VECTOR FIELDS ON POLYHEDRA

BY

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ABSTRACT. This paper presents a bundle theory for studying vector fields and their integral flows on polyhedra, and applications.

Every polyhedron has a tangent object in the category of simplicial bundles in much the same way as every smooth manifold has a tangent object in the category of smooth vector bundles. One can show that there is a correspondence between piecewise smooth flows on a polyhedron P and sections of the tangent object of P (i.e., vector fields on P); using this result one can prove existence results for piecewise smooth flows on polyhedra. Finally an integral formula for the Euler characteristic of a closed, oriented, even-dimensional combinatorial manifold is given; as a consequence of this result one obtains a representation of Euler classes of such combinatorial manifolds in terms of piecewise smooth forms.

0. **Introduction.** The purpose of this paper is to describe a bundle theory for studying vector fields and their integral flows on polyhedra and, in particular, on combinatorial manifolds. Two applications of this theory are presented; these results are motivated by corresponding results for smooth manifolds.

Here are the results obtained:

THEOREM A. *The sum of the indices of any integrable vector field on a polyhedron P which has a finite number of zeroes is the Euler characteristic $\chi(P)$ of P. Furthermore, if M is a closed combinatorial manifold with $\chi(M) = 0$, then there is a nowhere vanishing integrable vector field on M.*

Theorem A corresponds to a classical result for smooth manifolds due to Hopf (see [10]). Actually Hopf's original result (see [7]) is also valid for a particular type of vector field on a combinatorial manifold. It will be seen that even though there are similarities between Hopf's original result and Theorem A, there are also major differences. As a special case of Theorem A one also obtains an analog to a calculation of Whitney (see [14]) giving the Euler characteristic of a triangulated smooth manifold using the singularities of vector fields.

Now let M be a closed, oriented, $2n$-dimensional combinatorial manifold.

From the proof of Theorem A it follows that there is an integrable vector field s on M with a finite number of zeroes interior to the $2n$-simplices σ_α^{2n} of M. On each σ_α^{2n} one can define smooth forms $e_\alpha^{2n} \in \Lambda^{2n} E(\sigma_\alpha^{2n})$ and $\eta_\alpha^{2n-1} \in \Lambda^{2n-1} E(T\sigma_\alpha^{2n})$ for which

$$\int_{\sigma_\alpha^{2n}} e_\alpha^{2n} - \int_{\partial\sigma_\alpha^{2n}} s^* \eta_\alpha^{2n-1} = \sum i(s, x_k)$$

where $i(s, x_k)$ is the index of s at the zero x_k of s, and the summation is taken over all zeroes $x_k \in \sigma_\alpha^{2n}$.

THEOREM B. *On a closed, oriented, $2n$-dimensional combinatorial manifold M the forms $e_\alpha^{2n} - ds^* \eta_\alpha^{2n-1}$, for σ_α^{2n} a $2n$-simplex of M and s an integrable vector field on M, define a global closed piecewise smooth $2n$-form $e(M) \in \Lambda^{2n} E(M)$ on M, and $\int_M e(M) = \chi(M)$.*

Theorem B is closely related to a result of Allendoerfer and Weil (see [1]) which gives a similar integral formula for the Euler characteristic of a smooth manifold with piecewise smooth boundary. The proof of Theorem B is similar to Chern's intrinsic proof of the Gauss-Bonnet formula for closed Riemannian manifolds (see [5]).

The bundle theory which yields these results is the theory of simplicial bundles. There are two major reasons why this theory is important in the study of polyhedral differential geometry. First this theory helps relate piecewise smooth forms on a polyhedron with vector fields, and more specifically piecewise smooth flows, on a polyhedron. Second, in analogy with smooth manifolds, this theory illustrates a global differential geometry on polyhedra which arises naturally from local constructions on "standard" open sets (*small open sets*) which can be coordinatized. This global theory is furthermore intrinsic: The differential geometry of a polyhedron in a neighborhood of a point depends only on what that neighborhood looks like.

The theory of simplicial bundles is the dual of a corresponding bundle theory which is used in studying piecewise smooth forms on polyhedra (see [11]); some of the terminology and results of [11] are briefly reviewed in §1. Simplicial bundles are defined in §2: every polyhedron has a tangent object in the category of simplicial bundles in much the same way every smooth manifold has a tangent object in the category of smooth vector bundles. In §3 piecewise smooth flows are defined, and the relation between piecewise smooth flows and integrable sections of the tangent object is described; using this relation one can prove existence results for piecewise smooth flows. In §4, Theorem A is proved. In §5, Riemannian metrics on polyhedra are considered. Finally in §6, Theorem B is proved.

1. Simplicial spaces. This section is devoted to a brief review of some of the terminology and results of [11].

For $n = 0, 1, 2, \ldots$, the *n-corner* $s^n = \{x = (x_k) \in R^n: x_k \geqslant 0 \; \forall \; k\}$.

1.1. DEFINITION. A *simplicial chart* on the topological space X is a pair (U, ψ) consisting of a nonempty subset (not necessarily an open subset) U of X and a homeomorphism $\psi: U \to V$ where V is an open subset of some n-corner s^n. Two simplicial charts $\psi_\alpha: U_\alpha \to V_\alpha \subseteq s_\alpha^m$ and $\psi_\beta: U_\beta \to V_\beta \subseteq s_\beta^n$ are *related* iff $\psi_\alpha(U_\alpha \cap U_\beta)$ and $\psi_\beta(U_\alpha \cap U_\beta)$ are open subsets of k-faces of s_α^m and s_β^n, for some k depending on U_α and U_β, and for which

$$\psi_\beta \circ \psi_\alpha^{-1}: \psi_\alpha(U_\alpha \cap U_\beta) \to \psi_\beta(U_\alpha \cap U_\beta)$$

is an invertible affine transformation (both domain and range being restricted). A *simplicial atlas* on X is a collection of related simplicial charts on X whose domains cover X and which is maximal with respect to the inclusion of related simplicial charts. A *simplicial space* X is a separable Hausdorff topological space equipped with a locally finite, finite-dimensional simplicial atlas.

Observe that in the definition of "related", m and n are not required to be equal, but k is required to be the dimension of both $\psi_\alpha(U_\alpha \cap U_\beta)$ and $\psi_\beta(U_\alpha \cap U_\beta)$.

The main examples of simplicial spaces considered in this paper are *polyhedra*, i.e., locally finite, finite-dimensional simplicial complexes.

A simplicial space is a space equipped with an atlas of related simplicial charts, and different atlases on a space determine different simplicial spaces. There is, for example, a natural atlas on s^n, for $n \geqslant 1$, generated by the identity id: $s^n \to s^n$, and the resulting simplicial space is different than any simplicial space determined by a triangulation of s^n. (Similar remarks hold for R^n, for $n \geqslant 1$, and many CW complexes.) Thus the category of polyhedra can be considered as a full subcategory of the category of simplicial spaces.

Many of the following results also hold for triangulable spaces and, in particular, for PL manifolds as a consequence of naturality of the appropriate constructions with respect to subdivision. In this paper, however, we consider only simplicial spaces since these spaces are in many ways more natural to use.

A simplicial space X may be described equivalently as a topological space X equipped with an atlas of related, locally finite, finite dimensional "packets" on X.

1.2. DEFINITION. A *packet* on a topological space X is a pair (U, Ψ) consisting of a nonempty open subset U of X and a collection Ψ of related simplicial charts (U_α, ψ_α) on X, α in some finite index set A, where $U = \bigcup_{\alpha \in A} U_\alpha$ and for which:

(1) U_α is an open subset of U_β iff $\alpha = \beta$, and

(2) for α, $\beta \in A$ there is a $\gamma \in A$ for which $U_\gamma = U_\alpha \cap U_\beta$.

The "domains" U of packets (U, Ψ) may be characterized intrinsically; such sets are called small open subsets: Let $x_0 \in X$. For every simplicial chart $\psi_\alpha \colon U_\alpha \to V_\alpha \subseteq s_\alpha^n$ on X for which $x_0 \in U_\alpha$, $\psi_\alpha(x_0)$ is contained in the interior of a unique face $s_\alpha^n(x_0)$ of s^n.

1.3. DEFINITION. An open neighborhood U of x_0 is *small* iff U may be written as the finite union of domains U_α of simplicial charts $\psi_\alpha \colon U_\alpha \to V_\alpha \subseteq s_\alpha^n$ on X, U_α containing x_0, such that V_α meets no face of s_α^n of dimension less than or equal to that of $s_\alpha^n(x_0)$, except $s_\alpha^n(x_0)$ itself, of course. A *small open subset* U of X is an open subset of X which is a small open neighborhood of some $x_0 \in U$.

The set of small open subsets of a simplicial space X forms a basis for the topology of X.

Every small open subset has a natural decomposition into "wedges".

1.4. DEFINITION. A *wedge* U_α of the small open subset U is a subset of U which is the domain of a simplicial chart (U_α, ψ_α) in a packet structure Ψ on U.

We now do analysis on simplicial spaces.

A *smooth function* $f \colon s^n \to R$ on the n-corner s^n is a map which is smooth when restricted to the interior of each face of s^n and such that for each x_0 on the boundary of s^n, the limit of $Df(x)$ as x approaches x_0 is finite for every differential operator

$$D = \partial^{q_1 + \cdots + q_n} / \partial x_1^{q_1} \cdots \partial x_n^{q_n}.$$

A *smooth p-form* on s^n is any differential form $\sum f_k \, df_{k_1} \cdots df_{k_p}$ where $f_k, f_{k_1}, \ldots, f_{k_p}$ are smooth functions on s^n.

1.5. DEFINITION. For $p = 0, 1, 2, \ldots$ and any small open subset U of the simplicial space X, the set $\Lambda^p E(U)$ of *piecewise smooth p-forms* on U is the set of compatible tuples $\theta = (\theta_\alpha)$ of smooth p-forms θ_α defined on the wedges U_α of U.

By "compatible" we mean that if U_α and U_β are two wedges of U then $\theta_{\alpha|U_\alpha \cap U_\beta} = \theta_{\beta|U_\alpha \cap U_\beta}$ where, for a smooth p-form $\theta = \sum f_k \, df_{k_1} \cdots df_{k_p}$ on U_α,

$$\theta_{|U_\alpha \cap U_\beta} = \sum (f_{k|U_\alpha \cap U_\beta}) d(f_{k_1|U_\alpha \cap U_\beta}) \cdots d(f_{k_p|U_\alpha \cap U_\beta}).$$

In particular $(p = 0)$, the ring $A(U)$ of piecewise smooth functions on U consists of those continuous functions $f \colon U \to R$ for which f is smooth on each wedge of U. The set of piecewise smooth p-forms on U becomes an $A(U)$-module by using wedgewise operations.

Wedgewise differentiation of piecewise smooth forms gives a cochain complex $(\Lambda^* E(U), d(U))$. This construction is natural, and hence defines a cochain complex of presheaves $(\Lambda^* \underline{E}(X), \underline{d}(X))$ on X.

1.6. DEFINITION. The *de Rham complex* $(\Lambda^* E(X), d(X))$ of piecewise smooth forms on the simplicial space X is the cochain complex of global sections of the cochain complex of sheaves associated to $(\Lambda^* \underline{E}(X), \underline{d}(X))$.

Again, in particular $(p = 0)$, the ring $A(X)$ of piecewise smooth functions on the simplicial space X consists of all those continuous functions $f: X \to R$ for which $f \circ \psi^{-1}: V \to R$ is smooth on V for every simplicial chart $\psi: U \to V$ on X. Furthermore the set of global piecewise smooth p-forms on X naturally becomes an $A(X)$-module.

1.7. THEOREM (DE RHAM). *The homology of the de Rham complex $(\Lambda^* E(X),$ $d(X))$ of piecewise smooth forms on the simplicial space X is isomorphic to the real singular cohomology of X.*

It is not difficult to define the integral of a piecewise smooth form on a simplicial space. Integration yields a map from the graded module of piecewise smooth forms on a simplicial space X to the graded module of singular simplicial cochains on X. Integration is a cochain map (Stokes's theorem), and hence induces a map in cohomology; de Rham's theorem states that this cohomology map is a ring isomorphism.

