{y \in M \mid d(f^m(x), f^m(y)) \to 0 \text{ as } m \to \infty\}$$

and

$$W^u(x) = \{y \in M \mid d(f^{-m}(x), f^{-m}(y)) \to 0 \text{ as } m \to \infty\}.$$

If for all $x, y \in \Omega(f)$, $W^s(x)$ and $W^u(y)$ have transversal intersection then f is said to satisfy the *strong transversality condition*.

THEOREM. *If f satisfies Axiom A and the strong transversality condition then f is structurally stable.*

This theorem was conjectured by Smale and proven by Robbin and Robinson [11], [13]. It is unknown if the converse is true, but hyperbolicity of periodic points is a necessary condition for structural stability [3]. In this paper we will characterize the structurally stable maps in a certain set of diffeomorphisms. The following corollary will simplify this problem:

COROLLARY. *If f, g, $h \in \mathrm{Diff}(M)$ and h is a conjugacy for f and g then*
(1) *f is Morse-Smale if and only if g is Morse-Smale*;
(2) *f is structurally stable if and only if g is structurally stable.*

PROOF OF (1). Suppose f is Morse-Smale and $g = h \circ f \circ h^{-1}$. Then $\Omega(g)$ is finite since $\Omega(g) = h(\Omega(f))$. Suppose $g^m(p) = p$ and produce $q \in M$ such that $p = h(q)$ and $f^m(q) = q$.
Then

$$T_p g^m = T_p(h \circ f^m \circ h^{-1})$$

$$= T_{h(q)}(h \circ f^m \circ h^{-1})$$

$$= (T_q h)(T_q f^m)(T_q h)^{-1}.$$

Thus $T_p g^m$ and $T_q f^m$ have the same eigenvalues and p is a hyperbolic periodic point of g.

Suppose $h(p), h(q) \in \Omega(g)$ and $h(x) \in W_g^u(h(p)) \cap W_g^s(h(q))$. Then $x \in W_f^u(p) \cap W_f^s(q)$.

$$T_{h(x)} M = T_x h(T_x(M))$$

$$= T_x h(T_x W_f^u(p) + T_x W_f^s(q))$$

$$= (T_x h)(T_x W_f^u(p)) + (T_x h)(T_x W_f^s(q))$$

$$= T_{h(x)} h(W_f^u(p)) + T_{h(x)} h(W_f^s(q))$$

$$= T_{h(x)} W_g^u(h(p)) + T_{h(x)} W_g^s(h(q)).$$

Therefore g is Morse-Smale. The proof of part (2) is similar.

Let $G(k, n; F)$ denote the *Grassmann manifold* of k-dimensional subspaces of F^n where F is either **R** or **C**. $\mathrm{Gl}(n; F)$ will be the *general linear group* of nonsingular $n \times n$ matrices over F. Each $A \in \mathrm{Gl}(n; F)$ induces a (*real or complex*) *Grassmann transformation* $A_k \in \mathrm{Diff}(G(k, n; F))$. Let $\mathfrak{T}(k, n; F)$ be the manifold of Grassmann transformations on $G(k, n; F)$. Our goals are the following:
(1) Characterize and classify the structurally stable maps in $\mathfrak{T}(k, n; F)$.
(2) Analyze an open, dense set in $\mathfrak{T}(k, n; F)$.

THEOREM A. *Let $A \in \mathrm{Gl}(n; F)$. The following are equivalent:*
I. A_k *is Morse-Smale.*
II. A_k *is structurally stable.*
III. *The eigenvalues of A have distinct moduli.*

COROLLARY. *The structurally stable maps form an open, dense, submanifold in* $\mathfrak{T}(k, n; \mathbf{C})$ *and any two structurally stable maps are topologically conjugate.*

The above results (whose proofs are contained in the following section) provide a complete description of the complex case. The structurally stable real Grassmann transformations are neither dense in $\mathfrak{T}(k, n; \mathbf{R})$ nor are they all topologically conjugate to each other. In classifying these maps we use a modification of Smale's definition of labelled diagram. In Theorem C we obtain a classification based on the ordering (relative to modulus) of the signs of the eigenvalues of the inducing matrix.

Suppose $A, B, P \in \mathrm{Gl}(n; F)$ and $B = PAP^{-1}$. Then $B_k = P_k A_k (P_k)^{-1}$ and thus the similarity transformation induces a differentiable conjugacy. Consequently in characterizing structurally stable maps and in classifying $\mathfrak{T}(k, n; F)$ it will suffice to consider matrices in the canonical form described below.

An *irreducible block* is, for some $j \geqslant 1$, a $j \times j$ matrix in one of the following two forms:

$$
\begin{pmatrix}
a & 1 & & & \\
 & a & 1 & & \\
 & & \ddots & \ddots & \\
 & & & \ddots & 1 \\
 & & & & a
\end{pmatrix}
\qquad
\begin{pmatrix}
\alpha & \beta & 1 & 0 & & & & \\
-\beta & \alpha & 0 & 1 & & & & \\
 & & \alpha & \beta & 1 & 0 & & \\
 & & -\beta & \alpha & 0 & 1 & & \\
 & & & & \cdot & & \ddots & \\
 & & & & & \cdot & 1 & 0 \\
 & & & & & \cdot & 0 & 1 \\
 & & & & & & \alpha & \beta \\
 & & & & & & -\beta & \alpha
\end{pmatrix}
\qquad
\begin{array}{l}
a \in F \\
\alpha, \beta \in \mathbf{R} \\
a \neq 0 \\
|\alpha + \beta i| \neq 0
\end{array}
$$

CANONICAL FORM LEMMA. *If $A \in \mathrm{Gl}(n; F)$ then A is similar to a matrix of the form*

$$
\begin{pmatrix}
\sigma_1 & & \\
 & \ddots & \\
 & & \sigma_m
\end{pmatrix}
$$

where for each i:
(1) *all eigenvalues of σ_i have the same modulus $|\sigma_i|$,*
(2) $|\sigma_i| < |\sigma_{i+1}|$,

(3) σ_i *consists of irreducible blocks along its diagonal and 0's elsewhere.*

Matrices in this form will be said to be in *canonical form.*

N. Kuiper has obtained a virtually complete classification of $\mathfrak{T}(1, n; \mathbf{R})$ [6]. We state his results in terms of canonical forms.

THEOREM. *If two real projective transformations A_1 and B_1 can be put in the canonical forms $A = \sigma_1 \oplus \cdots \oplus \sigma_m$ and $B = \tau_1 \oplus \cdots \oplus \tau_m$ and there exist $\lambda_1, \ldots, \lambda_m > 0$ such that, for each i, $\sigma_i = \lambda_i \tau_i$, then $A_1 \sim B_1$.*

Kuiper has also proven the converse of this theorem for the transformations in $\mathfrak{T}(1, n; \mathbf{R})$ whose periodic points all have period q or $2q$ where q is a prime power or 1.

