THE CLOSED LEAF INDEX OF FOLIATED MANIFOLDS

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ABSTRACT. For $M$ a closed, connected, oriented 3-manifold, a topological invariant is computed from the cohomology ring $H^*(M; \mathbb{Z})$ that provides an upper bound to the number of topologically distinct types of closed leaves any smooth transversely oriented foliation of $M$ can contain. In general, this upper bound is best possible.

Introduction. Let $M$ be a closed, connected, oriented $n$-manifold, $F$ a smooth transversely oriented foliation of codimension one on $M$. Throughout this paper, the term "foliated $n$-manifold" will mean such a pair $(M, F)$.

Definition. The closed leaf index of $F$ is the number $\gamma(F)$ of distinct topological types of closed leaves in $F$.

It is easy to see that $\gamma(F)$ is finite. By a theorem of S. P. Novikov [3], all foliations of $S^3$ have $\gamma(F) = 1$, and those of $S^1 \times S^2$ have $\gamma(F) > 1$. It is also known [2] that foliations of $T^2$ satisfy $0 < \gamma(F) < 2$ and that those of $S^1 \times S^2$ satisfy $1 < \gamma(F) < 2$, each of these possibilities being realized by a suitable foliation. By [1], the greatest lower bound of $\gamma(F)$ on any closed 3-manifold $M$ is always 0 or 1. It is a result of P. Schweitzer [5] that, for $n > 5$, every $n$-manifold that can be foliated admits a $C^0$ foliation $F$ (with $C^\infty$ leaves) having $\gamma(F) = 0$. In this paper we will produce a fairly severe upper bound to the closed leaf index of foliated 3-manifolds. Analogous but weaker results will be obtained in higher dimensions.

Definition. The symbol $\alpha(M)$ denotes the largest integer such that some basis $x_1, \ldots, x_a$ of $H^1(M; \mathbb{Z}) = \mathbb{Z}^a$ satisfies $x_i \cup x_j = 0$, $1 < i, j < \alpha(M)$.

For instance, $\alpha(S^3) = 0$, $\alpha(S^1 \times S^2) = 1$, and $\alpha(T^3) = 1$. Indeed, we will compute $\alpha$ for all oriented $S^1$-bundles over closed oriented surfaces ($\S 3$).

It is convenient to define a strictly increasing function $\varphi: \mathbb{Z}^+ \to \mathbb{Z}^+$ by

$$\varphi(k) = \begin{cases} k + 1, & k = 0, 1, \\ 3k - 2, & k > 2. \end{cases}$$

**Theorem A.** If $(M, F)$ is a foliated 3-manifold, then $\gamma(F) < \varphi(\alpha(M))$.

**Theorem B.** For each integer $k > 0$, there is a foliated 3-manifold $(M_k, F_k)$ such that $\alpha(M_k) = k$ and $\gamma(F_k) = \varphi(k)$.

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We do not know, however, whether Theorem A gives the best possible upper bound for $\gamma(F)$ on every 3-manifold. We conjecture that this fails, in fact, for the "least twisted" nontrivial oriented $S^1$-bundle over a closed, oriented surface of genus $g > 0$.

In higher dimensions it is not reasonable to expect such sharp information on $\gamma(F)$, but the following invariants are manageable.

**Definition.** The *homology index* of $F$ is the number $h(F)$ of classes in $H_{n-1}(M^n; \mathbb{Z})$, distinct modulo sign, that can be represented by closed leaves of $F$. The mod 2 homology index $h_2(F)$ is the number of distinct classes in $H_{n-1}(M^n; \mathbb{Z}_2)$ so represented.

**Definition.** The *Euler index* of $F$ is the number $e(F)$ of distinct integers $|\chi(L)|$ for closed leaves $L$ of $F$.

On even dimensional manifolds, the Euler index is always 0 or 1; hence this invariant is of real significance only on odd dimensional manifolds. For foliated 3-manifolds, the equality $\gamma(F) = e(F)$ is immediate from the classification of closed surfaces and the Reeb stability theorem, so Theorem A and Theorem B are specializations respectively of Theorem A' and Theorem B' below.

**Theorem A'.** If $(M, F)$ is a foliated $n$-manifold, then $e(F) < h(F) = h_2(F) < \varphi(\alpha(M))$.

**Theorem B'.** For each pair of integers $(n, k) \neq (4, 0)$ with $n \geq 3$ and $k > 0$, there is a foliated $n$-manifold $(M^n_k, F_k)$ with $\alpha(M^n_k) = k$ and $h(F_k) = \varphi(k)$. If $n$ is odd, then also $\gamma(F_k) > e(F_k) = \varphi(k)$.