A cosimplicial bundle is a continuous surjection $\xi: E \to X$ from a topological space E to a simplicial space X for which $\xi^{-1}(x)$ is a finite-dimensional real vector space for each $x \in X$, and which satisfies certain other criteria. There is a category of cosimplicial bundles which contains a subcategory of vector bundles. In general, however, cosimplicial bundles differ from vector bundles since fiber dimensions in cosimplicial bundles are allowed to vary. A section s of a cosimplicial bundle $\xi: E \to X$ is a (not necessarily continuous) function $s: X \to E$ for which $\xi \circ s = \mathrm{id}_X$ and which satisfies certain other criteria; the set $\Gamma(X, \xi)$ of global sections of ξ forms an $A(X)$-module.

In much the same way every smooth manifold has a cotangent object in the category of smooth vector bundles, every simplicial space X has a related cotangent object $\tau^*(X)$ in the category of cosimplicial bundles. The cosimplicial bundle $\tau^*(X)$ is a cotangent object for X since there is an $A(X)$-module isomorphism between the module $E(X)$ of global piecewise smooth 1-forms on X and the module $\Gamma(X, \tau^*(X))$ of global sections of $\tau^*(X)$. Such cotangent objects are not usually vector bundles.

2. Simplicial bundles. The category of simplicial bundles is the dual of the category of cosimplicial bundles. Every simplicial space has a tangent object in the category of simplicial bundles in much the same way every smooth manifold has a tangent object in the category of smooth vector bundles.

Henceforth all vector spaces are finite-dimensional, real, and carry the usual topology.

2.1. DEFINITION. A *simplicial bundle* is a continuous surjection $\xi: E \to X$

from a topological space E to a simplicial space X such that $\xi^{-1}(x)$ is a vector space for each $x \in X$, and:

(1) for every simplicial chart (U_α, ψ_α) on X there is a vector space F_α and a fiber preserving embedding $\phi_\alpha : U_\alpha \times F_\alpha \to E$ which is a linear injection on each fiber,

(2) if (U_α, ψ_α) and (U_β, ψ_β) are simplicial charts on X for which $U_\alpha \subseteq U_\beta$, then $\phi_\alpha(x, -) = \phi_\beta(x, -) \circ l_{\beta\alpha}$ for every $x \in U_\alpha$ and some fixed linear embedding $l_{\beta\alpha} : F_\alpha \to F_\beta$,

(3) if (U_α, ψ_α), (U_β, ψ_β), and (U_γ, ψ_γ) are simplicial charts on X for which $U_\alpha \subseteq U_\beta \subseteq U_\gamma$, then $l_{\gamma\alpha} = l_{\gamma\beta} \circ l_{\beta\alpha}$,

(4) if (U_α, ψ_α) and (U_β, ψ_β) are simplicial charts on X for which U_α is an open subset of U_β, then $l_{\beta\alpha}$ is a linear isomorphism, and

(5) for each $x \in X$, $\xi^{-1}(x)$ is spanned by the set of all $\phi_\alpha(x, -)F_\alpha$ such that (U_α, ψ_α) is a simplicial chart on X and $x \in U_\alpha$.

Simplicial bundles and the appropriate morphisms form a category.

2.2. EXAMPLE. Let $X = P$ be the polyhedron which has three vertices x_0, x_1, and x_2, and which has two 1-simplices (x_0, x_1) and (x_0, x_2). Let $\pi_1 : X \times R^2 \to X$ be the product vector bundle, R^2 being equipped with basis vectors $\underline{e}_1$ and $\underline{e}_2$. Let E be the subspace of $X \times R^2$ consisting of all those (x, v) for which v is a multiple of $\underline{e}_k$ if $x \in (x_0, x_k) - \{x_0\}$, and $v \in R^2$ if $x = x_0$. If $\xi : E \to X$ is the map given by $\xi(x, v) = x$ for $(x, v) \in E$, then ξ is a simplicial bundle. The structure fibers F_α of ξ are as follows: F_α is the zero vector space R^0 if $U_\alpha = \{x_0\}$ and otherwise F_α is the vector space on one generator $\underline{e}_k$ whenever $U_\alpha \subseteq (x_0, x_k)$. The structure maps $\phi_\alpha : U_\alpha \times F_\alpha \to E$ of ξ are as follows: $\phi_\alpha(x_0, 0) = (x_0, 0)$ if $U_\alpha = \{x_0\}$ and otherwise $\phi_\alpha(x, c\underline{e}_k) = (x, c\underline{e}_k)$ whenever $U_\alpha \subseteq (x_0, x_k)$. Notice that ξ is clearly not a vector bundle since $\xi^{-1}(x_0) = R^2$ and $\xi^{-1}(x) = R^1$ for all other $x \in X$. (See Diagram 1.)
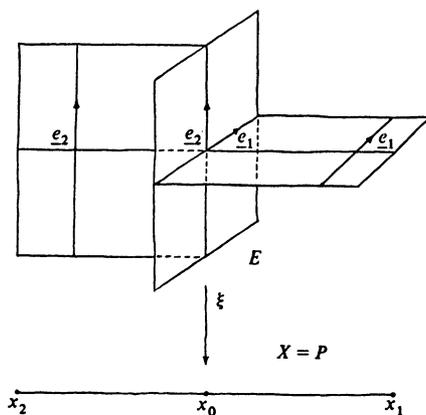

DIAGRAM 1

For any simplicial bundle $\xi: E \to X$, the inverse image $\xi^{-1}(x)$ of any $x \in X$ is given by the data $\{\phi_\alpha(x, -)F_\alpha; l_{\beta\alpha}\}$ (Axiom 5 of Definition 2.1). This is why in Example 2.2 the structure fiber $F_\alpha = R^0$ of ξ associated to the simplicial chart with domain $U_\alpha = \{x_0\}$ and the inverse image $\xi^{-1}(x_0) = R^2$ of x_0 are so radically different.

The following result helps motivate Definition 2.1 (see [13]). First, however, observe that a small open subset U of a simplicial space X may be viewed as the free union $\coprod V_\alpha$ of copies of open subsets $V_\alpha \subseteq s_\alpha^n$ (n may vary with α) under the identification $R: x_\alpha R x_\beta$ iff x_β is the image of x_α under $\psi_\beta \circ \psi_\alpha^{-1}$, i.e., $U = \coprod V_\alpha/R$.

2.3. THEOREM. *There is a functor τ from the category of simplicial spaces to the category of simplicial bundles*

$$\tau: X \rightsquigarrow (\tau(X): TX \to X)$$

which satisfies the following criteria:

(1) $\tau(V) = \pi_1: TV = V \times R^n \to V$ *for every open subset $V \subseteq s^n$, for any n, π_1 denoting projection onto the first factor,*

(2) *if $U = \coprod V_\alpha/R$ is a small open subset of the simplicial space X as described above, then $\tau(U)$ is the simplicial bundle generated by*

$$\pi: \coprod TV_\alpha/R_* \to U$$

where R_* is the identification: $v_\alpha R_* v_\beta$ iff v_β is the image of v_α under the Jacobian $(\psi_\beta \circ \psi_\alpha^{-1})_*$ of $\psi_\beta \circ \psi_\alpha^{-1}$, and where π is the map induced by the maps $\psi_\alpha^{-1} \circ \tau(V_\alpha)$, and

(3) *if U is a small open subset of the simplicial space X then $\tau(U)$ is the restriction of $\tau(X)$ to $TU = \tau(X)^{-1}U$.*

(*Note.* In criterion (2) above, "generated" means that, for each $x \in U$, $\tau(U)^{-1}x$ is the smallest vector space which spans $\coprod \tau(V_\alpha)^{-1}x/R_*$.)

PROOF. First let us recall that if X is a simplicial space and U is a small open neighborhood of $x_0 \in X$, then a *coordinate system u on U* (with origin x_0) is a collection of piecewise affine functions $u_k: U \to R$ such that, for each wedge U_α of U, $u_{k|U_\alpha} = 0$ except for a subset of $\dim U_\alpha$ functions, and the set $\{u_\alpha^*\}$ of functions $u_\alpha^* = u_{k|U_\alpha}$ for u_k in this subset forms a coordinate system on U_α (with origin $x_0 \in U_\alpha$). If x_0 is the origin of the coordinate system u on U then U is necessarily a small open neighborhood of x_0.

If X is a simplicial space and U is a small open neighborhood of $x_0 \in X$, let V be the vector space which has for a basis the vectors $\{(\partial/\partial u_k)_{x_0}: u_k \in u\}$. Let TU be the subspace of $U \times V$ consisting of all those (x, v) for which the kth component c_k of $v = \sum c_k(\partial/\partial u_k)_{x_0}$ is nonzero only if there is a wedge U_α

of U for which $x \in U_\alpha$ and $u_{k|U_\alpha} \in \{u_\alpha^*\}$, and let $\tau(U)$: $TU \to U$ be projection onto the first factor. Then:

(1) if U_α is a wedge of U and $\tilde{F}_\alpha$ is the vector space with basis $\{(\partial/\partial u_\alpha^k)_{x_0} : u_\alpha^k \in \{u_\alpha^*\}\}$, then the map $\tilde{\phi}_\alpha : U_\alpha \times \tilde{F}_\alpha \to TU$ given by

$$\tilde{\phi}_\alpha\left(x, \sum c_k(\partial/\partial u_\alpha^k)_{x_0}\right) = \left(x, \sum c_k(\partial/\partial u_k)_{x_0}\right)$$

is a fiber preserving embedding which is a linear injection on each fiber,

(2) if U_α and U_β are wedges of U for which $U_\alpha \subseteq U_\beta$, then $\tilde{\phi}_\alpha(x, -) = \tilde{\phi}_\beta(x, -) \circ \tilde{l}_{\beta\alpha}$ for every $x \in U_\alpha$ where $\tilde{l}_{\beta\alpha}$ is the (Jacobian) map given on generators by $\tilde{l}_{\beta\alpha}(\partial/\partial u_\alpha^k)_{x_0} = (\partial/\partial u_\beta^k)_{x_0}$, and

(3) if U_α, U_β, and U_γ are wedges of U for which $U_\alpha \subseteq U_\beta \subseteq U_\gamma$, then $\tilde{l}_{\gamma\alpha} = \tilde{l}_{\gamma\beta} \circ \tilde{l}_{\beta\alpha}$.

The global construction of τ follows from the following facts (see [11]):

2.4. *Fact.* If u is a coordinate system on the small open subset U with origin x_0, then for any other $x_1 \in U$, a coordinate system on a small open neighborhood $W \subseteq U$ of x_1 with origin x_1 may be obtained by translating coordinate functions in some subset of u.

2.5. *Fact.* If u' and u'' are coordinate systems on the small open subset U both with origin $x_0 \in U$, then for every $u_j \in u'$ there are unique real constants c_{jk} for which $u_j = \sum c_{jk} u_k$, $u_k \in u''$.

The total space TX of $\tau(X)$ is defined by covering X with all possible combinations of small open subsets U equipped with coordinate systems u and letting TX be the free union of all the spaces TU as constructed above modulo the identification T: $(x', v') T(x'', v'')$ iff $x' = x''$ and, after translating coordinate systems u' and u'' to x' and x'', respectively, $v' = \sum_{jk} c_k c_{jk}(\partial/\partial u_j)_x$ for $u_j \in u'$ where $x = x' = x''$, where $v'' = \sum_k c_k(\partial/\partial u_k)_x$, $u_k \in u''$, and where the constants c_{jk} are given by $u_j = \sum c_{jk} u_k$. Let $\tau(X)$: $TX \to X$ be the induced projection.