2. Structurally stable Grassmann transformations. The purpose of this section is to characterize and classify the structurally stable diffeomorphisms in $\mathfrak{T}(k, n; F)$.

DEFINITION. Let $c_{r_1, \ldots, r_k}$ be the coordinate plane spanned by the $r_1, \ldots, r_k$ axes in F^n. We will assume $r_i < r_{i+1}$.

If A is a diagonal matrix, then each $c_{r_1, \ldots, r_k}$ is a fixed point of A_k. We will compute the derivative of A_k at $c_{r_1, \ldots, r_k}$.

Consider the set of $n \times k$ matrices: $\mathcal{C} = \mathcal{C}_{r_1, \ldots, r_k} = \{\{x_{ij}\} | x_{r_i i} = 1$ and $x_{r_i j} = 0$ if $i \neq j\}$. The column vectors of each $\{x_{ij}\}$ span a k-dimensional subspace of F^n. $\mathcal{C}$ is a chart for $c_{r_1, \ldots, r_k}$ with the coordinates of $\{x_{ij}\}$ being the $k(n - k)$ values of the matrix in the non-r_i rows. We will call $\mathcal{C}$ the *standard chart* for $c_{r_1, \ldots, r_k}$.

EXAMPLE. $A = \mathrm{dg}(1\ 2\ 3\ 4)$,

$$
\mathcal{C}_{1,3} = \left\{ \begin{pmatrix} 1 & 0 \\ x_1 & x_2 \\ 0 & 1 \\ x_3 & x_4 \end{pmatrix} \right\}.
$$

LEMMA. *Let $A = \mathrm{dg}(a_1 \cdots a_n)$ and $c_{r_1, \ldots, r_k}$ be a coordinate plane in F^n. Then in $\mathcal{C}$-coordinates the derivative of A_k at $c_{r_1, \ldots, r_k}$ is*

$$
DA_k(c_{r_1, \ldots, r_k}) = \mathrm{dg} \left\{ \frac{a_i}{a_{r_l}} \right\}^{1 \leqslant i \leqslant n}_{1 \leqslant j \leqslant k} \quad i \neq r_l, l = 1, \ldots, k.
$$

Note. The derivative is a diagonal matrix consisting of all quotients of $\{a_1, \ldots, a_n\}$ where the subscript of the denominator element is an r_j from $c_{r_1, \ldots, r_k}$ and the subscript of the numerator is not an r_j.

EXAMPLE. $A = \mathrm{dg}(1\ 2\ 3\ 4)$,

$$DA_2(c_{1,3}) = \mathrm{dg}(2\ 2/3\ 4\ 4/3).$$

PROOF OF LEMMA.

$$A(\{x_{ij}\}) = \begin{pmatrix} a_1\,x_1 \\ a_2\,x_2 \\ \vdots \\ \vdots \\ a_n\,x_n \end{pmatrix},$$

x_i denotes ith row vector of $\{x_{ij}\}$. Then $(A(\{x_{ij}\}))_{r_i i} = a_{r_i}$ and $(A(\{x_{ij}\}))_{r_i j} = 0$ if $i \neq j$.

Let $\{y_{ij}\}$ be the matrix which results from dividing the ith column vector $(i = 1, 2, \ldots, k)$ of $A(\{x_{ij}\})$ by a_{r_i}.

Then $y_{r_i i} = 1$ and $y_{r_i j} = 0$ if $i \neq j$. Also $y_{lj} = a_l/a_{r_j}$ if $i \neq r_l$, $l = 1, \ldots, k$. So $\{y_{ij}\} \in \mathcal{C}$.

Each coordinate x_{ij} of $\{x_{ij}\}$ is mapped by A_k to the coordinate $(a_i/a_{r_j})x_{ij}$. Q.E.D.

In proving the lemma we have also shown that if A is diagonal and $\mathcal{C}$ is a standard chart then $A_k(\mathcal{C}) = \mathcal{C}$ and A_k is linear in standard coordinates on $\mathcal{C}$. Thus, if p is a hyperbolic fixed point in $\mathcal{C}$, $W^s(p) \subset \mathcal{C}$ and $W^u(p) \subset \mathcal{C}$.

LEMMA. *If* $p \in G(k, n; F)$, $A = \mathrm{dg}(a_1 \cdots a_n)$, *and* $|a_i| < |a_{i+1}|$ *then there exist coordinate k-planes* $q, q' \in G(k, n; F)$ *such that p is forward asymptotic to q and backward asymptotic to q'.*

PROOF. Let $p \in G(k, n; F)$.

If $\vec{v} = (v_1, \ldots, v_n) \neq \vec{0}$, define $s(\vec{v}) = \max\{j \mid v_j \neq 0\}$. Choose a basis $\{\vec{v}^i\}_{i=1}^k$ for p such that $s(\vec{v}^i) < s(\vec{v}^{i+1})$. Then $A^m(\vec{v}^i) \to c_{s(\vec{v}^i)}$ as $m \to \infty$. So $(A_k)^m(p) \to c_{s(\vec{v}^i), \ldots, s(\vec{v}^k)}$ as $m \to \infty$.

Similarly p is backward asymptotic to a coordinate plane.

The following theorem characterizes the structurally stable diffeomorphisms in $\mathfrak{T}(k, n; \mathbf{R})$ and $\mathfrak{T}(k, n; \mathbf{C})$.

THEOREM A. *Let* $A \in \mathrm{Gl}(n; F)$. *The following are equivalent*:
I. A_k *is Morse-Smale.*
II. A_k *is structurally stable.*
III. *The eigenvalues of A have distinct moduli.*

COROLLARY 1. A_k *is structurally stable if and only if A is similar to a diagonal matrix whose entries are of increasing modulus.*

PROOF OF THEOREM A. It suffices to consider canonical forms.
I $\Rightarrow$ II follows from the theorem of Palis and Smale.
III $\Rightarrow$ I. To show A_k is Morse-Smale we must prove:
(1) $\Omega(A_k)$ is finite.

(2) The periodic points of A_k are hyperbolic.

(3) For each $p, q \in \Omega(A_k)$, $W^u(p)$ and $W^s(q)$ have transversal intersection.

Suppose $A = \mathrm{dg}(a_1 \cdots a_n)$ and $0 < |a_i| < |a_{i+1}|$. Let $p = c_{r_1, \ldots, r_k}$ be a coordinate plane. Then p is a fixed point of A_k.

By the lemma

$$DA_k(p) = \mathrm{dg}\left\{ \frac{a_i}{a_{r_j}} \right\} \begin{matrix} 1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant k \end{matrix} i \neq r_l, \qquad l = 1, \ldots, k.$$

Since $|a_i| \neq |a_{r_j}|$ (if $i \neq r_j$), p is a hyperbolic fixed point of A_k.

