PRODUCTS OF SEQUENTIALLY COMPACT SPACES AND THE V-PROCESS

BY

M. RAJAGOPALAN[1] AND R. GRANT WOODS[2]

ABSTRACT. In this paper we produce a family of sequentially compact, locally compact, T_2 first countable, scattered and separable spaces whose product is not countably compact and thus answer a problem of C. T. Scarborough and A. H. Stone [11] in the negative. We do this using the continuum hypothesis. We also produce a completely regular, T_2, sequentially compact space K which is not p-compact for any $p \in \beta N - N$.

NOTATION 1.1. N denotes the discrete space of integers $\{1, 2, 3, \ldots\}$. All spaces in this paper are Hausdorff. If X is a topological space and π is a partition of X then X/π denotes both the quotient set and quotient space. We use the continuum hypothesis (CH) throughout. Ω denotes the first uncountable ordinal. We reserve greek symbols like $\alpha, \beta, \ldots$, etc. to denote a countable ordinal. English letters like m, n are usually members of N. βN denotes the Stone-Čech compactification of N, and if $X \subset \beta N$ then we give X its subspace topology. Notions about filters and βN can be found in [4]. We follow [5] for notions about scattered spaces. Hereafter, an element $P \in \beta N - N$ is considered to be given in advance.

DEFINITION 1.2. Let Y be an open subset of βN, $P \in \beta N - N$ and π a partition of Y. We say that the pair (Y, π) satisfies the condition V if the following hold:

(i) $P \notin Y$.

(ii) $N \subset Y$ and $\{n\} \in \pi$ for all $n \in N$.

(iii) π is a partition of Y by compact sets of βN.

(iv) Y/π is countable, locally compact and T_2.

(v) Given a member $A \in \pi$ there is a compact open set V of βN so that $A \subset V \subset Y$ and V is saturated under π.

LEMMA 1.3. *Let (Y, π) be as in Definition 1.2 and satisfy* V. *Let F_1, $F_2, \ldots, F_n, \ldots$ be a sequence of distinct members of π such that*

$$\left(\overline{\bigcup_{n=1}^{\infty} F_n} - \bigcup_{n=1}^{\infty} F_n \right) \cap Y = \varnothing \quad and \quad F = \overline{\bigcup_{n=1}^{\infty} F_n} - \bigcup_{n=1}^{\infty} F_n.$$

Let $P \notin F$. Then there exists a compact, open set W of βN so that $P \notin W$ and $W \supset F$ and $W \cap Y$ is saturated under π.

PROOF. From (iv) and (v) of Definition 1.2 we get that Y can be expressed as a pairwise disjoint union $\bigcup_{n=1}^{\infty} M_n$ of compact, open sets M_1, $M_2, \ldots, M_n, \ldots$ of βN so that M_n is saturated under π for all $n \in N$. If we put $M_n/\pi = A_n$ for all $n \in N$ then Y/π is the disjoint union $\bigcup_{n=1}^{\infty} A_n$ of compact, open sets $A_1, A_2, \ldots, A_n, \ldots$ of Y/π. Let $q: Y \to Y/\pi$ be the canonical quotient map. Then the fact that M_n/π is T_2 (from (iv) of Definition 1.2) gives that the restriction of q to M_n is closed. Hence $q: Y \to Y/\pi$ is a closed map. Thus if $B \subset Y$ is open and $\mathcal{C} \subset B$ is the union of all members of π contained in B then $\mathcal{C}$ is open in Y and hence in βN. Now $\overline{\bigcup_{n=1}^{\infty} F_n} = (\bigcup_{n=1}^{\infty} F_n) \cup F$ and $F \cap Y = \varnothing$. So

$$\left(\overline{\bigcup_{n=1}^{\infty} F_n} \right) \cap M_k = \left(\bigcup_{n=1}^{\infty} F_n \cap M_k \right)$$

is compact for all $k \in N$. Put $(\bigcup_{n=1}^{\infty} F_n) \cap M_k = D_k$ for $k \in N$. Since $P \notin F$ and F is closed we have that there is a compact, open set S of βN so that $F \subset S$ and $P \notin S$. Now

$$\bigcup_{k=1}^{\infty} D_k \cup F = \bigcup_{n=1}^{\infty} F_n \cup F = \overline{\bigcup_{n=1}^{\infty} F_n}$$