To show that $\tau(X)$ is a simplicial bundle, for every simplicial chart (U_α, ψ_α) on X let U be a small open subset of X for which U_α is a wedge of U, and let u be a coordinate system on U. Let F_α be the structure fiber $\tilde{F}_\alpha$ of $\tau(U)$: $TU \to U$ associated to U_α, and let $\phi_\alpha : U_\alpha \times F_\alpha \to TX$ be $\tilde{\phi}_\alpha$ followed by inclusion. If (U_α, ψ_α) and (U_β, ψ_β) are simplicial charts on X for which $U_\alpha \subseteq U_\beta$, let $l_{\beta\alpha} = \tilde{l}_{\beta\alpha}$.

Finally, if u' and u'' are two different coordinate systems on U both with origin x_0 then there is a linear isomorphism l: $F'' \to F'_\alpha$ for which $\phi_\alpha''(x, -) = \phi_\alpha'(x, -) \circ l$ for each $x \in U_\alpha$: l is described on generators by $l(\partial/\partial u_\alpha^k) = \sum_j c_{jk}(\partial/\partial u_\alpha^j)$. To verify that $\phi_\alpha''(x, -) = \phi_\alpha'(x, -) \circ l$, observe that $\phi_\alpha'(x, l(\partial/\partial u_\alpha^k)) = (x, \sum_j' c_{jk}(\partial/\partial u_j))$ where the summation is taken over all j for which $u_{j|U_\alpha} \in \{u_\alpha^*\}'$, and that $\phi_\alpha''(x, \partial/\partial u_\alpha^k) = (x, \sum_j c_{jk}(\partial/\partial u_j))$ where the

summation is taken over all j. The conclusion follows since $c_{jk} = 0$ if $u_{j|U_\alpha} = 0$ and $u_{k|U_\alpha} \in \{u_\alpha^*\}''$.   Q.E.D.

2.6. DEFINITION. The functor τ is the *tangent functor*, and the total space TX of the tangent bundle $\tau(X): TX \to X$ of the simplicial space X is the *tangent space* of X.

2.7. EXAMPLE. If $X = R^n$ (R^n being equipped with the trivial simplicial structure), then the tangent bundle of X is precisely the classical smooth tangent (vector) bundle of R^n, namely $\tau(X) = \pi_1: TX = X \times R^n \to X$, π_1 denoting projection onto the first factor.

2.8. EXAMPLE. The simplicial bundle $\xi: E \to X$ constructed in Example 2.2 is the tangent object $\tau(X): TX \to X$ of the simplicial space $X = P$.

2.9. REMARK. The construction of tangent objects for simplicial spaces is natural in the sense that if $f: X_1 \to X_2$ is a map of simplicial spaces, then there is a natural way to define the *Jacobian* $f_*: TX_1 \to TX_2$ of f and a corresponding morphism $\tau(X_1) \to \tau(X_2)$ of simplicial bundles. It is easy to show that $\mathrm{id}_* = \mathrm{id}: TX \to TX$ for id: $X \to X$ the identity map, and that given simplicial maps $f: X_1 \to X_2$ and $g: X_2 \to X_3$, $(g \circ f)_* = g_* \circ f_*$.

It is Theorem 2.3 which will clarify the relation between de Rham theory and vector field theory for simplicial spaces (see Remark 2.10 and Definition 2.11), and the relation between vector fields and piecewise smooth flows on simplicial spaces (see Proposition 3.6).

2.10. REMARK. Having introduced Riemannian metrics on cosimplicial bundles, one can associate a simplicial bundle ξ to each metrizable cosimplicial bundle ξ^*; ξ is the *dual* of ξ^*. Dualization is a natural equivalence between the categories of metrizable simplicial and metrizable cosimplicial bundles. The tangent objects and cotangent objects of simplicial spaces are metrizable and correspond to each other under dualization (see §5).

We now define sections of simplicial bundles. A section of a simplicial bundle $\xi: E \to X$ will be a function (not necessarily a continuous function) $s: X \to E$ for which $\xi \circ s = \mathrm{id}_X$ and which satisfies certain criteria. Sections are not required to be continuous since continuity implies excessive restrictions: in Example 2.2, for instance, a continuous function $s: X \to E$ for which $\xi \circ s = \mathrm{id}_X$ would necessarily be zero at x_0.

2.11. DEFINITION. A *section* s of the simplicial bundle $\xi: E \to X$ is a (not necessarily continuous) function $s: X \to E$ such that $\xi \circ s = \mathrm{id}_X$ and such that for every small open subset U of X there are piecewise smooth sections s_α of some of the product vector bundles $U_\alpha \times F_\alpha \to U_\alpha$, U_α a wedge of U and F_α the structure space of ξ associated to U_α, for which:

(1) for every wedge U_β of U there is a wedge U_α of U containing U_β for which s_α is defined,

(2) if U_α and U_β are wedges of U for which $U_\alpha \subseteq U_\beta$ and for which sections

s_α and s_β are defined, then $s_{\beta|U_\alpha} = (\text{id} \times l_{\beta\alpha}) \circ s_\alpha$, and

(3) for each $x \in U$, $s(x) = \Sigma\ (\phi_\alpha \circ s_\alpha)x$, the summation taken over all α for which $x \in U_\alpha$, s_α is defined, and the $(\phi_\alpha \circ s_\alpha)x$ are distinct.

A section of the tangent object $\tau(X)$ of the simplicial space X is a *vector field* on X.

The remainder of this paper is concerned with studying vector fields on simplicial spaces.

In [7], Hopf obtains results for "vector fields" on a class of spaces which can be embedded in Euclidean space. If, for example, P is a polyhedron embedded in R^N in general position, then a "vector field" on P in Hopf's sense is an association of a tangent vector in $T_x R^N$ to each point $x \in P$ which satisfies two conditions. Such "vector fields" can naturally be constructed in either of two ways: First a "vector field" can be constructed from a small transformation $f: P \to P$ of P; such a map is a "vector field" in the sense that it corresponds to a right inverse $s: P \to P \times P \subseteq P \times R^N$ to projection $\pi_1: P \times R^N \to P$ onto the first factor: $s(x) = (x, f(x))$. Second, in order to obtain results which are local (i.e., only depend on behavior in arbitrarily small neighborhoods of points), one can consider sequences $f_k: P \to P$, $k = 1, 2, \ldots$, of small transformations of P which converge uniformly to the identity $\text{id}: P \to P$; the value at the point $x \in P$ of the "vector field" s associated to such a sequence of small transformations is given by

$$s(x) = \lim_{k \to \infty} \frac{(x, f_k(x))}{|(x, f_k(x))|},$$

if this limit exists.

With respect to this latter construction, the point in talking about simplicial bundles is that the set of "tangent vectors" produced in this manner span a subspace of TR^N which is intrinsic to P, i.e., which can be defined independently of an embedding of P in R^N. In fact, if $r: P \to R^N$ is any embedding of P in R^N in general position, then this subspace is precisely the image $r_* TP$ of the tangent space TP of P under the Jacobian r_* of r, and the following diagram commutes:

$$
\begin{array}{ccc}
TP & \xrightarrow{\ r_*\ } & TR^N \\
\Big\downarrow{\scriptstyle \tau(P)} & & \Big\downarrow{\scriptstyle \tau(R^N)} \\
P & \xrightarrow[\ r\ ]{} & R^N
\end{array}
$$

Furthermore, the constructions in [7] can be performed without embedding P in R^N.

3. Piecewise smooth flows. After defining piecewise smooth flows on simplicial spaces we will show that every piecewise smooth flow on the simplicial space X generates a vector field on X, and conversely that every vector field on X which satisfies certain integrability conditions generates a piecewise smooth flow on X.

3.1. DEFINITION. A *preflow* F on the n-corner s^n is a continuous function $F: \underline{s^n \times R} \to s^n$, where $\underline{s^n \times R}$ is a (not necessarily open) subset of $s^n \times R$ containing $s^n \times \{0\}$, such that:

(1) for each x_0 in the interior of s^n there is a neighborhood $N \times J \subseteq \underline{s^n \times R}$ of $(x_0, 0)$ which is open in $s^n \times R$ and over which F is a smooth flow,

(2) if $F(x, t) = (F_k(x, t))$ and $\overset{\circ}{F}_k(x, t)$ is the restriction of $F_k(x, t)$ to the interior of $\underline{s^n \times R}$, then $(\partial \overset{\circ}{F}_k(x, t)/\partial t)_{|t=0}$ extends to a (necessarily unique) smooth function $(\partial F_k(x, t)/\partial t)_{|t=0}$ on s^n, and

(3) if $F': (s^n \times R)' \to s^n$ is any map satisfying the above conditions and $F(x, t) = F'(x, t)$ for all (x, t) in $(s^n \times R) \cap (s^n \times R)'$ then $\underline{s^n \times R} = \underline{(s^n \times R)'}$.

The relative version of a preflow is a map $F: \underline{W \times I} \to s^n$, where W is some small open subset of s^n and I is some interval neighborhood of $0 \in R$, for which the relative versions of the above criteria are met.

If $\partial/\partial x_k$, $k = 1, 2, \ldots$, are the standard tangent vector fields on s^n then a preflow F on the n-corner s^n naturally induces the vector field

$$\sum (\partial F_k(x, t)/\partial t)_{|t=0}(\partial/\partial x_k)$$

on s^n. Preflows are introduced to establish a converse to this construction. In general the concept of flow will not suffice: On s^2, for example, consider the vector field $v = \partial/\partial x_1 - \partial/\partial x_2$. If this vector field is extended to the smooth vector field $\underline{v} = \partial/\partial x_1 - \partial/\partial x_2$ on R^2, then the smooth flow $\underline{F}$ on R^2 which generates $\underline{v}$ is given by $\underline{F}(x_1, x_2, t) = (x_1 + t, x_2 - t)$. Restricting $\underline{F}$ to

$$\underline{s^2 \times R} = \{(x_1, x_2, t) \in s^2 \times R: \underline{F}(x_1, x_2, t) \in s^2\}$$

we obtain a preflow F on s^2 which generates v. Observe that the only value of t for which $(0, 0, t)$ is in the domain of F is $t = 0$.

3.2. LEMMA. *Let U and W be small open neighborhoods of the origin $0 \in s^n$ which have compact closure and for which the closure of W is contained in U. If $\sum f_k(\partial/\partial x_k)$ is any vector field on U (where $f_k \in A(U)$ for each k) then there is a unique preflow $F: \underline{W \times I} \to U$ for which $f_k = (\partial F_k(x, t)/\partial t)_{|t=0}$ for each k.*

PROOF. For $c \in R$ let $\underline{c} = (c + |c|)/2 \in R^+$, and for $x = (x_k) \in R^n$ let $\underline{x} = (\underline{x}_k) \in s^n$. On the open neighborhood

$$\underline{U} = \{x \in R^n: \underline{x} \in U \text{ and } x_k > -1 \ \forall \ k\}$$

of $0 \in R^n$ define $\underline{f}_k \colon \underline{U} \to R$ by $\underline{f}_k(x) = f_k(\underline{x})$, for each k, and $\underline{f} = (\underline{f}_k) \colon \underline{U} \to R^n$; let $\underline{W} = \{x \in \underline{U} \colon \underline{x} \in W$ and $x_k > -\frac{1}{2} \forall k\}$. Since $\underline{U}$ and $\underline{W}$ have compact closure it is not difficult to verify the Lipschitz condition

$$|\underline{f}(x_0) - \underline{f}(x_1)| < K|x_0 - x_1|$$

for $x_0, x_1 \in \underline{W}$ and some fixed constant K.

Now apply Picard's method of successive approximations (see [6]) to the equations

$$\partial \underline{F}(x, t)/\partial t = \underline{f}(\underline{F}(x, t)), \qquad \underline{F}(x, 0) = x,$$

to obtain a unique flow $\underline{F} \colon \underline{W} \times I \to \underline{U}$, and define the preflow F on W by restricting $\underline{F}$ to

$$\underline{W \times I} = \{(x, t) \in W \times I \colon F(x, t) \in U\}.$$

The remainder of the lemma is immediate.   Q.E.D.