Pick $p \in G(k, n; F)$ and produce $q = c_{r_1, \ldots, r_k}$ such that $p \in W^s(q)$. If $\mathcal{C}$ is a standard chart for q, then $p \in \mathcal{C}$. But A_k is linear in standard coordinates on the invariant chart $\mathcal{C}$. Thus $\Omega(A_k) \cap \mathcal{C} = \{q\}$. So if p is not a coordinate plane, then p wanders.

It remains to prove the transversality condition.

Let $p = c_{r_1, \ldots, r_k}$ and $q = c_{s_1, \ldots, s_k}$ and let $\mathcal{C}$ be the standard chart for p, i.e., $\mathcal{C} = \{\{x_{ij}\} | x_{r_i i} = 1 \text{ and } x_{r_i j} = 0 \text{ if } i \neq j\}$ where each $\{x_{ij}\}$ is an $n \times k$ matrix.

$$W^u(p) = \{\{x_{ij}\} | \{x_{ij}\} \in \mathcal{C} \text{ and } x_{ij} = 0 \text{ if } i < r_j\} \subset \mathcal{C}.$$

Let $\{b_{ij}\} \in W^u(p) \cap W^s(q)$ and suppose $\{b_{ij}\}$ is in $\mathcal{C}$ coordinates.

For transversality it suffices to show:

$$W^s(q) \supset \{\{x_{ij}\} | \{x_{ij}\} \in \mathcal{C} \text{ and } x_{ij} = b_{ij} \text{ if } i \geqslant r_j\}.$$

Suppose $\{x_{ij}\}$ is in the above set. Let x_j and b_j denote the jth column vectors of $\{x_{ij}\}$ and $\{b_{ij}\}$.

Then $\{b_j\}_{j=1}^k$ is a basis for $\{b_{ij}\} \in G(k, n; F)$. Since $\{b_{ij}\} \in W^s(q)$, there exists another basis $\{f_j\}_{j=1}^k$ such that $A^m(f_j) \to c_{s_j}$ as $m \to \infty$.

Each f_j can be written as a linear combination,

$$f_j = \sum_{l=1}^{m_j} \beta_{lj} b_l \quad \text{where } \beta_{m_j j} \neq 0.$$

Since $\{b_{ij}\} \in \mathcal{C}$, $b_{r_{m_j} m_j} = 1$ and $b_{r_{m_j} l} = 0$ if $l \neq m_j$. So the r_{m_j} coordinate of the vector f_j is

$$(f_j)_{r_{m_j}} = \sum_{l=1}^{m_j} \beta_{lj} b_{r_{m_j} l} = \beta_{m_j j} \neq 0.$$

In general a nonzero vector is forward asymptotic under A to c_{s_j} if and only if it is nonzero in the s_j place and zero beyond the s_j place.

Therefore $s_j \geqslant r_{m_j}$.

Let $y_j = \sum_{l=1}^{m_j} \beta_{lj} x_l$. Then $\{y_j\}_{j=1}^k$ is a basis for $\{x_{ij}\}$.

The column vectors x_l and b_l have the same coordinates in the r_l place and

below the r_j place. So f_j and y_j have the same coordinates in the r_{m_j} place and below the r_{m_j} place. Since $s_j \geqslant r_{m_j}$, f_j and y_j are both forward asymptotic to c_{s_j}.

Therefore $\{x_{ij}\} \in W^s(q)$.

To prove II $\Rightarrow$ III, suppose condition III is not satisfied.

Case 1. A is diagonal and $|a_i| = |a_{i+1}|$ for some i. Let $p = c_{r_1, \ldots, r_k}$ be a coordinate plane with $r_j = i$, but $r_{j+1} \neq i + 1$.

By the lemma $DA_k(p)$ is a diagonal matrix one of whose entries is a_{i+1}/a_i. Thus p is a nonhyperbolic fixed point. So A_k is not structurally stable.

Case 2. $F = \mathbf{R}$ and A has complex eigenvalues, but all canonical blocks are trivial (i.e., no ones on the superdiagonal).

A can be approximated by a matrix B which is also of the form of Case 2, but whose complex eigenvalues are roots of real numbers. Then for some m, B^m is diagonal and at least two of its eigenvalues are of the same modulus.

By Case 1, $(B_k)^m$ has a nonhyperbolic fixed point. So B_k has a nonhyperbolic periodic point. Thus B_k is not structurally stable. Since B_k approximates A_k, A_k is not structurally stable.

Case 3. All nontrivial canonical blocks are triangular.

We will approximate A by a matrix whose canonical form is of either Case 1 or 2.

Suppose A contains the m-block (a may be real or complex)

$$\begin{pmatrix} a & 1 & & & \bigcirc \\ & a & 1 & \cdot & \cdot \\ & & \cdot & \cdot & \cdot \\ \bigcirc & & & \cdot & a & 1 \\ & & & & & a \end{pmatrix}.$$

Perturb this block to

$$\begin{pmatrix} a + \varepsilon_1 & 1 & & & & \bigcirc \\ & a + \varepsilon_2 & 1 & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot & \\ \bigcirc & & & a + \varepsilon_{m-2} & 1 & \\ & & & & a & 1 \\ & & & & -\varepsilon_m^2 & a \end{pmatrix}$$

where the ε_i's are small positive numbers.

If $F = \mathbf{R}$ the canonical form for the perturbation is

$$\begin{pmatrix} a + \varepsilon_1 & & & & & \bigcirc \\ & a + \varepsilon_2 & \cdot & & & \\ & & \cdot & a + \varepsilon_{m-2} & & \\ & & & & a & \varepsilon_m \\ \bigcirc & & & & -\varepsilon_m & a \end{pmatrix}.$$

If $F = \mathbf{C}$ the canonical form for the perturbation is (the order may actually be different)

$$\begin{pmatrix} a + \varepsilon_1 & & & & & \bigcirc \\ & a + \varepsilon_2 & \cdot & & & \\ \bigcirc & & \cdot & \cdot & a - \varepsilon_m i & \\ & & & & & a + \varepsilon_m i \end{pmatrix}.$$

In the above manner, perturb each block of the form

$$\begin{pmatrix} a & \cdot & 1 & \cdot & \bigcirc \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & 1 \\ \bigcirc & & & a & 1 \\ & & & & a \end{pmatrix}.$$

Let B be the matrix which results from the perturbations and let C be the canonical form for B. Then C is of the form of Case 1 or 2 and thus C_k is not structurally stable. Since C_k is differentiably conjugate to B_k and B_k approximates A_k, A_k is not structurally stable.

Case 4. $F = \mathbf{R}$ and A contains nontrivial complex canonical blocks.