is compact and the collection $\{ S, M_1, M_2, \ldots, M_n, \ldots \}$ is an open cover for $\overline{\bigcup_{n=1}^{\infty} F_n}$. So there is an integer $k \geqslant 1$ so that $D_l \subset S$ if $l > k$ and $l \in N$. It is now clear that $\overline{\bigcup_{i=k+1}^{\infty} D_i} \supset F$. Using the closedness of the quotient map $q: Y \to Y/\pi$ we get that the union V of all members of π contained in $Y \cap S \cap (\bigcup_{i=k+1}^{\infty} M_i)$ is open in βN. It follows from (iv) and (v) of Definition 1.2 that given $l > k$ and $l \in N$ there is a compact open set V_l of βN so that $D_l \subset V_l \subset M_l \cap V$ and V_l is saturated under π since D_l is compact. Put $L = \bigcup_{i=k+1}^{\infty} V_l$. Then L is an open set of βN so that $\bigcup_{i=k+1}^{\infty} D_i \subset L \subset S \cap Y$ and L is saturated under π. Then $\bar{L} \supset F$ and $\bar{L} \subset S$ and $\bar{L} \cap Y$ is saturated under π. Then $\bar{L} = W$ is the required open set of the theorem.

COROLLARY 1.4. *Let Y, π be as in Lemma 1.3 and (Y, π) satisfy the condition* V. *Let $F_1, F_2, \ldots, F_n, \ldots$ be a sequence of distinct members of π so that $\overline{\bigcup_{n=1}^{\infty} F_n} - \bigcup_{n=1}^{\infty} F_n = F$ is disjoint with Y and $P \notin F$. Then there exists an*

open set Y_0 and a partition π_0 of Y_0 so that the following hold:

(a) (Y_0, π_0) *satisfies* V.

(b) $Y_0 \supset F$.

(c) $Y_0 \supset Y$ *and* $\pi_0 \supset \pi$.

PROOF. Let W be as in the conclusion of Lemma 1.3. Let $Y_0 = Y \cup W$, $\pi_0 = \pi \cup \{W - Y\}$ if $F \neq \varnothing$ and $Y_0 = Y$, $\pi_0 = \pi$ otherwise. Then we get the corollary.

LEMMA 1.5. *Let* $Y_n \subset \beta N$ *be an open set*, π_n *a partition of* Y_n *so that* (Y_n, π_n) *satisfies* V, *and* $Y_{n+1} \supset Y_n$ *and* $\pi_{n+1} \supset \pi_n$ *for* $n \in N$. *Let* $Y = \bigcup_{n=1}^{\infty} Y_n$ *and* $\pi = \bigcup_{n=1}^{\infty} \pi_n$. *Then* (Y, π) *satisfies* V.

PROOF. Easy to see.

COROLLARY 1.6. *Let* (Y, π) *be as in Lemma 1.3 and satisfy condition* V. *Let* $\mathscr{F}$ *be a countable family of subsets of* βN *so that if* $A \in \mathscr{F}$ *then there is a countable collection* $\{G_1, G_2, \ldots, G_n, \ldots\}$ *of distinct members of* π *so that* $A = (\overline{\bigcup_{n=1}^{\infty} G_n} - \bigcup_{n=1}^{\infty} G_n)$ *and* $P \notin A$. *Then there is an open set* Y_0 *of* βN *and a partition* π_0 *of* Y_0 *so that the following hold*:

(a) $Y_0 \supset Y$ *and* $\pi_0 \supset \pi$.

(b) (Y_0, π_0) *satisfies the condition* V.

(c) *If* $A \in \mathscr{F}$ *and* $A \neq \varnothing$ *then* $Y_0 \cap A \neq \varnothing$.

PROOF. Write the members of $\mathscr{F}$ in a sequence $A_1, A_2, \ldots, A_n, \ldots$. If $Y \cap A_1 \neq \varnothing$ or $A_1 = \varnothing$ then put $Y_1 = Y$ and $\pi_1 = \pi$. If $Y \cap A_1 = \varnothing$ then use Corollary 1.4 to get an open set Y_1 and a partition π_1 of Y_1 so that $Y_1 \supset Y$; $\pi_1 \supset \pi$; (Y_1, π_1) satisfies V and $Y_1 \supset A_1$. By induction suppose we have defined Y_i and π_i for $i = 1, 2, \ldots, n$ where $n \in N$. Assume further that the following hold:

(i) $Y \subset Y_i \subset Y_j$ for all $i, j = 1, 2, \ldots, n$ so that $i < j$.