It is not difficult to extend Lemma 3.2 to the case in which U and W are small open neighborhoods of any point $x_0 \in s^n$.

3.3. DEFINITION. A *piecewise smooth flow* F on the simplicial space X is a map $F \colon X \times I \to X$, I an interval neighborhood of the origin $0 \in R$, such that:

(1) $F(x, 0) = x$ and $F(x, t_1 + t_2) = F(F(x, t_1), t_2)$ for all $x \in X$ and $t_1, t_2 \in I$ for which $t_1 + t_2 \in I$, and

(2) if U and W are small open subsets of X for which the closure of W is contained in U, then for every wedge U_β of U there is a wedge U_α of U for which $U_\beta \subseteq U_\alpha$ and $F \colon W_\alpha \times I \to U_\alpha$ is a preflow on W_α where

$$\underline{W_\alpha \times I} = \{(x, t) \in W_\alpha \times I \colon F(x, t) \in U_\alpha\}$$

and $W_\alpha = W \cap U_\alpha$.

The purpose of the following result is to show existence of piecewise smooth flows.

3.4. PROPOSITION. *Let* $T \colon X \to R^n$ *be a triangulation of* R^n *by a simplicial space* X. *If* $F \colon R^n \times R \to R^n$ *is a smooth flow on* R^n, *then the map*

$$T^{-1} \circ F \circ (T \times \mathrm{id}) \colon X \times R \to X$$

is a piecewise smooth flow on X.

PROOF. The first condition of Definition 3.3 is immediate. To verify the second condition, let U and W be small open subsets of X as indicated. Then for every wedge U_β of U there is a wedge U_α of U of dimension n containing U_β, and it is easy to show that $F \colon \underline{W_\alpha} \times R \to U_\alpha$ is a preflow.   Q.E.D.

3.5. EXAMPLE. Consider the triangulation $T \colon X \to R^1$ of R^1 illustrated in Diagram 2 which is induced by the identity $\mathrm{id} \colon R^1 \to R^1$. Equip X and R^1 with

coordinate systems as illustrated. If $F: R^1 \times R \to R^1$ is the smooth flow given by $F(x,t) = x + t$, then $T^{-1} \circ F \circ (T \times \mathrm{id}): X \times R \to X$ is a piecewise smooth flow on X. In this case

$$F_{|\underline{U_\alpha \times R}}(u_\alpha, t) = u_\alpha + t \quad \text{where } \underline{U_\alpha \times R} = \{(u_\alpha, t) \in U_\alpha \times R: u_\alpha + t \geqslant 0\},$$

and

$$F_{|\underline{U_\beta \times R}}(u_\beta, t) = u_\beta + t \quad \text{where } \underline{U_\beta \times R} = \{(u_\beta, t) \in U_\beta \times R: u_\beta - t \geqslant 0\}.$$
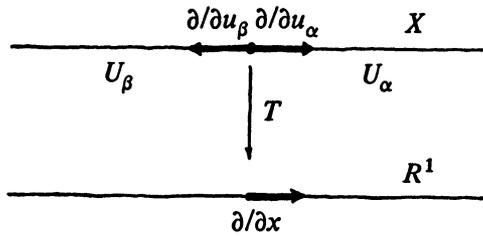

DIAGRAM 2

The next result relates piecewise smooth flows and vector fields on simplicial spaces.

Let U and W be small open subsets of the simplicial space X which have compact closure and for which the closure of W is contained in U, and for every wedge U_α of U let $W_\alpha = W \cap U_\alpha$.

3.6. PROPOSITION. *If $F: W \times I \to U$ is a piecewise smooth flow on W then*:

(1) *for every wedge U_β of U there is a wedge U_α of U containing U_β for which* $F_\alpha = F_{|\underline{W_\alpha \times I}}: W_\alpha \times I \to U_\alpha$ *is a preflow on W_α where*

$$\underline{W_\alpha \times I} = \{(x, t) \in W_\alpha \times I: F(x, t) \in U_\alpha\},$$

and

(2) *if U_α and U_β are wedges of U for which $U_\alpha \subseteq U_\beta$ and for which preflows F_α and F_β are defined, then $F_{\beta|\underline{W_\alpha \times I}} = F_\alpha$.*

Conversely, a collection of preflows $F_\alpha: \underline{W_\alpha \times R} \to U_\alpha$ defined on some of the wedges W_α of W such that:

(3) *there is a positive number ε such that, for each $(x,t) \in W \times (-\varepsilon, \varepsilon)$, $(x, t) \in \underline{W_\alpha \times R}$ for some wedge W_α of W (W_α may vary with t), and*

(4) *whenever $(x, t) \in (\underline{W_\alpha \times R}) \cap (\underline{W_\beta \times R})$, $F_\alpha(x, t) = F_\beta(x, t)$, generates a piecewise smooth flow $F: \overline{W \times I} \to U$, I an interval neighborhood of $0 \in R$. Furthermore, if $F: W \times I \to U$ is a piecewise smooth flow and the corresponding collection of preflows $F_\alpha: W_\alpha \times I \to U_\alpha$ satisfies conditions (3) and (4), then the piecewise smooth flow constructed from the F_α's is F: and if F*

is a piecewise smooth flow constructed from a collection of preflows satisfying conditions (3) *and* (4) *then the collection of preflows generated by F contains each original preflow.*

PROOF. The first part of this result is immediate.

Given a collection of preflows $F_\alpha\colon \underline{W_\alpha \times R} \to U_\alpha$ defined on some of the wedges W_α of W which satisfy conditions $\overline{(3)}$ and (4), let I denote the largest possible open interval $(-\varepsilon, \varepsilon)$ for which condition (3) holds. For $(x, t) \in W \times I$ we define $F(x, t)$ as $F_\alpha(x, t)$ for any α for which $(x, t) \in \underline{W_\alpha \times R}$; this α exists by condition (3), and $F(x, t)$ is well defined by condition $\overline{(4)}$.

The remainder of this result is immediate.   Q.E.D.

3.7. COROLLARY. *Every piecewise smooth flow F: $X \times I \to X$ on the simplicial space X generates a vector field on X. Conversely let s: $X \to TX$ be a vector field on the compact simplicial space X. If, for all choices of U and W as above, the piecewise smooth sections $s_\alpha\colon U_\alpha \to TU_\alpha = U_\alpha \times F_\alpha$ defined on the wedges U_α of U generate preflows $F_\alpha\colon \underline{W_\alpha \times R} \to U_\alpha$ on W_α which satisfy conditions* (3) *and* (4) *of Proposition* 3.6, *then s generates a piecewise smooth flow on X.*

PROOF. If U and W are small open neighborhoods of $x \in U$ for which the closure of W is contained in U, then, with the notation of Proposition 3.6, each preflow $F_\alpha\colon W_\alpha \times I \to U_\alpha$ generates a vector field $s_\alpha\colon W_\alpha \to TW_\alpha$ on W_α. We define $s\colon X \to TX$ by $s(x) = \sum (\phi_\alpha \circ s_\alpha)x$, the summation taken over all α for which $x \in U_\alpha$, s_α is defined, and the $(\phi_\alpha \circ s_\alpha)x$ are distinct. The remainder of the corollary follows immediately from Lemma 3.2.   Q.E.D.

Conditions (3) and (4) correspond closely to the two conditions in [7] mentioned at the end of §2 which are necessary for a map $s\colon P \to P \times R^N$ to be a "vector field".

We will now use Corollary 3.7 to construct a piecewise smooth flow on an arbitrary polyhedron. In this construction, and frequently throughout the sequel, we will assume the polyhedron P is described by cells $B^n = [-1, 1]^n$ $\subseteq R^n$ instead of by simplices

$$\Delta^n = \left\{ x = (x_k) \in R^n \colon x_k \in [0, 1] \;\forall k \text{ and } \sum x_k = 1 \right\} \subseteq R^n.$$

This is done strictly for mechanical convenience: All pertinent results are valid for simplicial polyhedra as well as for cell polyhedra.

To be precise: An *n-cell B^n in R^N* is the image of the standard *n*-cell $B^n = [-1, 1]^n \subseteq R^n$ under an injective linear transformation $T\colon R^n \to R^N$. A *cell polyhedron P* is a collection of cells $\{B_\alpha^n\}$ in R^N, for some N, such that:

(1) if B_α^n is in P then each (closed) face of B_α^n is in P, and

(2) if B_α^m and B_β^n are in P, then $B_\alpha^m \cap B_\beta^n$ is a (unique) face of both B_α^m and B_β^n.

Every cell polyhedron may be topologized as a subset of Euclidean space,

and with this topology is combinatorially equivalent to a simplicial polyhedron.

3.8. COROLLARY. *Every compact (or locally finite) cell polyhedron P admits a nontrivial piecewise smooth flow.*

PROOF. Let $\phi: [-1, 1] \to [0, 1]$ be a smooth function for which $\phi(t) = -t^2 + 1$ for $t \in [0, 1/3)$, $\phi(t) = (t - 1)^2$ for $t \in (2/3, 1]$, $\phi(t) = \phi(-t)$ for $t \in [-1, 1]$, and $\phi'(t) = 0$ only for $t = 0, \pm 1$. Define the (piecewise) smooth real valued function f^n on the n-cell $B^n = [1, 1]^n$ by $f^n(x) = \sum_{k=1}^{n} \phi(x_k)$.

Since $\phi(\pm 1) = 0$, the functions $f_\alpha^n: B_\alpha^n \to R$ defined on the n-cells B_α^n of P define a global piecewise smooth function $f: P \to R$. Observe that if (U_α, ψ_α) is a simplicial chart on P then $U_\alpha \subseteq B_\alpha^n$ for some n-cell B_α^n, and thus $f_{|U_\alpha} = f_{\alpha|U_\alpha}^n$.

For every simplicial chart (U_α, ψ_α) on P let $s_\alpha: U_\alpha \to TU_\alpha = U_\alpha \times F_\alpha$ be the vector field given by $s_\alpha = \text{grad} f_\alpha^n$ where $U_\alpha \subseteq B_\alpha^n$ (and grad = gradient). If (U_α, ψ_α) and (U_β, ψ_β) are simplicial charts on P for which $U_\alpha \subseteq U_\beta$, then it is easy to verify that $s_{\beta|U_\alpha} = s_\alpha$. Define the *continuous* vector field (i.e., map) $s: P \to TP$ by letting $s(x) = s_\alpha(x)$ for (U_α, ψ_α) any simplicial chart for which $x \in U_\alpha$ and $s_\alpha = \text{grad} f_\alpha^n$. It is not difficult to verify conditions (3) and (4) of Proposition 3.6, and hence the conclusion follows.   Q.E.D.

For example, the graphs of $f^1: B^1 \to R$ and $f^2: B^2 \to R$ are illustrated in Diagram 3.

B^1

B^2
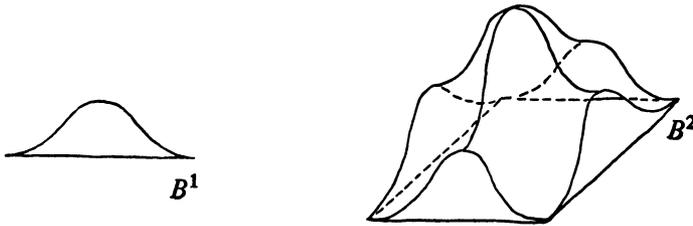
DIAGRAM 3

The flow lines for the corresponding piecewise smooth flows are illustrated in Diagram 4.

B^1

B^2

DIAGRAM 4

3.9. REMARK. Let x_0 be the barycenter of the m-face B^m of the n-cell $B^n = [-1, 1]^n$. Let u be the coordinate system obtained by translating the standard coordinate functions on R^n to x_0, and order $u = \{u_*\}$ so that $\{u_1, \ldots, u_m\}$ forms a coordinate system on B^m. Then in a neighborhood of x_0,

$$f^n(x) = \sum_{k=m+1}^{n} u_k^2 - \sum_{k=1}^{m} u_k^2$$

so that

$$\operatorname{grad} f^n(x) = \sum_{k=m+1}^{n} 2u_k \frac{\partial}{\partial u_k} - \sum_{k=1}^{m} 2u_k \frac{\partial}{\partial u_k}$$

and $F(x, t) = (u_k \exp c_k t)$ where $c_k = -2$ if $k = 1, \ldots, m$, and $c_k = 2$ if $k = m + 1, \ldots, n$.