Consider the canonical $2m$-block

$$\begin{pmatrix} a & b & 1 & 0 & & & & & \\ -b & a & 0 & 1 & & & \bigcirc & & \\ & & a & b & 1 & 0 & & & \\ & & -b & a & 0 & 1 & & & \\ & & & & \cdot & & \cdot & & \\ & & & & & \cdot & & \cdot & \\ & & & & & & a & b & 1 & 0 \\ \bigcirc & & & & & & -b & a & 0 & 1 \\ & & & & & & & & a & b \\ & & & & & & & & -b & a \end{pmatrix}.$$

Perturb this block to

$$\begin{pmatrix} a + \varepsilon_1 & b & 1 & 0 & & & & & \\ -b & a + \varepsilon_1 & 0 & 1 & & & & \bigcirc & \\ & & a + \varepsilon_2 & b & 1 & 0 & \cdot & & \\ & & -b & a + \varepsilon_2 & 0 & 1 & & \cdot & \\ & & & & \cdot & & \cdot & & \\ & & & & & \cdot & \cdot & & \\ & & & & & & a + \varepsilon_{m-1} & b & 1 & 0 \\ \bigcirc & & & & & & -b & a + \varepsilon_{m-1} & 0 & 1 \\ & & & & & & & & a & b \\ & & & & & & & & -b & a \end{pmatrix}$$

where the ε_i's are small distinct positive numbers. The canonical form for the perturbation is

$$
\begin{pmatrix}
a + \varepsilon_1 & b \\
-b & a + \varepsilon_1 \\
& & a + \varepsilon_2 & b \\
& & -b & a + \varepsilon_2 & & & \bigcirc \\
& & & & \ddots \\
& & & & & a + \varepsilon_{m-1} & b \\
& & & & & -b & a + \varepsilon_{m-1} \\
& & \bigcirc & & & & & a & b \\
& & & & & & & -b & a
\end{pmatrix}.
$$

Perturbing each nontrivial complex block in the above manner yields a matrix whose canonical form is that of either Case 2 or Case 3. Therefore A_k is not structurally stable.   Q.E.D.

Let $\mathcal{S}(k, n; F)$ denote the set of structurally stable diffeomorphisms in $\mathcal{T}(k, n; F)$. For the remainder of this section we shall be concerned with the classification of $\mathcal{S}(k, n; F)$ up to topological conjugacy.

LEMMA. *$A_k = B_k$ if and only if $A = \lambda B$ for some $0 \neq \lambda \in F$.*

PROOF. If $A = \lambda B$ then clearly $A_k = B_k$. To prove the "only if" part, we first consider the case where $B = I$.

Suppose $A_k = I_k$ and $A = \{a_{ij}\}$.

Let e_i be the vector $(0, \ldots, 0, 1, 0, \ldots, 0)$ where the 1 occurs in the ith place. Then $A(\langle e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_{k+1} \rangle) = \langle e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_{k+1} \rangle$ so

$$
A(e_{k+1}) \in \bigcap_{i=1}^{k} \langle e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_{k+1} \rangle = \langle e_{k+1} \rangle.
$$

Similarly A maps each e_i into $\langle e_i \rangle$. Therefore A is a diagonal matrix. If two of the diagonal entries are not equal, then there is a coordinate plane at which the derivative of A_k is not the identity. Thus there exists $0 \neq \lambda \in F$ such that $A = \lambda I$.

Suppose $A_k = B_k$. Then $(AB^{-1})_k = A_k(B_k)^{-1} = I_k$. Produce $0 \neq \lambda \in F$ such that $AB^{-1} = \lambda I$. Then $A = \lambda B$.   Q.E.D.

If $Z = \{\lambda I | \lambda \in F\}$, then $\mathrm{Gl}(n; F)/Z$ is a manifold. Define Φ: $\mathrm{Gl}(n; F)/Z \to \mathcal{T}(k, n; F)$ by $\Phi([A]) = A_k$. The map Φ is a diffeomorphism (it is injective and well defined by the preceding lemma). Let $S(n; F)$ be the set of all $A \in \mathrm{Gl}(n; F)$ such that the eigenvalues of A have distinct moduli. The set $S(n; F)$ is open in $\mathrm{Gl}(n; F)$ and thus projects to an open set in $\mathrm{Gl}(n; F)/Z$. Since $\mathcal{S}(k, n; F)$ is the image under Φ of this projection, $\mathcal{S}(k, n; F)$ is an open

submanifold of $\mathfrak{T}(k, n; F)$. If $F = \mathbf{C}$ then $S(n; \mathbf{C})$ is dense in $\text{Gl}(n; \mathbf{C})$ and thus $\mathfrak{S}(k, n; \mathbf{C})$ is dense in $\mathfrak{T}(k, n; \mathbf{C})$.

DEFINITION. If $A_k \in \mathfrak{S}(k, n; \mathbf{R})$ define $\text{Sgn}(A): \{1, \ldots, n\} \to \{-1, 1\}$ to be the finite sequence given by $\text{Sgn}(A)(i) = \text{sgn}(\lambda_i)$ where λ_i is the eigenvalue of A of ith largest modulus.

Then $\text{Sgn}(A)$ gives an ordering of the signs of A's eigenvalues. We now state a corollary to Theorem A.

COROLLARY 2. *If* A_k, $B_k \in \mathfrak{S}(k, n; \mathbf{R})$ *and either* $\text{Sgn}(A) = \text{Sgn}(B)$ *or* $\text{Sgn}(A) = -\text{Sgn}(B)$ *then* $A_k \sim B_k$.

Corollary 2 does not give a complete classification of $\mathfrak{S}(k, n; \mathbf{R})$. Later in this section we will show that if $k > 1$, there exist A_k, $B_k \in \mathfrak{S}(k, 2k; \mathbf{R})$ with $A_k \sim B_k$ but $\text{Sgn}(A) \neq \text{Sgn}(B)$ and $\text{Sgn}(A) \neq -\text{Sgn}(B)$. For structurally stable transformations on complex Grassmann manifolds the situation is simpler.

COROLLARY 3. $\mathfrak{S}(k, n; \mathbf{C})$ *is an open, dense submanifold of* $\mathfrak{T}(k, n; \mathbf{C})$ *and there is exactly one topological conjugacy class in* $\mathfrak{S}(k, n; \mathbf{C})$.

PROOF OF COROLLARY 2. We follow the method of Kuiper [6]. Suppose

$$A = \text{dg}(a_1, \ldots, a_n), \qquad |a_j| < |a_{j+1}|,$$
$$B = \text{dg}(b_1, \ldots, b_n), \qquad |b_j| < |b_{j+1}|,$$

and $\text{Sgn}(A) = \text{Sgn}(B)$.