The nonexistence of $(M^5_0, F_0)$ is seen by noting that $H^1(M^5_0) = 0$. By Poincaré duality the relation $\chi(M^5_0) \geq 2$ must hold and, consequently, there can be no foliation of $M^5_0$ of codimension one.

Theorem A' involves a series of topological and combinatorial considerations. Theorem B' is a consequence of recent deep results of W. Thurston [6]. The authors are grateful to Thurston for preprints of his work.

1. Submanifold systems of codimension one. Let $M$ be a closed, connected, oriented $n$-manifold. A set $\mathcal{L} = \{L_1, \ldots, L_r\}$ of disjoint, imbedded, closed, connected, oriented submanifolds of $M$ of codimension one will be called a *submanifold system*. In this section we assemble a number of facts about such systems, applications being made later to foliations.

We denote by $[L_i]$ the element of $H_{n-1}(M)$ represented by the fundamental cycle of $L_i$, and by $[\mathcal{L}]$ the set $\{[L_1], \ldots, [L_r]\}$. Here and elsewhere, unless otherwise specified, all homology and cohomology will have integral coefficients. Note that $H_{n-1}(M) \cong H^1(M)$ and that this group is free abelian by the universal coefficient theorem.
For the usual reasons, any two maximal linearly independent subsets of \([\mathcal{E}]\) have the same cardinality, namely the rank of \(\text{span}_g(\mathcal{E})\). The geometric interpretation of this fact is that any two subsets of \([\mathcal{E}]\), maximal with respect to the property of not disconnecting \(M\), have the same cardinality. We suppose that \([L_1], \ldots, [L_k]\) is a maximal linearly independent subset and define \(\text{rank}(\mathcal{E}) = k\).

**Lemma 1.1.** The set \([L_1], \ldots, [L_k]\) spans a direct summand of \(H_{n-1}(M)\). An element \(x \in H_{n-1}(M)\) which is linearly dependent on \([L_1], \ldots, [L_k]\) can be represented by a closed, connected, oriented submanifold \(L \subset M\) disjoint from \(L_i\), \(1 < i < k\), if and only if \(x = \epsilon_1[L_1] + \cdots + \epsilon_k[L_k]\) where each \(\epsilon_i \in \{0, 1, -1\}\).

**Proof.** Since \(W = M - \bigcup_{i=1}^{k} L_i\) is an open connected subspace of \(M\), we can let \(a_1, \ldots, a_k\) be disjoint smoothly imbedded circles in \(M\) transverse to \(\bigcup_{i=1}^{k} L_i\) with \(a_i \cap L_j\) a set of \(\delta_{ij}\) points, \(1 < i, j < k\). By suitably orienting these circles we obtain the homology intersection products \([a_i] \ast [L_j] = \delta_{ij}\). In particular, \([L_1], \ldots, [L_k]\) spans a direct summand of \(H_{n-1}(M)\).

If \(L \subset W\) is a closed, connected, oriented submanifold of \(M\) which does not disconnect \(W\), then the above argument shows that \([L]\) is linearly independent of \([L_1], \ldots, [L_k]\). But if \(L\) does disconnect \(W\), then it is clear that \([L] = \sum_{i=1}^{k} \epsilon_i[L_i]\), where each \(\epsilon_i \in \{0, 1, -1\}\).

If \(x = \epsilon_1[L_1] + \cdots + \epsilon_k[L_k]\) and each \(\epsilon_i \in \{0, 1, -1\}\), we can suppose \(x - [L_1] + \cdots + [L_h] - [L_{h+1}] - \cdots - [L_{h+q}]\) (where either \(h\) or \(q\) may be 0). Let \(L'_1\) be obtained by displacing \(L_i\) a distance \(\epsilon > 0\) along the positive normal trajectories to \(L_i\), \(1 < i < h\), and along the negative normal trajectories if \(h + 1 < i < h + q\). Using the fact that \(W\) is connected, we construct simple arcs \(\tau_i: [0, 1] \rightarrow M\) meeting the following specifications. No \(\tau_i\) meets an \(L_j\), \(1 < j < k\), and the interior of no \(\tau_i\) meets an \(L'_j\), \(1 < j < h + q\). For \(1 < i < h - 1\), the arc \(\tau_i\) joins the positive side of \(L'_i\) to the positive side of \(L'_{i+1}\). The arc \(\tau_h\) joins the positive side of \(L'_h\) to the negative side of \(L_{h+1}\). For \(h + 1 < i < h + q - 1\), the arc \(\tau_i\) joins the negative side of \(L'_i\) to the negative side of \(L'_{i+1}\). The connected sum of the manifolds \(L'_i\) via small tubes along these arcs is a closed, connected, oriented manifold \(L\) meeting no \(L_j\), \(1 < j < k\), such that \(x = [L]\). □

In particular, we can now write \([L_j] = \sum_{i=1}^{h} \epsilon_j^i[L_i]\), \(1 < j < r\), where each \(\epsilon_j^i \in \{0, 1, -1\}\). Let \(E_j\) denote the coefficient \(k\)-tuple \((\epsilon_1^j, \epsilon_2^j, \ldots, \epsilon_k^j)\) of \([L_j]\).