(ii) $\pi \subset \pi_i \subset \pi_j$ for all $i, j = 1, 2, \ldots, n$ so that $i < j$.

(iii) $Y_i \cap A_i \neq \varnothing$ if $A_i \neq \varnothing$ where $i = 1, 2, 3, \ldots, n$.

(iv) (Y_i, π_i) satisfies condition V for all $i = 1, 2, \ldots, n$.

If $A_{n+1} = \varnothing$ or $A_{n+1} \cap Y_n \neq \varnothing$ then put $Y_{n+1} = Y_n$ and $\pi_{n+1} = \pi_n$. If $A_{n+1} \neq \varnothing$ and $A_{n+1} \cap Y_n = \varnothing$ then use Corollary 1.4 to get (Y_{n+1}, π_{n+1}) so that (Y_{n+1}, π_{n+1}) satisfies V; $Y_{n+1} \supset Y_n$; $\pi_{n+1} \supset \pi_n$; and $Y_{n+1} \supset A_{n+1}$. Then the statements (i), (ii), (iii), (iv) above are true for $n + 1$ also. Now put $Y_0 = \bigcup_{n=1}^{\infty} Y_n$ and $\pi_0 = \bigcup_{n=1}^{\infty} \pi_n$. Then this (Y_0, π_0) satisfies the (a), (b), (c) of this corollary.

REMARK 1.7. Let $P \in \beta N - N$. We are going to give below a method for constructing an open set Y of βN and a partition π of Y by compact sets so that $P \notin Y$ and Y/π is first countable, locally compact, T_2, locally count-

able, $Y \supset N$ and $\{n\} \in \pi$ for all $n \in N$ and whenever $F_1, F_2, \ldots, F_n, \ldots$ is a sequence of distinct members of π and $(\overline{\bigcup_{n=1}^{\infty} F_n} - \bigcup_{n=1}^{\infty} F_n) = F \neq \emptyset$ then $Y \cap F \neq \emptyset$. This method was introduced by the first author in [6] and was used in [7], and [9] to solve some problems of W. W. Comfort; Z. Semadeni; R. Telgarsky; P. Nyikos and S. P. Franklin.

DEFINITION 1.8 (THE V-PROCESS). The following method of constructing an open set Y of βN and a partition π of Y is called the V-process. Put $Y_1 = N$ and $\pi_1 = \{\{n\} | n \in N\}$. Put $\mathcal{F}_1$ to be the collection of all sequences of distinct member of π_1. Put $\mathcal{G}_1$ to be the collection of all sets A of the form

$$A = \left(\overline{\bigcup_{n=1}^{\infty} F_n} - \bigcup_{n=1}^{\infty} F_n \right)$$

where the sequence $(F_1, F_2, \ldots, F_n, \ldots)$ is a member of $\mathcal{F}_1$ and $P \notin A$. Then $\mathcal{G}_1$ is uncountable. Let $\{A_{11}, A_{12}, \ldots, A_{1\alpha}, \ldots \}$ be as well ordering of $\mathcal{G}_1$ (using (CH)) where $1 \leqslant \alpha < \Omega$. Now assume that $\theta \in [1, \Omega)$ is a successor ordinal equal to $\gamma + 1$ and that we have defined Y_α, π_α and $A_{\alpha\delta}$ for all α so that $1 \leqslant \alpha \leqslant \gamma$ and δ so that $1 \leqslant \delta < \Omega$. Assume further that (Y_α, π_α) satisfies the condition V for all $\alpha \in [1, \gamma]$ and $Y_\alpha \supset Y_\beta$ and $\pi_\alpha \supset \pi_\beta$ for all α, β in $[1, \gamma]$ and $\beta \leqslant \alpha$. Assume that given $\alpha \in [1, \gamma]$ and $\delta \in [1, \Omega)$ there is a sequence $F_1, F_2, \ldots, F_n, \ldots$ of distinct members of π_α so that $A_{\alpha\delta} = \overline{\bigcup_{n=1}^{\infty} F_n} - \bigcup_{n=1}^{\infty} F_n$ and $P \notin A_{\alpha\delta}$. Conversely assume that given $\alpha \in [1, \gamma]$ and a sequence $G_1, G_2, \ldots, G_n, \ldots$ of distinct members of π_α so that $P \notin \bigcup_{n=1}^{\infty} G_n$ there is an $\eta \in [1, \Omega)$ so that $A_{\alpha\eta} = (\overline{\bigcup_{n=1}^{\infty} G_n} - \bigcup_{n=1}^{\infty} G_n)$. Then using Corollary 1.6 construct an open set $Y_\theta = Y_{\gamma+1}$ and a partition $\pi_\theta = \pi_{\gamma+1}$ of $Y_{\gamma+1}$ so that the following conditions (a), (b), (c), (d) hold:

(a) $Y_{\gamma+1}$ is an open subset of βN and $\pi_{\gamma+1}$ is a partition of $Y_{\gamma+1}$ by compact sets.

(b) $(Y_{\gamma+1}, \pi_{\gamma+1})$ satisfies condition V.

(c) $Y_{\gamma+1} \supset Y_\gamma$ and $\pi_{\gamma+1} \supset \pi_\gamma$.

(d) $Y_{\gamma+1} \cap A_{\eta\delta} \neq \emptyset$ provided $1 \leqslant \eta \leqslant \gamma$ and $1 \leqslant \delta \leqslant \omega^\gamma$ and $A_{\eta\delta} \neq \emptyset$.

Now put $\mathcal{F}_{\gamma+1}$ to be the collection of all sequences $\{F_1, F_2, \ldots, F_n, \ldots \}$ of distinct members of $\pi_{\gamma+1}$ so that $P \notin \overline{\bigcup_{n=1}^{\infty} F_n}$. Put $\mathcal{G}_{\gamma+1}$ to be the collection of all sets of the form $A = (\overline{\bigcup_{n=1}^{\infty} F_n} - \bigcup_{n=1}^{\infty} F_n)$ where the sequence $F_1, F_2, \ldots, F_n, \ldots$ is in $\mathcal{F}_{\gamma+1}$. We well order $\mathcal{G}_{\gamma+1}$ as $A_{(\gamma+1)1}, A_{(\gamma+1)2}, \ldots, A_{(\gamma+1)\delta}, \ldots$ with $\delta \in [1, \Omega)$ by using (CH). (At this point we remark that it is easy to see that $\mathcal{G}_{\gamma+1}$ is uncountable and has cardinality c and hence $\mathcal{G}_{\gamma+1}$ can be well ordered by $[1, \Omega)$.)

Now assume that θ is a limit ordinal in $[1, \Omega)$ and we have defined Y_α, π_α and $A_{\alpha\delta}$ for all α, δ where $\alpha \in [1, \theta)$ and $\delta \in [1, \Omega)$. Put $Y_\theta = \bigcup_{\alpha < \theta} Y_\alpha$ and $\pi_\theta = \bigcup_{\alpha < \theta} \pi_\alpha$. Let $\mathcal{F}_\theta$ denote the collection of all sequences $(F_1, F_2, \ldots, F_n, \ldots)$ of distinct members of π_θ so that $P \notin \overline{\bigcup_{n=1}^{\infty} F_n}$. Let $\mathcal{G}_\theta$ be

the collection of all sets A of the form $\overline{\bigcup_{n=1}^{\infty} F_n} - \bigcup_{n=1}^{\infty} F_n$ where the sequence $(F_1, F_2, \ldots, F_n, \ldots)$ is in $\mathcal{F}_\theta$. Then $\mathcal{G}_\theta$ is uncountable. Well order $\mathcal{G}_\theta$ as $A_{\theta 1}, A_{\theta 2}, \ldots, A_{\theta \delta}, \ldots$ where $1 \leqslant \delta < \Omega$.

Finally put $Y_p = \bigcup_{\theta \in [1, \Omega)} Y_\theta$ and $\pi_p = \bigcup_{\theta \in [1, \Omega)} \pi_\theta$ and $X_p = Y_p / \pi_p$. The V-process ends here.

The spaces X_p constructed above will be the spaces that will answer a problem of C. Scarborough and A. H. Stone [11]. We proceed below to show that X_p is a first countable, locally compact, T_2 sequentially compact space for all $P \in \beta N - N$.