LEMMA. *If* $|a_{j-1}| < \beta |a_j|$ *then*

$$A_k = (\text{dg}(a_1, \ldots, a_n))_k \sim (\text{dg}(a_1, \ldots, a_{j-1}, \beta a_j, \beta a_{j+1}, \ldots, \beta a_n))_k.$$

Before proving the lemma we will complete the proof of the corollary. Produce $\beta_1, \ldots, \beta_n > 0$ such that

$$b_1 = \beta_1 a_1,$$
$$b_2 = \beta_1 \beta_2 a_2,$$
$$\vdots$$
$$b_n = \beta_1 \beta_2 \cdots \beta_n a_n;$$

then applying the lemma:

$$A_k = (\mathrm{dg}(a_1, \ldots, a_n))_k$$

$$= (\mathrm{dg}(\beta_1 a_1, \ldots, \beta_1 a_n))_k$$

$$= (\mathrm{dg}(b_1, \beta_1 \alpha_2, \ldots, \beta_1 a_n))_k$$

$$\sim (\mathrm{dg}(b_1, \beta_1 \beta_2 a_2, \ldots, \beta_1 \beta_2 a_n))_k$$

$$\text{since } |b_1| < |b_2| = |\beta_1 \beta_2 a_2| = \beta_2 |\beta_1 a_2|$$

$$= (\mathrm{dg}(b_1, b_2, \ldots, \beta_1 \beta_2 a_n))_k$$

$$\vdots$$

$$\sim (\mathrm{dg}(b_1, b_2, \ldots, b_{n-1}, \beta_1 \beta_2 \cdots \beta_n a_n))_k$$

$$= (\mathrm{dg}(b_1, \ldots, b_n))_k$$

$$= B_k.$$

If $\mathrm{Sgn}(A) = -\mathrm{Sgn}(B)$, then $A_k \sim (-B)_k = B_k$.

It remains to prove the lemma.

Suppose $|a_{j-1}| < \beta|a_j|$. For each real number t between 1 and β, define $A^t = \mathrm{dg}(a_1, \ldots, a_{j-1}, ta_j, ta_{j+1}, \ldots, ta_n)$.

Note. $A^1 = A$ and $(A^t)_k \in \mathbb{S}(k, n; \mathbf{R})$.

Thus $\{(A^t)_k\}$ is a path of structurally stable diffeomorphisms between the two diffeomorphisms which we want to show are conjugate. Since structural stability is an open condition, the two diffeomorphisms are topologically conjugate.

In proving Corollary 2 we have actually shown that if $\mathrm{Sgn}(A) = \mathrm{Sgn}(B)$ then A_k and B_k are in the same path component of $\mathbb{S}(k, n; \mathbf{R})$. The proof of the second part of Corollary 3 is similar and amounts to showing that $\mathbb{S}(k, n; \mathbf{C})$ is path connected. The hypothesis of the lemma is changed to $|a_{j-1}| < |\beta a_j|$ where β is a complex number. To prove the new lemma, define A^t by choosing a path of t's in $\mathbf{C}$ between one and β such that the modulus of each point on the path is between 1 and $|\beta|$.

For the remainder of this section we shall be concerned with the topological classification of $\mathbb{S}(k, n; \mathbf{R})$. We begin by showing the relationship between $\mathbb{S}(k, n; \mathbf{R})$ and $\mathbb{S}(n - k, n; \mathbf{R})$. Recall that if $A_k \in \mathbb{S}(k, n; \mathbf{R})$ then $\mathrm{Sgn}(A)$: $\{1, \ldots, n\} \to \{-1, 1\}$ gives the ordering of the signs of the eigenvalues (from the smallest to largest). Note that $\mathrm{Sgn}(A^{-1})$ merely reverses the ordering of $\mathrm{Sgn}(A)$.

THEOREM B. *If* $A_k \in \mathbb{S}(k, n; \mathbf{R})$ *and* $B_{n-k} \in \mathbb{S}(n - k, n; \mathbf{R})$ *and either* $\mathrm{Sgn}(A^{-1}) = \mathrm{Sgn}(B)$ *or* $\mathrm{Sgn}(A^{-1}) = -\mathrm{Sgn}(B)$ *then* $A_k \sim B_{n-k}$.

EXAMPLES.

$$(\mathrm{dg}(1\ 2\ 3\ 4\ -5))_2 \sim (\mathrm{dg}(-1\ 2\ 3\ 4\ 5))_3,$$

$$(\mathrm{dg}(1\ -2\ 3\ 4))_2 \sim (\mathrm{dg}(1\ 2\ -3\ 4))_2.$$

PROOF. We can assume $A = \mathrm{dg}(a_1 \cdots a_n)$, $|a_i| < |a_{i+1}|$. Let $d_k \colon G(k, n; \mathbf{R})$ $\to G(n - k, n; \mathbf{R})$ be the diffeomorphism that maps a k-dimensional subspace of $\mathbf{R}^n$ to its $(n - k)$-dimensional orthogonal complement.

LEMMA. d_k *is a conjugacy for* A_k *and* $(A^{-1})_{n-k}$.

PROOF OF LEMMA. $A^{-1} = \mathrm{dg}(a_1^{-1} \cdots a_n^{-1})$.
Let $p \in G(k, n; \mathbf{R})$ and suppose $\vec{u}$ is a vector in p. Suppose $\vec{r}$ is a vector in $d_k(p) \in G(n - k, n; \mathbf{R})$. If $\vec{u} = (u_1, \ldots, u_n)$ and $\vec{r} = (r_1, \ldots, r_n)$ then $A(\vec{u}) = (a_1 u_1, \ldots, a_n u_n)$ and $A^{-1}(\vec{r}) = (a_1^{-1} r_1, \ldots, a_n^{-1} r_n)$, and $\vec{u} \cdot \vec{r} = 0$.

$$A(\vec{u}) \cdot A^{-1}(\vec{r}) = (u_1 r_1, \ldots, u_n r_n) = \vec{u} \cdot \vec{r} = 0.$$

So A applied to a vector $\vec{u}$ in p is orthogonal to A^{-1} applied to a vector which is orthogonal to $\vec{u}$. But the choice of vectors was arbitrary. Therefore the diagram commutes:

$$
\begin{array}{ccc}
G(k,\ n;\ \mathbf{R}) & \xrightarrow{\ \ A_k\ \ } & G(k,\ n;\ \mathbf{R}) \\
\downarrow{\scriptstyle d_k} & & \downarrow{\scriptstyle d_k} \\
G(n - k,\ n;\ \mathbf{R}) & \xrightarrow{(A^{-1})_{n-k}} & G(n - k,\ n;\ \mathbf{R})
\end{array}
$$

Returning to the theorem, suppose $\mathrm{Sgn}(A^{-1}) = \mathrm{Sgn}(B)$. Then by Corollary 2, $(A^{-1})_{n-k} \sim B_{n-k}$. So by the lemma, $A_k \sim B_{n-k}$. If $\mathrm{Sgn}(A^{-1}) = -\mathrm{Sgn}(B)$, then $A_k \sim (-B)_{n-k} = B_{n-k}$.