4. A Hopf theorem. In this section we will associate indices to fixed points of piecewise smooth flows. We will show that if F is a piecewise smooth flow on the cell polyhedron P which has only a finite set of fixed points then the sum of the indices of these fixed points is the Euler characteristic of P. Finally we will show that if $P = M$ is a closed polyhedral manifold whose Euler characteristic is zero then there is a fixed point free piecewise smooth flow on P.

There is an index (see [3]) which associates to every self-map $f\colon P \to P$ of the cell polyhedron P, and small open subset U of P on whose boundary there are no fixed points of f, an integer $i(f, U)$ as follows: Suppose r is a realization of P in $R^N = S^N - \{(0, \ldots, 0, 1)\}$. Let $\rho\colon W \to rP$ be a retraction of an open subset $W \subseteq R^N$ onto rP. Let $V = \rho^{-1}(rU)$ and $\operatorname{Fix} rU$ denote the set of fixed points of f in rU (by abuse of notation we write f for rfr^{-1}). Define $d\colon V \to R^N$ by $d(x) = x - f\rho(x)$. Since ρ is a retraction, $d(x) = 0$ implies that $x = f(x)$ so that $d(V - \operatorname{Fix} rU) \subseteq R^N - \{0\}$. The inclusion j of $(V, V - \operatorname{Fix} rU)$ into $(S^N, S^N - \operatorname{Fix} rU)$ induces an excision isomorphism. The *index* $i(f, U)$ is the degree of the map

$$k^* \circ (j^*)^{-1} \circ d^* \colon H^N(R^N, R^N - \{0\}) \to H^N(S^N)$$

where the map k^* from $H^N(S^N, S^N - \operatorname{Fix} rU)$ to $H^N(S^N)$ is the connecting map from the exact sequence of the pair $(S^N, S^N - \operatorname{Fix} rU)$. A pair such as (f, U) is an *admissible pair*.

This index is independent of realization and retraction, and furthermore satisfies the following axioms:

AXIOM 1 (LOCALIZATION). If (f, U) is admissible and $g\colon P \to P$ is a map for which $g = f$ on the closure of U then $i(f, U) = i(g, U)$.

AXIOM 2 (HOMOTOPY). If, for a homotopy $f_t\colon P \times [0, 1] \to P$, (f_t, U) is

admissible for each $t \in [0, 1]$ then $i(f_0, U) = i(f_1, U)$.

AXIOM 3 (ADDITIVITY). *If (f, U) is admissible and $U_1, \ldots, U_m$ is a set of mutually disjoint small open subsets of U for which $f(x) \neq x$ for $x \in U - \bigcup_{k=1}^{m} U_k$, then $i(f, U) = \sum_{k=1}^{m} i(f, U_k)$.*

AXIOM 4 (NORMALIZATION). *If $\mathfrak{U}$ is a cover of P by small open sets for which (f, U_k) is admissible for all $U_k \in \mathfrak{U}$ and for which there are no fixed points of f in the overlap of sets of $\mathfrak{U}$ then $\sum_{U_k \in \mathfrak{U}} i(f, U_k)$ is the Lefshetz number of f.*

4.1. DEFINITION. Let $F: X \times I \to X$ be a piecewise smooth flow on the simplicial space X. The point $x_0 \in X$ is a *fixed point* of F iff $F(x_0, t) = x_0$ for all $t \in I$; x_0 is an *ordinary point* of F iff $F(x_0, -): I \to X$ is injective. If x_0 is a fixed point of F which has a neighborhood U in which there are no nonordinary points of F other than x_0, then the *index $i(F, x_0)$* of F at x_0 is $i(F(-, t_0), U)$ for any positive $t_0 \in R$.

This index is independent of the particular open neighborhood U as a result of the additivity axiom. It is also independent of the particular positive $t_0 \in R$ chosen: Given any other positive $t_1 \in R$, $F(-, st_1 + (1 - s)t_0)$ for $s \in [0, 1]$ is a homotopy between $F(-, t_0)$ and $F(-, t_1)$ to which the homotopy axiom applies.

The following lemma normalizes this index in another way (i.e., other than Axiom 4). First, however, observe that a piecewise smooth flow $F: X \times I \to X$ on the simplicial space $X = R^n$ (R^n being equipped with the trivial simplicial structure) is a smooth flow on R^n. Also recall that if $s: R^n \to TR^n$ is a smooth vector field on R^n which has an isolated zero at x_0, then the standard index $i(s, x_0)$ of s at x_0 is defined as follows:

Introduce coordinates $x_1, \ldots, x_n$ about x_0 with origin x_0 so that $s = \sum_{k=1}^{n} f_k(\partial/\partial x_k)$ where each $f_k: R^n \to R$ is smooth. Let $S^n(r)$ be a sphere of radius r centered at x_0, and map $S^n(r)$ to itself by

$$x \to r \left(\sum_{k=1}^{n} f_k(x)^2 \right)^{-1/2} (f_1(x), \ldots, f_n(x)).$$

The *index $i(s, x_0)$* is the degree of this map. If the determinant $|\partial f_j / \partial x_k|$ is nonzero at x_0 then x_0 is a nondegenerate zero of s, and $i(s, x_0)$ is plus or minus one as the determinant $|\partial f_j / \partial x_k|$ is positive or negative at x_0.

4.2. LEMMA. *Let $F: X \times R \to X$ be a piecewise smooth flow on the simplicial space $X = R^n$ and let x_0 be a fixed point of F for which there are no nonordinary points of F in a neighborhood U of x_0 other than x_0. If $s: R^n \to TR^n$ is the smooth vector field on R^n generated by F then $i(F, x_0) = i(s, x_0)$.*

PROOF. Assume that $x_0 = 0 \in R^n$. Let $W \subseteq U$ be a neighborhood of x_0 and I an interval neighborhood of $0 \in R$ for which $F: W \times I \to U$. Pick a positive $t_0 \in I$ and let $f: W \to U$ be given by $f(-) = F(-, t_0)$. Letting $\rho: W$

$\rightarrow W$ be the retraction given by the identity we have $x - f\rho(x) = x - f(x)$
$= F(x, 0) - F(x, t_0)$; using the homotopy $W \times [0, 1] \rightarrow U$ given by

$$(x, s) \rightarrow (F(x, 0) - F(x, st_0))/s$$

we may conclude that $x - f\rho(x)$ is homotopic to $(\partial F(x, t)/\partial t)_{|t=0}$ on W, and hence on $W - \{0\}$. Since $(\partial F(x, t)/\partial t)_{|t=0}$ has a zero only at the origin $0 \in R^n$, $x - f\rho(x)$ is homotopic to

$$(\partial F(x, t)/\partial t)_{|t=0}/|(\partial F(x, t)/\partial t)_{|t=0}|$$

on $W - \{0\}$.   Q.E.D.

4.3. THEOREM (HOPF). *If* $F: P \times R \rightarrow P$ *is a piecewise smooth flow on the cell polyhedron* P *which is ordinary except for a finite set* Fix F *of fixed points then* $\sum_{x_k \in \mathrm{Fix}\, F} i(F, x_k)$ *is the Euler characteristic* $\chi(P)$ *of* P.

PROOF. As a result of the index axioms, if $f: P \rightarrow P$ is a map which is homotopic to the identity id: $P \rightarrow P$, and if $\mathfrak{U}$ is a cover of P by small open subsets U_k for which (f, U_k) is admissible for each k and for which there are no fixed points of f in the overlap of sets of $\mathfrak{U}$, then $\sum_{U_k \in \mathfrak{U}} i(f, U_k) = \chi(P)$. The result follows since, for a fixed positive $t_0 \in R$, $F(-, st_0)$ for $s \in [0, 1]$ is a homotopy between $f(-) = F(-, t_0)$ and the identity id: $P \rightarrow P$.   Q.E.D.

In the next example we consider a special case of the Hopf theorem. First, however, we prove a preliminary result.

4.4. LEMMA. *If* $F: P \times R \rightarrow P$ *is the piecewise smooth flow on the compact cell polyhedron* P *constructed in Corollary 3.8 then the barycenters of* P *are the fixed points of* F *and, for every barycenter* x_0 *of* P, $i(F, x_0) = (-1)^p$ *where* p *is the dimension of the cell of* P *whose barycenter is* x_0.

PROOF. The first part of the lemma is obvious.

To prove the second part, let N be the cardinality of any coordinate system on a small open neighborhood of x_0 with origin x_0. Take the p-cell B^p of P whose barycenter is x_0 linearly to

$$B_0^p = \left\{ x = (x_k) \in R^N : x_k \in [-1, 1] \text{ for } k = 1, \dots, p \right.$$

$$\left. \text{and } x_k = 0 \text{ for } k = p + 1, \dots, N \right\},$$

thus taking x_0 to $0 \in R^N$. Take each $(p + 1)$-cell B_α^{p+1} in P which contains B^p linearly to a cell in R^N whose p-face corresponding to B^p coincides with B_0^p, and whose opposite p-side lies on the cylinder

$$\left\{ x = (x_k) \in R^N : x_k \in [-1, 1] \text{ for } k = 1, \dots, p \text{ and } \sum_{k=p+1}^{N} x_k^2 = 1 \right\};$$

we may assume that the images of the B_α^{p+1} lie in distinct coordinate planes

of R^N. Extend this map to a realization r of the closed star St x_0 of x_0 in R^m by using linearity on each cell.

Let $U = D^N(1/3) \subseteq R^N$ be the open N-ball of radius 1/3 centered at the origin. For some neighborhood $W \subseteq U$ of $r(x_0) = 0 \in R^N$ and some interval neighborhood I of $0 \in R$, define the smooth flow $\underline{F}: W \times I \to U$ by

$$\underline{F}(x, t) = (x_1 \exp c_1 t, \ldots, x_N \exp c_N t)$$

where $c_k = -2$ if $k = 1, \ldots, p$, and $c_k = 2$ if $k = p + 1, \ldots, N$. Clearly

$$\underline{F}_{|(W \cap r(\mathrm{St}\,x_0)) \times I} = r \circ F \circ (r^{-1} \times \mathrm{id})_{|(W \cap r(\mathrm{St}\,x_0)) \times I}$$

(see Remark 3.9).

For a positive $t_0 \in I$ define $f: W \cap r(\mathrm{St}\,x_0) \to U \cap r(\mathrm{St}\,x_0)$ by $f(\text{--})$ $= (r \circ F \circ (r^{-1} \times \mathrm{id}))(\text{--}, t_0)$ and $\underline{f}: W \to U$ by $\underline{f}(\text{--}) = \underline{F}(\text{--}, t_0)$. It is not difficult to construct a retraction $\rho: W \to W \cap r(\mathrm{St}\,x_0)$ for which $x - f\rho(x)$ is homotopic to $x - \underline{f}x$ (such a ρ may be defined by collapsing the wedges of W not belonging to $r(\mathrm{St}\,x_0)$ down to $W \cap r(\mathrm{St}\,x_0)$). Using the homotopy $W \times [0, 1] \to U$ given by

$$(x, s) \to (\underline{F}(x, 0) - \underline{F}(x, st_0))/s$$

we may conclude that $x - f\rho(x)$ is homotopic to $(\partial \underline{F}(x, t)/\partial t)_{|t=0}$ on W, hence on $W - \{0\}$. Since $(\partial \underline{F}(x, t)/\partial t)_{|t=0}$ has a zero only at the origin $0 \in R^N$, $x - f\rho(x)$ is homotopic to

$$\frac{(\partial \underline{F}(x, t)/\partial t)_{|t=0}}{|(\partial \underline{F}(x, t)/\partial t)_{|t=0}|}$$

on $W - \{0\}$.   Q.E.D.