THEOREM 1.9. *Let $P \in \beta N - N$, and let Y_p, π_p, X_p, Y_α, π_α, $A_{\alpha\delta}$ be as in the definition of V-process for all $\alpha \in [1, \Omega)$ and $\delta \in [1, \Omega)$. Then the space X_p obtained in the V-process is locally compact, T_2, locally countable and hence first countable. Moreover if F_1, F_2, \ldots, F_n, \ldots is a sequence of distinct members of π_p and $\overline{\bigcup_{n=1}^{\infty} F_n} - \bigcup_{n=1}^{\infty} F_n = A \neq \varnothing$ then $A \cap Y_p \neq \varnothing$.*

PROOF. Now Y_α is open in βN, $Y_\alpha \supset Y_\beta$ and $\pi_\alpha \supset \pi_\beta$ for all α, $\beta \in [1, \Omega)$ and $\alpha > \beta$. (These assertions are easy to see from the definition of Y_α and π_α in the V-process.) Put $X_\alpha = Y_\alpha / \pi_\alpha$ for all $\alpha \in [1, \Omega)$. Then $X_\alpha \subset X_\beta$ in a natural way and X_α is open in X_β because $\pi_\alpha \subset \pi_\beta$ for all α, $\beta \in [1, \Omega)$ and $\alpha < \beta$. Moreover, $X_p = \bigcup_{\alpha \in [1, \Omega)} X_\alpha$. Since (Y_α, π_α) satisfies condition V we have that X_α is countable, locally compact, T_2 and hence first countable for all $\alpha \in [1, \Omega)$. Since $X_p = \bigcup_{\alpha \in [1, \Omega)} X_\alpha$ and X_α is open in X_p for all $\alpha \in [1, \Omega)$ we get that X_p is T_2, locally compact, locally countable and first countable. Now let $F_1, F_2, \ldots, F_n, \ldots$ be a sequence of distinct members of π_p and let $\overline{\bigcup_{n=1}^{\infty} F_n} - \bigcup_{n=1}^{\infty} F_n = A \neq \varnothing$. We claim that $A \cap Y_p \neq \varnothing$. To see this we first of all see from the definition of π_p that given in $n \in N$ there is an $\alpha_n \in [1, \Omega)$ so that $F_n \in \pi_{\alpha_n}$. So, if $\alpha \in [1, \Omega)$ and $\alpha > \alpha_n$ for all $n \in N$ then $F_n \in \pi_\alpha$ for all $n \in N$. Now we have two cases.

Case (i). $P \notin \overline{\bigcup_{n=1}^{\infty} F_n} - \bigcup_{n=1}^{\infty} F_n = A$. Then there is a δ in $[1, \Omega)$ so that $A = A_{\alpha\delta}$. (This is from the definition of $A_{\alpha\delta}$'s.) Now there is a $\gamma \in [1, \Omega)$ so that $\gamma > \alpha$ and $\delta < \omega^\gamma$. Then $Y_\gamma \cap A_{\alpha\delta} \neq \varnothing$ from the construction of Y_γ in the V-process. So $Y_p \cap A_{\gamma\delta}$ is also $\neq \varnothing$.

Case (ii). $P \in A = (\overline{\bigcup_{n=1}^{\infty} F_n} - \bigcup_{n=1}^{\infty} F_n)$. Then we claim that there is a subsequence $F_{n_1}, F_{n_2}, \ldots, F_{n_r}, \ldots$ of $F_1, F_2, \ldots, F_n, \ldots$ so that $P \notin B = \overline{\bigcup_{r=1}^{\infty} F_{n_r}} - \bigcup_{r=1}^{\infty} F_{n_r}$ and $B \neq \varnothing$, and $B \cap Y_\alpha = \varnothing$. If this is proved then the Case (i) aboe shows that $Y_p \cap B \neq \varnothing$ and $B \subset \overline{\bigcup_{n=1}^{\infty} F_n} - \bigcup_{n=1}^{\infty} F_n = A$. Thus we will get $Y_p \cap A \neq \varnothing$ in this case too. Now (Y_α, π_α) satisfies condition V. So, as in the proof of Lemma 1.3 we can find pairwise disjoint compact, open subsets $M_1, M_2, \ldots, M_n, \ldots$ of βN so that $Y_\alpha = \bigcup_{n=1}^{\infty} M_n$ and each M_n is saturated under π_α. Since $P \in \overline{\bigcup_{n=1}^{\infty} F_n}$ we have that $\bigcup_{n=1}^{\infty} F_n$ is not compact. So $\bigcup_{n=1}^{\infty} F_n$ cannot be contained in a finite