For $n = 2k$ Theorem B gives some conditions sufficient for the topological equivalence of elements in $\mathcal{S}(k, n; \mathbf{R})$. We shall now show that these conditions combined with those given in Corollary 2 are necessary. This will complete the classification of $\mathcal{S}(k, n; \mathbf{R})$.

For the remainder of this chapter assume unless stated otherwise that A_k, $B_k \in \mathcal{S}(k, n; \mathbf{R})$ and that A and B are in canonical form. Since A_k and B_k are also Morse-Smale we can define their diagrams.

If $p, q \in \mathrm{Fix}(A_k)$ then $p \leqslant q$ if and only if $W^u(p) \cap W^s(q) \neq \varnothing$. If $p \leqslant q$ and $p \neq q$, then $p < q$. A *diagram* of A_k consists of vertices corresponding to fixed points and oriented segments from vertex p to q (denoted $p \to q$) when $p < q$, but there is no q' such that $p < q' < q$. If $p \to q$ then p is said to *give to* q and q is said to *receive from* p.

The following lemma describes the diagram structure of a canonical form

structurally stable transformation. Lemma a is also valid for $F = \mathbf{C}$.

LEMMA a. (1) $c_{r_1,\ldots,r_k} \leqslant c_{s_1,\ldots,s_k}$ *if and only if, for each i, $r_i \leqslant s_i$.*

(2) $c_{r_1,\ldots,r_k} \to c_{s_1,\ldots,s_k}$ *if and only if there exists $1 \leqslant j \leqslant k$ such that $r_j = s_j - 1$ and $r_i = s_i$ for $i \neq j$.*

(3) *A vertex can give to at most k other vertices and can receive from at most k other vertices.*

PROOF. For (1) look at $W^u(c_{r_1,\ldots,r_k}) \subset \mathcal{C}$ where $\mathcal{C}$ is the standard chart for $c_{r_1,\ldots,r_k}$. Consider what happens to a point in $W^u(c_{r_1,\ldots,r_k})$ under forward iteration.

(3) follows from (2) which follows from (1).

LEMMA b. *If $p \to q$, then $[W^u(p) \cap W^s(q)] \cup \{p,q\}$ is a circle which is invariant under A_k. Also p and q are the only fixed points of $A_k|([W^u(p) \cap W^s(q)] \cup \{p,q\})$.*

The diagrams that we have defined are special cases of the Morse-Smale diagrams which Smale describes in his survey article [15]. Our labelling of the diagrams will be different from that of Smale.

Label $p \to q$ with a "+" ("−") if A_k preserves (reverses) orientation on $([W^u(p) \cap W^s(q)] \cap \{p,q\})$. Lemma a implies that without labelling, the diagram is completely determined by n and k. The diagrams of A_k and B_k will be defined to be equal provided that they have the same labelling.

LEMMA c. (1) *The labelling of $c_{r_1,\ldots,r_k} \to c_{r_1,\ldots,r_{i-1},r_i+1,\ldots,r_k}$ is the sign of $a_{r_i} a_{r_i+1}$.*

(2) *A_k and B_k have equal diagrams if and only if either $\mathrm{Sgn}(A) = \mathrm{Sgn}(B)$ or $\mathrm{Sgn}(A) = -\mathrm{Sgn}(B)$.*

DEFINITION. The *jth row* in the diagram of A_k is the set of fixed points with stable manifolds of dimension j.
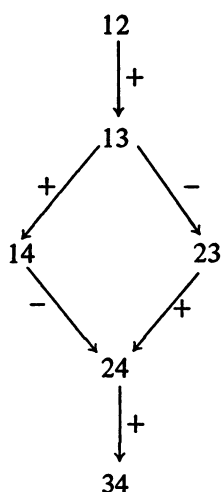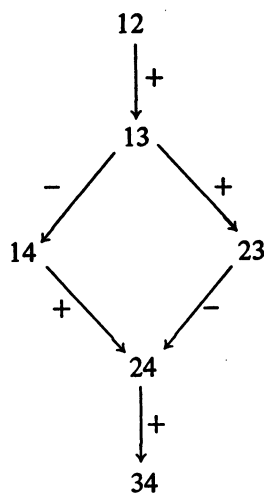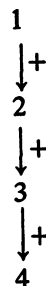
LEMMA d. *The fixed point $c_{r_1,\ldots,r_k}$ is in row $(\sum_{i=1}^{k} r_i - k(k+1)/2)$.*

DEFINITION. $H: \mathrm{Fix}(A_k) \to \mathrm{Fix}(B_k)$ is a *diagram isomorphism* if and only if:

(a) $\dim(W^s(H(p))) = \dim(W^s(p))$ for any $p \in \mathrm{Fix}(A_k)$.

(b) $p \to q \Leftrightarrow H(p) \to H(q)$.

(c) If $p \to q$ then $\mathrm{sgn}\{p,q\} = \mathrm{sgn}\{H(p), H(q)\}$ where $\mathrm{sgn}\{p,q\}$ is the sign (+ or −) of the diagram arrow.

For the definition of diagram isomorphism we will not require A and B to be in canonical form. If A and B are in canonical form then $\mathrm{Fix}(A_k) = \mathrm{Fix}(B_k)$. In this case we shall say that the diagram isomorphism is a *diagram automorphism* on the diagram of A_k.

EXAMPLES.

(1) $(dg(-1\ 2\ 3\ 4))_2$

12
↓ +
13
+   −
14   23
−   +
24
↓ +
34

(2) $(dg(1\ 2\ 3\ -4))_2$

12
↓ +
13
−   +
14   23
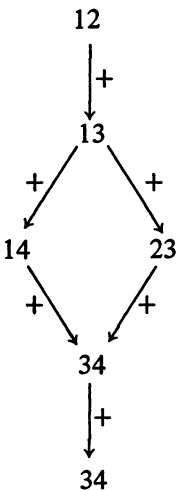+   −
24
↓ +
34

(3) $(dg(1\ 2\ 3\ 4))_1$

1
↓ +
2
↓ +
3
↓ +
4

The diagrams of examples (1) and (2) are isomorphic. The diagram isomorphism is also a diagram automorphism.

(4) $(dg(-1\ 2\ 3\ 4\ 5))_2 \in \mathcal{S}(2,\ 5;\ \mathbf{R})$

12
↓ +
13
+   −
14   23
+   −   +
15   24
−   +   +
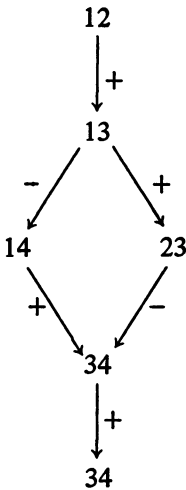25   34
+   +
35
↓ +
45
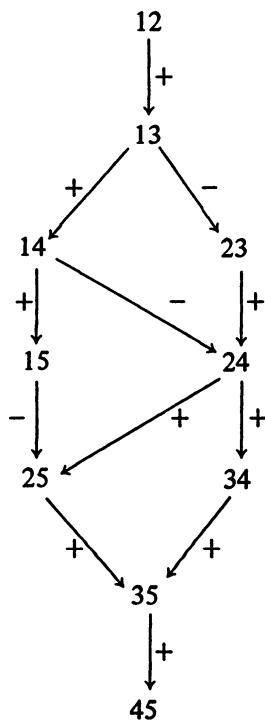
Note. The only diagram automorphism on the diagram of example (4) is the identity.