4.5. EXAMPLE. If $F: P \times R \to P$ is the piecewise smooth flow on the compact cell polyhedron P constructed in Corollary 3.8, then using Lemma 4.4, $\sum_{x_k \in \mathrm{Fix}\,F} i(F, x_k)$ is the number of even-dimensional cells of P minus the number of odd-dimensional cells of P; this is precisely the Euler characteristic of P. This is the calculation described in the Introduction which is similar to a calculation of Whitney (see [14]) giving the Euler characteristic of a smooth manifold using the singularities of vector fields.

4.6. DEFINITION. A *polyhedral m-manifold* M is a cell polyhedron for which the closed star of each point can be mapped homeomorphically into R^m so that the image of each cell of M is a cell in R^m.

For related definitions see [4].

4.7. COROLLARY (TO LEMMA 4.4). *If M is a closed polyhedral m-manifold then there is a piecewise smooth flow on M which is ordinary except for a finite number*

of fixed points which are on the interior of m-cells of M.

PROOF. Let x_0 be a barycenter of M and r any realization of the closed star of x_0 in R^m. Without loss of generality we may assume that the image of the p-cell B^p of M of which x_0 is the barycenter under r is

$$\left\{ x = (x_k) \in R^m\colon x_k \in [-1, 1] \text{ for } k = 1, \ldots, p \text{ and } x_k = 0 \text{ for all other } k \right\}.$$

There is now a decomposition of R^m into wedges W_α^n where

$$W_\alpha^n = \{ tx \in R^m\colon t \in R^+ \text{ and } x \in r(B_\alpha^n) \}$$

for B_α^n an n-cell of M containing B^p; i.e.,

(1) $R^m = \cup W_\alpha^n$, and

(2) W_α^m is an open subset of W_β^n iff $m = n$ and $\alpha = \beta$, and

(3) for each W_α^m and W_β^n, $W_\alpha^m \cap W_\beta^n$ is another wedge in the decomposition of R^m.

Let $g\colon R^m \to R^m$ be an invertible map such that:

(1) $g(W_\alpha^n) = W_\alpha^n$ for each wedge W_α^n in the decomposition of R^m,

(2) the restriction of g to each W_α^n is linear and $g\colon W_0^p \to W_0^p$ is the identity, and

(3) for each $(p + 1)$-cell B_α^{p+1} of M containing B^p,

$$grB_\alpha^p \subseteq \left\{ x = (x_k) \in R^m\colon x_k \in [-1, 1] \text{ for} \right.$$

$$\left. k = 1, \ldots, p \text{ and } \sum_{k=p+1}^{m} x_k^2 = 1 \right\}$$

where B_α^p is the p-face of B_α^{p+1} opposite B^p.

Finally we let r_0 denote the realization of the closed star of x_0 in R^m given by $r_0 = g \circ r$.

The piecewise smooth flow $F\colon M \times R \to M$ on M constructed in Corollary 3.8 is given on $r_0^{-1}(U) = r_0^{-1}(D^m(1/3))$ by pulling back the smooth flow on $U = D^m(1/3) \subseteq R^m$ associated to the gradient grad f of

$$f(x) = \sum_{k=p+1}^{m} x_k^2 - \sum_{k=1}^{p} x_k^2.$$

Let $x_1 \in U$ be any point which does not lie on the image of the $(m - 1)$-skeleton of M under r_0. Let $T\colon U \to U$ be an invertible smooth transformation which takes x_1 to the origin $0 \in R^m$ and which is the identity on $U - U_0$ where U_0 is an open neighborhood of the origin $0 \in R^m$ containing x_1 and whose closure is contained in U. Then $f \circ T\colon U \to R$ is a smooth function which has a critical point at x_1 and no other critical points in U. Thus $f \circ T$ generates a smooth flow F on U which has a fixed point at x_1 and which agrees

with the flow induced by f outside U_0. Pulling this flow back to M we obtain a piecewise smooth flow F' on M which agrees with F on $M - r_0^{-1}(U_0)$ and which has a fixed point only at $r_0^{-1}(x_1)$, a point which does not lie on the $(m - 1)$-skeleton of M. Performing this construction at each fixed point of F, i.e., at each barycenter of M, we obtain the desired result.   Q.E.D.

4.8. THEOREM (HOPF). *If M is a closed polyhedral m-manifold whose Euler characteristic is zero, then there is a piecewise smooth flow on M which is ordinary at every point.*

PROOF. Let $F: M \times I \to M$ be the piecewise smooth flow on M constructed in Corollary 4.7. Order the m-cells of M: $B_0^m, \ldots, B_N^m$ so that each B_k^m has an $(m - 1)$-face in common with a previous B_i^m; for each k pick one such $(m - 1)$-cell B_k^{m-1} and let x_k be the barycenter of B_k^{m-1}.

For each $k = 1, 2, \ldots, N$ there are realizations $r_k: \operatorname{St} x_k \to V_k \subseteq R^m$ and small open neighborhoods U_k and W_k of x_k for which the closure of W_k is contained in U_k and such that

$$r_k \circ F \circ (r_k^{-1} \times \operatorname{id}): r_k(W_k) \times I_k' \to r_k(U_k)$$

is a smooth flow for some interval neighborhood I_k' of $0 \in R$. For $k = 0, 1, \ldots, N$, let $\underline{U}_k$ and $\underline{W}_k$ be open subsets of the interior of B_k^m for which the closure of $\underline{W}_k$ is contained in $\underline{U}_k$, and:

(1) all the fixed points of F in B_k^m lie in $\underline{W}_k$ for $k = 0, 1, \ldots, N$,
(2) $\underline{W}_k \cap W_k \neq \varnothing$ and $\underline{W}_{k-1} \cap W_k \neq \varnothing$ for $k = 1, \ldots, N$, and
(3) there is an interval neighborhood I_k'' of $0 \in R$ for which $F: \underline{W}_k \times I_k'' \to \underline{U}_k$.
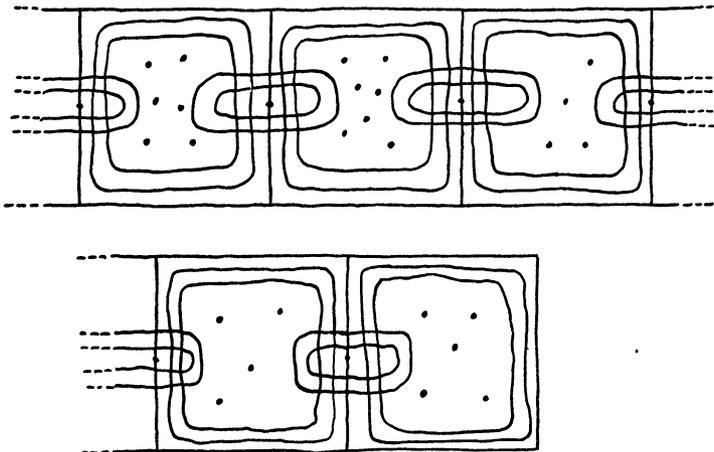(See Diagram 5.)

DIAGRAM 5

If $I_N = I'_N \cap I''_N$, then

$$r_N \circ F \circ (r_N^{-1} \times \mathrm{id}): r_N(\underline{W}_N \cup W_N) \times I_N \to r_N(\underline{U}_N \cup U_N)$$

is a smooth flow which has a finite number of fixed points which are in the interior of $r_N(\underline{W}_N \cup W_N)$. This flow may be smoothly altered to a smooth flow

$$r_N \circ F_N \circ (r_N^{-1} \times \mathrm{id}): r_N(\underline{W}_N \cup W_N) \times I_N \to r_N(\underline{U}_N \cup U_N)$$

which agrees with $r_N \circ F \circ (r_N^{-1} \times \mathrm{id})$ in a nonempty open neighborhood of the boundary of $r_N(\underline{W}_N \cup W_N)$ and which contains only a finite number of fixed points all of which are in $r_N(\underline{W}_{N-1} \cup W_N)$. Pulling this flow back to M we obtain a piecewise smooth flow $F_N: M \times I_N \to M$ which is ordinary except for a finite number of fixed points which are in the interior of the m-cells $B_0^m, \ldots, B_{N-1}^m$.

Proceeding inductively we obtain a piecewise smooth flow $F_1: M \times I_1 \to M$ which is ordinary except for a finite number of fixed points which are in the interior of B_0^m.

For a proof of the following lemma see [10].

4.9. LEMMA. *Let D^m be the closed unit disc in R^m with boundary S^{m-1} and let $s: D^m \to TD^m$ be a smooth vector field on D^m whose only zeroes occur at finitely many points $x_k \in D^m - S^{m-1}$. If $\sum i(s, x_k) = 0$ then there is a nowhere vanishing vector field $s': D^m \to TD^m$ which agrees with s in a neighborhood of S^{m-1}.*

Let D be a closed disc interior to B_0^m which contains all the fixed points of F_1. Since $\chi(M) = 0$ we may use Lemma 4.9 to alter the vector field on D generated by $F_1: M \times I_1 \to M$ to obtain a nowhere vanishing vector field on D and then integrate to obtain a nonvanishing piecewise smooth flow $F_0: M \times I_0 \to M$ on M.   Q.E.D.

Lemma 4.9 is also the key lemma in proving the classical smooth version of Theorem 4.8 (see [10]); in fact the proof of Theorem 4.8 closely parallels the proof of this classical result.

5. **Riemannian metrics.** In this section Riemannian metrics are defined on simplicial spaces. We will show that the topology on a connected simplicial space induced by such a metric is the topology of the simplicial space.

5.1. DEFINITION. The *Whitney sum* $\xi' \oplus \xi'': E' \oplus E'' \to X$ of the simplicial bundles $\xi': E' \to X$ and $\xi'': E'' \to X$ is the simplicial bundle whose total space $E' \oplus E''$ is the subspace of $E' \times E''$ consisting of all those (v', v'') for which $\xi'(v') = \xi''(v'')$, whose projection $\xi' \oplus \xi''$ is $(\xi' \oplus \xi'')(v', v'') = \xi'(v') = \xi''(v'')$, whose fiber at $x \in X$ is $(\xi' \oplus \xi'')^{-1}x = (\xi')^{-1}x \oplus (\xi'')^{-1}x$, and:

(1) for every simplicial chart (U_α, ψ_α) on X whose structure fiber F_α

$= F'_\alpha \oplus F''_\alpha$ and corresponding structure map $\phi_\alpha \colon U_\alpha \times F_\alpha \to (\xi' \oplus \xi'')^{-1} U$ is $\phi_\alpha = \phi'_\alpha \oplus \phi''_\alpha$, and

(2) for simplicial charts (U_α, ψ_α) and (U_β, ψ_β) on X for which $U_\alpha \subseteq U_\beta$, whose structure map $l_{\beta\alpha} = l'_{\beta\alpha} \oplus l''_{\beta\alpha}$.

A *Riemannian metric on the n-corner* s^n is a smooth, symmetric, bilinear, positive definite map $\langle -, - \rangle \colon Ts^n \oplus Ts^n \to R$.

5.2. DEFINITION. A *Riemannian metric on the simplicial space* X is a symmetric, bilinear, positive definite map $\langle -, - \rangle \colon TX \oplus TX \to R$.

If $\langle -, - \rangle$ is a Riemannian metric on the simplicial space X, then for every simplicial chart $\psi_\alpha \colon U_\alpha \to V_\alpha$ on X,

$$\langle (\psi_\alpha)_* -, (\psi_\alpha)_* - \rangle \colon TV_\alpha \oplus TV_\alpha \to R$$

is a Riemannian metric on V_α, $(\psi_\alpha)_*$ the Jacobian of ψ_α.

The existence of Riemannian metrics on simplicial spaces follows from the local construction of the tangent objects of simplicial spaces and the fact that piecewise smooth partitions of unity (subordinate to covers by small open subsets) can be defined on simplicial spaces.