union of the M_n's. So there exists an infinite sequence $n_1 < n_2 < n_3 < \cdots < n_k < \ldots$ of integers in N so that each M_{n_k} contains at least one F_n. Now βN is extremally disconnected. So P can belong to the closure of at most one of the sets $\bigcup_{k=1}^{\infty} M_{n_{(2k)}}$ or $\bigcup_{k=1}^{\infty} M_{n_{(2k+1)}}$.

Choose the sequence $(M_{n_{2k}})$ if $P \notin \bigcup_{k=1}^{\infty} M_{n_{2k}}$ and the sequence $(M_{n_{(2k+1)}})$ otherwise. Rename the chosen sequence as (M_{p_k}) for convenience. For each $k \in N$ choose one set F_{r_k} from the sequence (F_n) so that $F_{r_k} \subset M_{p_k}$ for all $k \in N$. Then $\bigcup_{k=1}^{\infty} F_{r_k}$ cannot be compact since the collection $\{M_{p_k} | k \in N\}$ is an open cover for $\bigcup_{k=1}^{\infty} F_{r_k}$ from which a finite cover cannot be extracted. So $\overline{\bigcup_{k=1}^{\infty} F_{r_k}} - \bigcup_{k=1}^{\infty} F_{r_k} = B \neq \varnothing$ and $P \notin B$. Clearly $B \cap Y_\alpha = \varnothing$ and hence $B \subset \overline{\bigcup_{n=1}^{\infty} F_n} - Y_\alpha \subset \overline{\bigcup_{n=1}^{\infty} F_n} - \bigcup_{n=1}^{\infty} F_n = A$. So the subsequence (F_{r_k}) is the required subsequence and we have the theorem.

THEOREM 1.10. *Let $P \in \beta N - N$, and let X_p be the space obtained by the V-process. Then X_p is countably compact. Since it is also first countable, it is sequentially compact.*

PROOF. Let $Y_p, \pi_p, Y_\alpha, \pi_\alpha$ be as in the definition of V-process for all α in $[1, \Omega)$. Let $q: Y_p \to X_p$ be the canonical quotient map. Let $\{x_1, x_2, \ldots, x_n, \ldots\}$ be a countable discrete infinite set in X_p. Choose $F_n \in \pi_p$ so that $q(F_n) = x_n$ for all $n \in N$. Now $Y_p \cap A \neq \varnothing$ by Theorem 1.9 where $A = \overline{\bigcup_{n=1}^{\infty} F_n} - \bigcup_{n=1}^{\infty} F_n$. Let $y \in Y_p \cap A$. Then $q(y) \in X_p$ is clearly a cluster point of the set $\{x_1, x_2, \ldots, x_n, \ldots\}$. So X_p is countably compact and Theorem 1.9 gives that X_p is also sequentially compact.

DEFINITION 1.11. Let $p \in \beta N - N$ be given. Let X be a topological space. X is called p-compact if given any 1-1 map $f: N \to X$ there is a continuous extension $g: N \cup \{p\} \to X$ of f from $N \cup \{p\}$ into X.

REMARK. The idea of p-compactness was first introduced in a different form by A. R. Bernstein in [2].

LEMMA 1.12. *Let $p \in \beta N - N$ be given. Then the space X_p obtained by the use of V-process is not p-compact. Moreover, X_p is scattered. So given an element $p \in \beta N - N$, there is a scattered, locally compact, T_2, locally countable, first countable, sequentially compact, non p-compact space.*