We shall use the diagrams to distinguish maps which are not topologically equivalent. We now give an example of two diffeomorphisms in $\mathcal{S}(2, 4; \mathbf{R})$ whose diagrams are not isomorphic. Using Smale's definition of labelling, these same maps have diagrams that are equal. The diffeomorphisms are not conjugate.
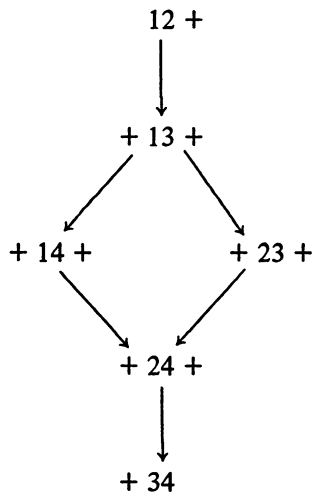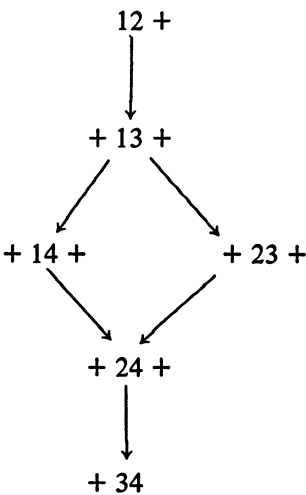
$(dg(1\ 2\ 3\ 4))_2$ $(dg(1\ 2\ 3\ -4))_2$

```
       12                              12
       |+                              |+
       v                               v
       13                              13
    +/   \+                         -/    \+
    v      v                        v       v
   14      23                      14        23
    +\    /+                        +\      /-
      v  v                            v    v
       34                              34
       |+                              |+
       v                               v
       34                              34
```

Same maps with Smale diagrams

               dim W^s dim W^u

```
     12 +              0   4              12 +
      |                                    |
      v                                    v
   + 13 +              1   3            + 13 +
   /    \                               /     \
  v      v                             v       v
+ 14 +  + 23 +         2   2        + 14 +    + 23 +
   \    /                               \     /
    v  v                                 v   v
   + 24 +              3   1            + 24 +
     |                                    |
     v                                    v
   + 34                4   0            + 34
```
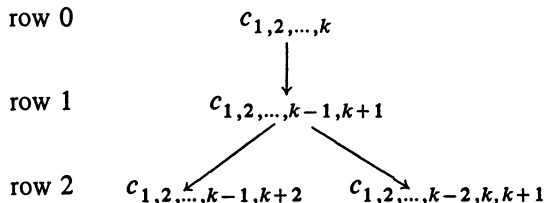
The sign on the left (right) side of the vertex indicates whether the map preserves or reverses orientation on the stable (unstable) manifold of the fixed point.

If $k \geqslant 2$ the first three rows (without labelling) in the diagram of A_k are:

row 0 $c_{1,2,\ldots,k}$

row 1 $c_{1,2,\ldots,k-1,k+1}$

row 2 $c_{1,2,\ldots,k-1,k+2}$ $c_{1,2,\ldots,k-2,k,k+1}$

LEMMA e. *If H is a diagram automorphism on the diagram of A_k, then H is determined by its restriction to row 2 (i.e., if A, B, C are in canonical form and A_k, B_k, $C_k \in \mathbb{S}(k,n; \mathbf{R})$ and H: $\mathrm{Fix}(A_k) \to \mathrm{Fix}(B_k)$, H': $\mathrm{Fix}(A_k) \to \mathrm{Fix}(C_k)$ are diagram automorphisms whose restrictions to row 2 are equal, then the diagrams of B_k and C_k are equal).*

PROOF. If $k = 1$, each row has only one element. Thus for $k = 1$, the identity is the only diagram automorphism.

Suppose $k > 1$ and H is a diagram automorphism. Then row 0 and row 1 each have only one element which therefore must be fixed by H.

To prove the lemma, it suffices to show that in each row j ($j \geqslant 3$) there are no two elements which receive from exactly the same vertices.

Suppose p and p' are in row $j \geqslant 3$ and that both p and p' receive from q in row $j - 1$.

Let $p = c_{r_1,\ldots,r_k}$. Then q is of the form $q = c_{r_1,\ldots,r_{m-1},r_m-1,r_{m+1},\ldots,r_k}$ and p' is of the form $p' = c_{r_1,\ldots,r_{l-1},r_l+1,r_{l+1},\ldots,r_{m-1},r_m-1,r_{m+1},\ldots,r_k}$.

So the indices of p and p' agree at all spots except for l and m where they differ by 1. It would be impossible to find a $q' \neq q$ that gives to both p and p'.

It remains to show that there cannot exist p and p' in row j ($j \geqslant 3$) which both receive from q and only q.

If p receives from exactly one vertex, it has one of the following two forms:

(i) $p = c_{r_1,r_1+1,\ldots,r_1+k-1}$;

(ii) $p = c_{1,\ldots,m-1,r_m,r_m+1,\ldots,r_m+k-m}$, $r_m > m$.

Clearly there do not exist p and p' in row j such that both p and p' are of form (i).

Suppose

$$p = c_{1,\ldots,m-1,r_m,r_m+1,\ldots,r_m+k-m}, \quad r_m > m.$$

Then

$$q = c_{1,\ldots,m-1,r_m-1,r_m+1,\ldots,r_m+k-m}.$$

Now p' can only have one gap in its sequence and p' must be derived from raising by one an element in the sequence for q. Since there is a gap between $r_m - 1$ and $r_m + 1$ in the sequence for q, that sequence must stop at $r_m - 1$ or else there would be two gaps in the sequence for p'. This forces $m = k$.

Thus $p = c_{1,\dots,k-1,r_k}$, $q = c_{1,\dots,k-1,r_k-1}$. Then $p' = c_{1,\dots,k-2,k,r_k-1}$.

Since p' can only have one gap in its sequence, $r_k - 1 = k + 1$. This means $p' = c_{1,\dots,k-2,k,k+1}$. But then p' is in row 2.   Q.E.D.

COROLLARY. *There are either* 1 *or* 2 *automorphisms on the diagram of* A_k.