A *smooth curve* $\gamma = (\gamma_k) \colon [a, b] \to s^n$ *in* s^n is a map for which each component function γ_k of γ is smooth on (a, b), and for which the one-sided limits at a and b of all the derivatives of each γ_k are finite; the *tangent vector* $T_\gamma(t_0)$ *of* γ *at* $t_0 \in [a, b]$ is

$$T_\gamma(t_0) = \gamma_*((d/dt)_{t_0}) = \sum (d\gamma_k/dt)_{|t_0}(\partial/\partial x_k)_{\gamma(t_0)}.$$

5.3. DEFINITION. A *piecewise smooth curve* $\gamma \colon [a, b] \to X$ *in the simplicial space* X is a map for which there is a finite partition $a = c_0 \leqslant c_1 \leqslant \cdots \leqslant c_n = b$ of $[a, b]$ and, for each $k = 0, \ldots, n - 1$, a simplicial chart $\psi_\alpha \colon U_\alpha \to V_\alpha$ on X with $\gamma([c_k, c_{k+1}]) \subseteq U_\alpha$ and

$$\psi_\alpha \circ \gamma \colon [c_k, c_{k+1}] \to V_\alpha$$

a smooth curve. If X is equipped with a Riemannian metric $\langle -, - \rangle$ and $|-|$ is the associated norm, then the *length* $L(\gamma)$ of γ is

$$L(\gamma) = \sum_{k=0}^{n-1} L(\gamma_k) = \sum_{k=0}^{n-1} \int_{c_k}^{c_{k+1}} T_{\gamma_k}(t) \, dt$$

where $\gamma_k = \gamma_{|[c_k, c_{k+1}]}$, and $T_{\gamma_k}(t) = (\gamma_k)_*((d/dt)_t)$.

The length $L(\gamma)$ of the piecewise smooth curve $\gamma \colon [a, b] \to X$ is clearly independent of the partition of $[a, b]$. Furthermore it follows from the chain rule that $L(\gamma)$ is independent of the parametrization of γ.

5.4. THEOREM. *If X is a connected simplicial space with Riemannian metric*

$\langle -, - \rangle$ *then the map d*: $X \times X \to R$ *given by*

$$d(p,q) = \inf\{L(\gamma): \gamma \text{ is a piecewise smooth curve from } p \text{ to } q\}$$

is a metric on X, and the metric topology on X is the topology of the simplicial space.

PROOF. The first assertion is immediate. The second assertion follows from the following result (see [13]).

5.5. LEMMA. *Let U and W be small open neighborhoods of $x_0 \in X$ which have compact closure and for which $W \subseteq U$. and let u be a coordinate system on U with origin x_0. If $|-|$ is the norm associated to the inner product $\langle -, - \rangle$ on U, and if $|-|_0$ is the norm associated to the inner product $\langle -, - \rangle_0$ on U given by*

$$\langle (\partial/\partial u_j)_x, (\partial/\partial u_k)_x \rangle_0 = \delta_{jk}$$

(δ_{jk} the Kronecker delta) then there are positive real constants m and M for which

$$m \cdot |-|_0 \leqslant |-| \leqslant M \cdot |-|_0$$

on W; consequently $mL_0(\gamma) \leqslant L(\gamma) \leqslant ML_0(\gamma)$ for any piecewise smooth curve $\gamma: [a,b] \to W$.

PROOF (OF LEMMA 5.5). Let V be the real vector space on generators $\partial/\partial u_k$, $u_k \in u$, and let

$$SU = \{(x,v) \in U \times V: |(x,v)|_0 = 1 \text{ and } (x,v) \in TU\}.$$

Define f: $SU \to R$ by $f(x,v) = |(x,v)|$; f is positive and continuous. Since W is compact there are positive constants m and M for which $m \leqslant f \leqslant M$ on $SW = \{(x,v) \in SU: x \in W\}$.

Now if $(x,v) \in TW$ and $v \neq 0$, let $\underline{v} = v/|(x_0,v)|_0 = v/|(x,v)|_0$. Then

$$m|(x,v)|_0 \leqslant |(x,v)|_0 f(x,\underline{v}) \leqslant M|(x,v)|_0.$$

The conclusion follows since

$$
\begin{aligned}
|(x,v)|_0 f(x,\underline{v}) &= |(x,v)|_0 |(x,\underline{v})| \\
&= |(x,v)|_0 (|(x,v)|/|(x,v)|_0) \\
&= |(x,v)|.
\end{aligned}
$$

(If $v = 0$, then $m|(x,v)|_0 = |(x,v)| = M|(x,v)|_0$.)   Q.E.D.

6. A Gauss-Bonnet formula for polyhedral manifolds. In this section we give an integral formula for the Euler characteristic of a closed, oriented, even-dimensional polyhedral manifold. As a consequence of this formula we obtain

a representation for the Euler class of such a polyhedral manifold in terms of piecewise smooth forms. This integral formula brings together everything which has been done in this paper.

Let us first recall the following construction of smooth Euler classes (see [2], [9], and [12]): Let $\underline{M}$ be a smooth manifold (with or without boundary), let $A(\underline{M})$ be the ring of smooth functions on $\underline{M}$, and let $(\Lambda^* E(\underline{M}), d(\underline{M}))$ be the smooth de Rham complex of $\underline{M}$. If $\xi \colon \underline{E} \to \underline{M}$ is a smooth, oriented, real $2n$-plane vector bundle over $\underline{M}$, let $\Gamma(\underline{M}, \xi)$ denote the $A(\underline{M})$-module of global smooth sections of ξ. A *connexion* on ξ is a real linear map

$$D \colon \Lambda^* E(\underline{M}) \underset{A(\underline{M})}{\otimes} \Gamma(\underline{M}, \xi) \to \Lambda^* E(\underline{M}) \underset{A(\underline{M})}{\otimes} \Gamma(\underline{M}, \xi)$$

of degree $+1$ for which $D(\theta \otimes s) = d\theta \otimes s + (-1)^p \theta \otimes Ds$ for $\theta \in \Lambda^p E(\underline{M})$; the *curvature* K of D is the $\Lambda^* E(\underline{M})$-linear map $K = D \circ D$. A Riemannian metric on ξ, which may be considered as an inner product

$$\langle -, - \rangle \colon \Gamma(\underline{M}, \xi) \otimes_{A(\underline{M})} \Gamma(\underline{M}, \xi) \to A(\underline{M})$$

induces an inner product

$$\langle -, - \rangle \colon (\Lambda^* E(\underline{M}) \otimes \Gamma(\underline{M}, \xi)) \underset{\Lambda^* E(\underline{M})}{\otimes} (\Lambda^* E(\underline{M}) \otimes \Gamma(\underline{M}, \xi)) \to \Lambda^* E(\underline{M}),$$

and the connexion D on ξ is *metric with respect to the inner product* $\langle -, - \rangle$ on ξ iff

$$\langle Ds', s'' \rangle + \langle s', Ds'' \rangle = d\langle s', s'' \rangle$$

for $s', s'' \in \Gamma(\underline{M}, \xi)$. If $\langle -, - \rangle$ is any metric on ξ then the *Euler class* $e(\xi)$ of ξ is

$$e(\xi) = (-1)^{n(n-1)/2} 4\pi^{-n} [\mathrm{Tr}(* \circ \Lambda^n K)] \in H^{2n}_{\mathrm{de\ Rham}}(\underline{M}),$$

$* \colon \Lambda^* E(\underline{M}) \to \Lambda^* E(\underline{M})$ denoting the Hodge star operator, and K the curvature of any metric connexion D on ξ. (It is not difficult to show that this definition of $e(\xi)$ is independent of the particular metric $\langle -, - \rangle$ and metric connexion D chosen on ξ.) If $\underline{M}$ is even-dimensional, the *Euler* class $e(\underline{M}) = e(\tau(\underline{M}))$, $\tau(\underline{M}) \colon T\underline{M} \to \underline{M}$ the tangent bundle of $\underline{M}$.

In the following result we let $e(\xi)$ denote the $2n$-form

$$e(\xi) = (-1)^{n(n-1)/2} 4\pi^{-n} \,\mathrm{Tr}\, (* \circ \Lambda^n K) \in \Lambda^{2n} E(\underline{M}),$$

K the curvature of any metric connexion D on ξ.

6.1. THEOREM. *Let $\underline{M}$ be a smooth $2n$-dimensional manifold with boundary $\partial \underline{M}$. Let $\xi \colon \underline{E} \to \underline{M}$ be a smooth, oriented, real $2n$-plane vector bundle over $\underline{M}$ with metric $\langle -, - \rangle$ and corresponding norm $|-|$. If $\underline{E}_0 = \{v \in \underline{E} \colon |v| > 0\}$ and*

$\xi_0\colon E_0 \to M$ *is the projection induced by* ξ, *then the inclusion* $E_0 \to E$ *induces a smooth nonzero section* s_0 *of* $\xi_0'\xi\colon E(\xi_0'\xi) \to E_0$. *Furthermore there is a smooth* $(2n-1)$*-form* $\eta(\xi)$ *defined on* E_0 *for which:*

(1) $\xi_0^* e(\xi) = d\eta(\xi)$,

(2) *if* $j\colon S^{2n}(\xi^{-1}(x_0)) \to E_0$ *is the inclusion of the unit sphere of* $\xi^{-1}(x_0)$ *into* E_0, *then* $-j^*\eta(\xi)$ *is the orientation class of* $S^{2n}(\xi^{-1}(x_0))$, *and*

(3) *a smooth nonzero section* $s\colon \partial M \to E_0 \subseteq E$ *of* ξ *defined on* ∂M *extends to a smooth nonzero section of* ξ *defined on* M *iff*

$$\int_M e(\xi) - \int_{\partial M} s^* \eta(\xi) = 0.$$

PROOF. The form $\eta(\xi)$ is defined by

$$\eta(\xi) = \pi^{-n} \sum_{p=0}^{n-1} (-1)^p (1 \cdot 3 \cdots \cdot (2n-2p-1) p! \, 2^{n+p})^{-1} \operatorname{Tr} \Phi_p$$

where, for $p = 0, 1, \ldots, n-1$,

$$\Phi_p\colon \Lambda^* E(M) \underset{A(M)}{\otimes} \Lambda^p \Gamma(M,\xi) \to \Lambda^* E(M) \underset{A(M)}{\otimes} \Lambda^p \Gamma(M,\xi)$$

is given on generators by

$$\Phi_p(e_1 \wedge \cdots \wedge e_p) = * (s_0 \wedge (Ds_0)^{2n-2p-1} \wedge \Lambda^p K(e_1 \wedge \cdots \wedge e_p)).$$

The third part of the conclusion follows from the fact that if s is a smooth section of ξ which is nonzero on ∂M and which has only a finite number of isolated zeroes then

$$\int_M e(\xi) - \int_{\partial M} s^* \eta(\xi) = \sum i(s, x_k)$$

where the summation is taken over the set $\{x_k\}$ of zeroes of s and $i(s, x_k)$ is defined as follows: Let $D^{2n}(\varepsilon)$ be a closed ball of radius ε about x_k relative to a local coordinate system in a neighborhood of x_k with origin x_k, and let $\phi\colon \xi^{-1}(D^{2n}(\xi)) \to \xi^{-1}(x_k)$ be a retraction which is an isomorphism on each fiber. Then for small enough positive $\delta < \varepsilon$,

$$\phi \circ s\colon \partial D^{2n}(\delta_0) \to \xi^{-1}(x_k) - \{0\}$$

for every positive $\delta_0 < \delta$; the *index* $i(s, x_k)$ is defined as the degree of $\phi \circ s$. (If $F\colon M \times R \to M$ is a smooth flow on M with an isolated fixed point at x_0 and s is the smooth vector field on M generated by F then $i(s, x_0)$ agrees with the index $i(s, x_0)$ as defined immediately before Lemma 4.2.)   Q.E.D.