PROOF. Let $X_p \cup \{\infty\}$ be the one point compactification of X_p. Let $q: Y_p \to X_p$ the quotient map and $f = q|N$. Then $f: N \to X_p$ is a function which is one-to-one on N. We are going to show that this f has no continuous extension from $N \cup \{p\} \to X_p$. Let $g: \beta N \to X \cup \{\infty\}$ be the function which maps $\beta N / Y_p$ onto $\{\infty\}$ and coincides with q on Y_p. Then g is continuous and is the unique continuous extension of f to βN. In particular $g(p) = \infty$. Now $\beta(N \cup \{p\}) = \beta N$ so every continuous function $\phi: N \cup \{p\} \to X_p \cup \{\infty\}$ must extend to a unique continuous function $\tilde{\phi}: \beta N \to X_p \cup \{\infty\}$. So if there

is a continuous extension $\tilde{f}$: $N \cup \{p\} \to X_p$ of f to $N \cup \{p\}$ then there will be a continuous extension $\tilde{g}$ of $\tilde{f}$ from $\beta N \to X_p \cup \{\infty\}$ and this $\tilde{g}$ will also be a continuous extension of f. Now $\tilde{g}(p) \in X_p$ and $g(p) = \infty$. So $\tilde{g} \neq g$. So, if an extension $\tilde{f}$ for f exists as a continuous function from $N \cup \{p\} \to X_p$ then f: $N \to X_p \cup \{\infty\}$ will have two continuous extensions g, $\tilde{g}$ from $\beta N \to X_p \cup \{\infty\}$ which is not possible. So X_p is not p-compact. The second part of the lemma now follows easily from Theorem 1.10.

LEMMA 1.13. *Let X be a topological space, and let f: $N \to X$ a one-to-one function from N into X. Then $f(N)$ has a cluster point in X if and only if there is at least one $p_0 \in \beta N - N$ so that f extends continuously to a function g: $N \cup \{p_0\} \to X$ from $N \cup \{p_0\}$ into X.*

PROOF. If f has a continuous extension $\tilde{f}$ from $N \cup \{p_0\}$ into X for some $p_0 \in \beta N - N$ then $\tilde{f}(p_0)$ is a cluster point of $f(N)$. Conversely let $f(N)$ have a cluster point l in X. Let $\mathcal{V}$ be the filter of all deleted neighbourhoods of l in X. Let $\mathcal{F}$ be the collection of subsets $A \subset N$ so that $A \in \mathcal{F} \Leftrightarrow \exists$ a member $V \in \mathcal{V}$ so that $A = \{i | f(i) \in V$ and $i \in N\}$. Then it is easily seen that $\mathcal{F}$ is a filter on N which is free. So $\mathcal{F}$ is contained in a free ultrafilter $p_0 \in \beta N$. Let g: $N \cup \{p_0\} \to X$ be the function so that $g(p_0) = l$ and $g|N = f$. Then g is a continuous extension of f to $N \cup \{p_0\}$.

THEOREM 1.14. *For every $p \in \beta N - N$ let Z_p be a T_2 topological space which is not p-compact. Then $\prod_{p \in \beta N - N} Z_p$ is not countably compact. Hence there exists a family of sequentially compact, T_2, locally compact, locally countable, scattered, first countable spaces whose product is not countably compact.*

PROOF. For every $p \in \beta N - N$ let f_p: $N \to Z_p$ be a one-to-one function from $N \to Z_p$ which does not extend continuously to a function g_p: $N \cup \{p\} \to Z_p$. Let f: $N \to \prod_{p \in \beta N - N} Z_p$ be the product function $\prod_{p \in \beta N - N} f_p$ defined by the rule that the pth coordinate of $f(n)$ is $f_p(n)$ for all $n \in N$ and $p \in \beta N - N$. Then f is a 1-1 map from N into $\prod_{p \in \beta N - N} Z_p$. Now let $p_0 \in \beta N - N$ be given. Then f cannot extend continuously to a function ϕ: $N \cup \{p_0\} \to \prod_{p \in \beta N - N} Z_p$. For if such a ϕ exists then the function $g = \phi_{p_0} \circ \phi$: $\prod_{p \in \beta N - N} \to Z_{p_0}$ will be a continuous extension of f_{p_0} to a function from $\prod_{p \in \beta N - N} \to Z_{p_0}$ where ϕ_{p_0} is the projection from $\prod_{p \in \beta N - N} Z_p$ onto Z_{p_0}. So $f(N)$ will be an infinite set which will have no cluster point in $\prod_{p \in \beta N - N} Z_p$ by Lemma 1.13. So $\prod_{p \in \beta N - N} Z_p$ is not countably compact.

Now taking $Z_p = X_p$ of Lemma 1.12 for each $p \in \beta N - N$ we get the last assertion of our theorem.