PROOF. The identity is always a diagram automorphism. There are only 1 or 2 elements in row 2, so there can be at most 2 automorphisms.

We are now ready to classify $\mathbb{S}(k, n; \mathbf{R})$.

THEOREM C. *Let* A_k, $B_k \in \mathbb{S}(k, n; \mathbf{R})$. *The following are equivalent*:
I. $A_k \sim B_k$.
II. *diagram* (A_k) *is isomorphic to diagram* (B_k).
III. *One of the following is true*:
  (i) $\mathrm{Sgn}(A) = \mathrm{Sgn}(B)$,
  (ii) $\mathrm{Sgn}(A) = -\mathrm{Sgn}(B)$,
  (iii) $n = 2k$ *and* $\mathrm{Sgn}(A^{-1}) = \mathrm{Sgn}(B)$,
  (iv) $n = 2k$ *and* $\mathrm{Sgn}(A^{-1}) = -\mathrm{Sgn}(B)$.

The theorem does not require that A and B be in canonical form.

PROOF. III $\Rightarrow$ I follows from Corollary 2 and Theorem B.

To prove I $\Rightarrow$ II we must show that diagram isomorphism is a topological invariant.

Let h be a conjugacy for A_k and B_k and let $H: \mathrm{Fix}(A_k) \to \mathrm{Fix}(B_k)$ be defined by $H = h|\mathrm{Fix}(A_k)$.

Claim. H is a diagram isomorphism.

(a) $W^s(H(p)) = h(W^s(p))$. So $\dim(W^s(H(p))) = \dim(W^s(p))$.

(b) Suppose $p \leqslant q$ (that is, $W^u(p) \cap W^s(q) \neq \varnothing$).

$$W^u(H(p)) \cap W^s(H(q)) = h(W^u(p)) \cap h(W^s(q))$$
$$= h(W^u(p) \cap W^s(q))$$
$$\neq \varnothing.$$

Thus $H(p) \leqslant H(q)$ and H preserves the partial ordering (as does H^{-1}). Therefore condition (b) is satisfied.

(c) Suppose $p \to q$. Then $h|(\{p, q\} \cup [W^u(p) \cap W^s(q)])$ is a conjugacy. So H must preserve $\mathrm{sgn}\{p, q\}$.

II $\Rightarrow$ III. Assume A_k and B_k are in canonical form.

Case 1. Suppose $n > 2k$ and neither $\mathrm{Sgn}(A) = \mathrm{Sgn}(B)$ nor $\mathrm{Sgn}(A)$

$= -\operatorname{Sgn}(B)$. To show diagram (A_k) is not isomorphic to diagram (B_k), it suffices to show that the identity is the only diagram automorphism on diagram (A_k).

Let H be a diagram automorphism on the diagram of A_k.

Claim. The restriction of H to the second row in the diagram of A_k is the identity.

Consider $c_{2,3,\ldots,k,k+1}$ in row k. This vertex receives from exactly 1 and gives to exactly 1 vertex. Because $n > 2k$, it is the only vertex in row k which has those properties. Consequently $c_{2,3,\ldots,k,k+1}$ must be fixed under H.

Define a finite sequence $\{s_i\}_{i=1}^{k-1}$ from $s_1 = c_{2,3,\ldots,k,k+1}$ to $s_{k-1} = c_{1,\ldots,k-2,k,k+1}$ by letting s_{i+1} be the only vertex that gives to s_i (i.e., $s_i = c_{1,2,\ldots,i-1,i+1,\ldots,k,k+1}$).

Since s_1 is fixed under H, each element in the sequence is also fixed under H. In particular $s_{k-1} = c_{1,\ldots,k-2,k,k+1}$ is a fixed point. But s_{k-1} is one of the two elements in row 2. So H restricted to the second row in the diagram of A_k is the identity.

By Lemma e, H is the identity.

Case 2. $k < n < 2k$ and neither $\operatorname{Sgn}(A) = \operatorname{Sgn}(B)$ nor $\operatorname{Sgn}(A) = -\operatorname{Sgn}(B)$.

Let $k' = n - k$. Then $n > 2k'$. Let $A^- = \operatorname{dg}(a_n^{-1}, \ldots, a_1^{-1})$ and $B^- = \operatorname{dg}(b_n^{-1}, \ldots, b_1^{-1})$.

Then A^- and B^- are in canonical form and neither $\operatorname{Sgn}(A^-) = \operatorname{Sgn}(B^-)$ nor $\operatorname{Sgn}(A^-) = -\operatorname{Sgn}(B^-)$. By Theorem B, $A_k \sim (A^-)_{k'}$ and $B_k \sim (B^-)_{k'}$.

But from Case 1, diagram $(A^-)_{k'}$ is not isomorphic to diagram $(B^-)_{k'}$.

By a slight generalization of "diagram isomorphism" and an argument similar to I $\Rightarrow$ II, diagram $(A^-)_{k'}$ is isomorphic to diagram (A_k) and diagram $(B^-)_{k'}$ is isomorphic to diagram (B_k).

Therefore diagram (A_k) is not isomorphic to diagram (B_k).

Case 3. $n = 2k$.

Let $A = \operatorname{dg}(a_1, \ldots, a_n)$, $A^{-1} = \operatorname{dg}(a_1^{-1}, \ldots, a_n^{-1})$, $A^- = \operatorname{dg}(a_n^{-1}, \ldots, a_1^{-1})$. We will show that the conjugacy between A_k and $(A^-)_k$ induces a nonidentity diagram automorphism.

In the conjugacy d_k from A_k to $(A^{-1})_k$, d_k was defined by taking orthogonal complements. Thus $d_k(c_{1,\ldots,k-1,k+2}) = c_{k,k+1,k+3,k+4,\ldots,2k}$.

Let $P = \{P_{ij}\}$ be the $n \times n$ matrix defined by $P_{ij} = 1$ if $i + j = n + 1$ and $P_{ij} = 0$ otherwise. P_k is a differentiable conjugacy for $(A^{-1})_k$ and $(A^-)_k$.

$$P_k(c_{k,k+1,k+3,k+4,\ldots,2k}) = c_{1,\ldots,k-2,k,k+1}.$$

Thus there is a conjugacy between A_k and $(A^-)_k$ whose restriction H to the fixed point set has the property that $H(c_{1,\ldots,k-1,k+2}) = c_{1,\ldots,k-2,k,k+1}$. But H is a diagram automorphism. So we have a nonidentity diagram automor-

phism produced by the conjugacy from A_k to $(A^-)_k$.

Since $\mathrm{Sgn}(A^{-1}) = \mathrm{Sgn}(A^-)$, conditions (iii) and (iv) generate the only possible nonidentity automorphism.

Since $\mathrm{Sgn}(A)$ is invariant up to similarity of the matrix A, II $\Rightarrow$ III is true regardless of whether A_k and B_k are in canonical form.

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