In particular Theorem 6.1 is valid for M the closed unit $2n$-ball $D^{2n} \subseteq R^{2n}$

and ξ the tangent bundle $\tau(\underline{M}): \underline{TM} \to \underline{M}$ of $\underline{M} = D^{2n}$. But moreover the analogous result is valid for the $2n$-cell $B^{2n} = [-1, 1]^{2n} \subseteq R^{2n}$ and ξ the tangent object (= tangent vector bundle) $\tau(B^{2n}): TB^{2n} \to B^{2n}$ of B^{2n}: The *Euler class* $e(B^{2n}) = e(\tau(B^{2n}))$ of B^{2n} may be defined exactly as in the smooth case, and furthermore we have the following result.

6.2. LEMMA. *Let $\tau(B^{2n}): TB^{2n} \to B^{2n}$ be the tangent object of the oriented $2n$-cell B^{2n} with metric $\langle -, - \rangle$ and corresponding norm $|-|$. If $TB_0^{2n} = \{v \in TB^{2n}: |v| > 0\}$ and $\tau(B^{2n})_0: TB_0^{2n} \to B^{2n}$ is the projection induced by $\tau(B^{2n})$, then the inclusion $TB_0^{2n} \to TB^{2n}$ induces a smooth nonzero section s_0 of*

$$\tau(B^{2n})_0^! \tau(B^{2n}): E(\tau(B^{2n})_0^! \tau(B^{2n})) \to TB_0^{2n}.$$

Furthermore there is a smooth $(2n - 1)$-form $\eta(B^{2n})$ defined on TB_0^{2n} for which:
  (1) $\tau(B^{2n})_0^* e(B^{2n}) = d\eta(B^{2n})$,
  (2) *if $j: S^{2n}(\tau(B^{2n})^{-1}x_0) \to TB_0^{2n}$ is the inclusion of the unit sphere of $\tau(B^{2n})^{-1}x_0$ into TB_0^{2n} then $-j^*\eta(B^{2n})$ is the orientation class of $S^{2n}(\tau(B^{2n})^{-1}x_0)$, and*
  (3) *a piecewise smooth nonzero section $s: \partial B^{2n} \to TB_0^{2n} \subseteq TB^{2n}$ of $\tau(B^{2n})$ defined on ∂B^{2n} extends to a smooth nonzero section of $\tau(B^{2n})$ defined on B^{2n} iff*

$$\int_{B^{2n}} e(B^{2n}) - \int_{\partial B^{2n}} s^* \eta(B^{2n}) = 0.$$

PROOF. The proof of this result is identical to the proof of Theorem 6.1. Q.E.D.

Now let M be a closed, oriented, $2n$-dimensional polyhedral manifold. We obtain the integral formula for the Euler characteristic of M as follows: Let $F: M \times I \to M$ be any piecewise smooth flow on M which has only a finite number of fixed points which are all interior to $2n$-cells of M; the existence of such flows is insured by Corollary 4.6.

6.3. LEMMA. *Let B_α^{2n} be a $2n$-cell of M and let $s_\alpha: B_\alpha^{2n} \to TB_\alpha^{2n}$ be the section of the tangent object (= tangent vector bundle) $\tau(B^{2n})$ of B_α^{2n} induced by the preflow $F_\alpha: \underline{B_\alpha^{2n} \times I} \to B_\alpha^{2n}$ defined by restricting F to*

$$\underline{B_\alpha^{2n} \times I} = \{(x, t) \in B_\alpha^{2n} \times I: F(x, t) \in B_\alpha^{2n}\}.$$

Then s_α is nonzero on the boundary ∂B_α^{2n} of B_α^{2n}.

PROOF. Let us first observe that if U and W are small open subsets of the n-corner s^n which have compact closure and for which the closure of W is contained in U, and $F: \underline{W \times R} \to U$ is a preflow on W for which

$$\left(\sum \left(\partial F_k(x, t)/\partial t \right)_{|t=0} (\partial/\partial x_k) \right) x_0 = 0$$

then $\{x_0\} \times R \subseteq \underline{W \times R}$ and $F(x_0, t) = x_0$ for all $t \in R$: According to Lemma 3.2 the preflow F can be reconstructed from the piecewise smooth functions $f_k = (\partial F_k(x, t)/\partial t)_{|t=0}$ by first extending the functions f_k to functions $\underline{f_k}$ defined on an open subset of R^n which has compact closure, applying Picard's method, and then restricting the resulting flow $\underline{F}$ to $\underline{W \times R}$. However if $f_k(x_0) = 0$ for each k then every flow $\underline{F}$ on R^n constructed in this manner will have a fixed point at x_0. The desired conclusions now follow from the construction of the preflow F from the flow $\underline{F}$.

Now let x_0 be a point on the $(2n - 1)$-skeleton of M (hence an ordinary point of the piecewise smooth flow $F: M \times I \to M$), (U, Ψ) a packet on M for which U is a small open neighborhood of x_0 (with compact closure) and u a coordinate system on (U, Ψ). Let W be a small open neighborhood of x_0 for which the closure of W is contained in U and $J \subseteq I$ an open interval containing the origin for which $F: W \times J \to U$. If $\psi_\alpha: U_\alpha \to V_\alpha \subseteq s_\alpha^{2n}$ is any simplicial chart of Ψ then the restriction F_α of F to

$$\underline{(W \cap U_\alpha) \times J} = \{(x, t) \in (W \cap U_\alpha) \times J: F(x, t) \in U_\alpha\}$$

is a preflow on $W \cap U_\alpha$. If $((\partial(u_\alpha^k \circ F_\alpha)/\partial t)_{|t=0})x_0 = 0$ for each k then $\{x_0\} \times J \subseteq \underline{(W \cap U_\alpha) \times J}$ and $F(x_0, t) = x_0$ for each $t \in J$, contradicting the fact that x_0 is an ordinary point of F.   Q.E.D.

Now a brief digression.

Recall (see [11]) that there is a natural morphism ρ called *restriction* from the de Rham complex $(\Lambda^* E(s^n), d(s^n))$ of the n-corner s^n to the de Rham complex $(\Lambda^* E(s^m), d(s^m))$ of the face s^m of s^n defined by

$$\rho(\theta) = \rho(\textstyle\sum f_k \, dx_{k_1} \cdots dx_{k_p}) = \sum (f_{k|s} m) d(x_{k_1|s} m) \cdots d(x_{k_p|s} m).$$

This morphism induces a morphism, again called *restriction*, from the de Rham complex $(\Lambda^* E(X), d(X))$ of any simplicial space X to the de Rham complex $(\Lambda^* E(X^m), d(X^m))$ of the m-skeleton X^m of X.

6.4. LEMMA. *Any piecewise smooth p-form θ defined on the $(n - 1)$-skeleton of an arbitrary simplicial space X extends to a piecewise smooth p-form $\underline{\theta}$ defined on the n-skeleton of X (i.e., $\underline{\theta}$ restricts to θ).*

PROOF. Since piecewise smooth partitions of unity subordinate to covers of X by small open subsets exist, it suffices to prove this lemma locally. Proving the local version of this lemma reduces to proving this lemma in the special case $X = s^n$.

If f is a piecewise smooth function defined on the boundary of s^n then an extension $\underline{f} \in A(s^n)$ of f may be defined as follows: For each j-tuple $k = (k_1, \ldots, k_j), j = 1, \ldots, n$, define the projection π_k from s^n to the boundary of s^n by taking each $x \in s^n$ to the point whose coordinates in the k-slots

are zero, and whose coordinates are otherwise unchanged; then $\underline{f}$ $= \sum_k (-1)^{k+1}(f \circ \pi_k)$ where the summation is taken over all j-tuples k, $j = 1, \ldots, n$, and where for each j-tuple k, $(-1)^{k+1} = (-1)^{j+1}$.

Given $\theta = \sum f_k dx_{k_1} \cdots dx_{k_p}$ defined on the boundary of s^n, $\underline{\theta}$ may be defined by $\sum \underline{f_k} dx_{k_1} \cdots dx_{k_p}$. Q.E.D.

Now again let M be a closed, oriented, $2n$-dimensional polyhedral manifold. Let $\langle -, - \rangle$ be any Riemannian metric on M. Each cell B_α^{2n} of M inherits an orientation and a Riemannian metric naturally from M, and hence we may construct the smooth $2n$-form

$$O(B_\alpha^{2n}) = e(B_\alpha^{2n}) - d(B_\alpha^{2n})\underline{s_\alpha^* \eta(B_\alpha^{2n})} \in \Lambda^{2n} E(B_\alpha^{2n})$$

on B_α^{2n} where s_α is as in Lemma 6.3.

6.5. LEMMA. *The forms* $O(B_\alpha^{2n})$, *for* B_α^{2n} *a $2n$-cell of M, define a global closed piecewise smooth $2n$-form $e(M)$ on M.*

PROOF. The fact that the forms $O(B_\alpha^{2n})$ define a global piecewise smooth $2n$-form $e(M)$ on M follows immediately from the definition of a piecewise smooth form on M. The fact that $e(M)$ is closed follows from the fact that there are no nontrivial piecewise smooth $(2n + 1)$-forms on M. Q.E.D.

The definition of $e(M)$ depends a priori on many choices (such as the piecewise smooth flow $F: M \times I \to M$ on M, Riemannian metric, metric connexions, and extensions). It is not difficult to verify directly that the definition of $e(M)$ is independent of all such choices. However the same result follows immediately from the following result which is the main result of this paper. The proof of this theorem naturally parallels Chern's intrinsic proof of the Gauss-Bonnet formula for closed Riemannian manifolds (see [5]).

6.6. THEOREM (POLYHEDRAL GAUSS-BONNET). *If M is a closed, oriented, $2n$-dimensional polyhedral manifold and $e(M) \in \Lambda^{2n} E(M)$ is the piecewise smooth $2n$-form on M constructed above then*

$$\int_M e(M) = \chi(M),$$

where $\chi(M)$ is the Euler characteristic of M.

PROOF. Using Stokes's theorem,

$$\int_M e(M) = \sum \int_{B_\alpha^{2n}} O(B_\alpha^{2n})$$

$$= \sum \left(\int_{B_\alpha^{2n}} (e(B_\alpha^{2n}) - d(B_\alpha^{2n})\underline{s_\alpha^* \eta(B_\alpha^{2n})}) \right)$$

$$= \sum \left(\int_{B_\alpha^{2n}} e(B_\alpha^{2n}) - \int_{\partial B_\alpha^{2n}} s^* \eta(B_\alpha^{2n}) \right).$$

Using Lemma 6.2 and Lemma 6.3,

$$\int_M e(M) = \sum i(s_\alpha, x_k).$$

Using Lemma 4.2, $\int_M e(M) = \sum i(F, x_k)$, so that, by Theorem 4.3, $\int_M e(M) = \chi(M)$.   Q.E.D.

Theorem 6.6 is closely related to a similar integral formula given by Allendoerfer and Weil (see [1]) for the Euler characteristic of a smooth manifold with piecewise smooth boundary. This latter result is obtained as follows: one first proves a generalized Gauss-Bonnet formula for a piece of a Riemannian manifold isometrically embedded in Euclidean space, and then pieces manifolds and forms together; the generalized formula is proved by first verifying the formula in a special case (namely the case of an embedded curved simplex), and then proceeding to the general case by using the fact that the additivity of angles implies an additivity property for integrals.

6.7. REMARK. As a result of Theorem 6.6, the Euler class $e(M)$ $\in H_{\text{de Rham}}^{2n}(M)$ of the closed, oriented, $2n$-dimensional polyhedral manifold M is the class described by the piecewise smooth $2n$-form $e(M) \in \Lambda^{2n} E(M)$ constructed above.

6.8. REMARK. If $\underline{M}$ is a closed, oriented, $2n$-dimensional smooth manifold, $e(\underline{M})$ is the Euler class of $\underline{M}$, and $T: M \to \underline{M}$ is a triangulation of $\underline{M}$ by a polyhedral manifold M, then $T^* e(\underline{M}) = e(M)$.

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