REMARK. The V-process of M. Rajagopalan is very effective in solving

some problems in topology considered by M. E. Rudin [10], A. V. Arhangel'skiĭ [1] and others on constructing hereditarily separable, regular, non-Lindelöf spaces and compact c-spaces. These results obtained by the first author will appear elsewhere.

K. Kunen, and Eric Van Douwen informed the authors in a private communication that by a different method they also got spaces like X_p of Definition 1.8. Recently their technique was also used by J. E. Vaughn [12] to get the same type of spaces. Both J. E. Vaughn and K. Kunen use (CH). Eric Van Douwen uses an axiom weaker then (CH). Their results are very recent and the results of this paper were obtained in 1974 and were announced in [13]. Recently M. Rajagopalan got the solution to the above-mentioned problem of C. Scarborough and A. H. Stone using an axiom called ⊛ which is weaker than both (CH) and (MA) [14].

The following result is of independent interest.

THEOREM 1.15. *There exists a completely regular T_2 sequentially compact space M which is not p-compact for any $p \in \beta N - N$. Then M^{2^c} is not countably compact.*

PROOF. Let x_0 be a fixed element of $\Pi_{p \in \beta N - N} X_p$ where X_p is the space constructed by V-process above for each $p \in \beta N - N$. Let M be the set of all elements of $\Pi_{p \in \beta N - N} X_p$ which differ from x_0 in at most a countable number of coordinates. Then M is easily seen to be sequentially compact. Now M contains a closed subspace homeomorphic to X_p whatever $p \in \beta N - N$ may be. Since X_p is not p-compact, it follows that M is not p-compact where $p \in \beta N - N$. Let us put $M_p = M$ for all $p \in \beta N - N$. Then imitating the proof of Theorem 1.14 we get that $\Pi_{p \in \beta N - N} M_p = M^{2^c}$ is not countably compact.

REFERENCES

1. A. V. Arhangel'skiĭ, *On cardinal invariants*, General Topology and its Relations to Modern Analysis and Algebra III (Proc. Third Prague Topological Sympos., 1971), Academia, Prague; Academic Press, New York, 1972, pp. 37–46. MR 49 #3800.

2. A. R. Bernstein, *A new kind of compactness for topological spaces*, Fund. Math. **66** (1969/70), 185–193. MR 40 #4924.

3. J. Dugundji, *Topology*, Allyn and Bacon, Boston, Mass., 1966. MR 33 #1824.

4. L. Gillmann and M. Jerison, *Rings of continuous functions*, Van Nostrand, Princeton, N. J., 1960. MR 22 #6994.

5. V. Kannan and M. Rajagopalan, *On scattered spaces*, Proc. Amer. Math. Soc. **43** (1974), 402–408. MR 48 #12469.

6. _____, *Scattered spaces.* II, Illinois J. Math. (to appear).

7. M. Rajagopalan, *Scattered spaces.* III, J. Indian Math. Soc. **41** (1977).

8. _____, *$\beta N - N - \{p\}$ is not normal*, J. Indian Math. Soc. (N. S.) **36** (1972), 173–176. MR 47 #9545.

9. _____, *A chain compact space which is not strongly scattered*, Israel J. Math. **23** (1976), 117–125.

10. M. E. Rudin, *Lectures on set theoretic topology*, CBMS Regional Conf. Ser. in Math., no. 23. Amer. Math. Soc., Providence, R. I., 1975. MR **51** #4128.

11. C. T. Scarborough and A. H. Stone, *Products of nearly compact spaces*, Trans. Amer. Math. Soc. **124** (1966), 131–147. MR **34** #3528.

12. J. E. Vaughn, *Products of perfectly normal sequentially compact spaces* (to appear).

13. M. Rajagopalan and R. Grant Woods, *Products of sequentially compact spaces*, Notices Amer. Math. Soc. **22** (1975), A-333. Abstract #75T-G29.

14. M. Rajagopalan, *Some outstanding problems in topology and the V-process* (Proc. Mannheim Conf. in Topology, 1975), Lecture Notes in Math., vol. 540, Springer-Verlag, Berlin and New York,1976, pp. 500–517.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MANITOBA, WINNIPEG R3T 2N2, MANITOBA, CANADA (Current address of R. G. Woods)

Current address (M. Rajagopalan): Departamento de Matematicas, Universidad de Los Andes, Merida, Venezuela