**Lemma 1.2.** If \(([L_p], [L_q])\) is linearly independent, then \(E_p\) and \(E_q\) do not have exactly the same zero entries.

**Proof.** We can assume \(p, q > k\). The set consisting of \(L_p\) together with those \(L_i\) for which \(\epsilon_i^p \neq 0\) (say \(1 < i < h < k\) after suitable renumbering)
 disconnects the manifold into two components, $W_1$ and $W_2$, with common boundary $L_p \cup L_1 \cup \cdots \cup L_h$. The manifold $L_q$ lies in one of these components, say $W_1$, and if $E_q$ has exactly the same zero entries as $E_p$, then $L_q$ separates $W_1$ into two components, one of which has boundary $L_q \cup L_1 \cup \cdots \cup L_h$. It follows that the other component has boundary $L_p \cup L_q$, contradicting the linear independence of $\{[L_p], [L_q]\}$. □

**Definition.** $\mathcal{E}$ is an admissible system if $[\mathcal{E}]$ is pairwise linearly independent (or, equivalently, if $M - (L_1 \cup L_2)$ is connected, $1 < i, j < r$).

**Remark.** Denote by $[L_i]_2$ the element of $H_{n-1}(M; \mathbb{Z}_2)$ carried by $L_i$ and by $[\mathcal{E}]_2$ the set of these classes. The force of (1.2) is that, for $\mathcal{E}$ admissible, $[\mathcal{E}]$ and $[\mathcal{E}]_2$ have the same cardinality.

Let $\mathcal{E}$ be an admissible system. We can modify each $L_i$ by attaching small handles $S^{n-2} \times I$ to $L_i$ within Euclidean neighborhoods of $M$. If $L'_i$ denotes the resulting submanifold, then $[L'_i] = [L_i]$ and the resulting admissible system $\mathcal{E}'$ has exactly the same separation relations in $M$ as did $\mathcal{E}$. Furthermore, if $U_1, \ldots, U_q$ are the components of $M - \bigcup L_i$ and $U'_1, \ldots, U'_q$ the corresponding components of $M - \bigcup L'_i$, then each $U'_j$ is obtained from $U_j$ by adjoining and/or cutting out solid handles. If a solid handle was adjoined, then $x(U'_j) = x(U_j) - 1$. If $n$ is even and a solid handle was cut out, then $x(U'_j) = x(U_j) + 1$.

**Definition.** $\mathcal{E}'$ as above is said to be a surgical modification of $\mathcal{E}$.

The principal results of this section will be the following propositions.

**Proposition 1.3.** If $x(M) = 0$ and $\mathcal{E} = \{L_1, \ldots, L_r\}$ is an admissible system, then suitable surgical modifications on $\mathcal{E}$ produce an admissible system $\mathcal{E}' = \{L'_1, \ldots, L'_r\}$ with the following properties:

1. The tangent bundle to $\bigcup_{i=1}^r L'_i$ extends to an $(n-1)$-plane field on $M$,
2. $H^1(L'_i; \mathbb{R}) \neq 0$, $1 < i < r$,
3. if $n$ is odd, $(x(L'_1), \ldots, x(L'_r))$ is a set of distinct negative integers.

**Proposition 1.4.** If $\mathcal{E} = \{L_1, \ldots, L_r\}$ is an admissible system, then $r < \varphi(x(M)) - 1$.

The second of these propositions is the heart of the proof of Theorem A', while the first, together with Thurston's results [6], makes possible a very straightforward construction of the foliated $n$-manifolds $(M_k^n, F_k)$ for Theorem B'.

First we will need a technical lemma. The notation $E \cdot N$ in the statement of the lemma denotes the usual dot product of $k$-tuples.

**Lemma 1.5.** Let $\mathcal{S}^k = \{ (e_1, \ldots, e_k) : e_i \in \{0, 1, -1\} \}$. Given any $R > 0$, any integer $k > 1$, and any $\eta = (\eta_1, \ldots, \eta_k) \in (\mathbb{Z}_2)^k$, there exists $N = (n_1, \ldots, n_k) \in \mathbb{Z}^k$ with all $n_i < 0$ such that, for all $E, E' \in \mathcal{S}^k$,
(1) \( E \cdot N = E' \cdot N \) if and only if \( E = E' \),
(2) \( |E \cdot N| > R \) if \( E \neq (0, \ldots, 0) \),
(3) \( N \equiv \eta \mod 2 \).

**Proof.** This is trivial for \( k = 1 \), so we argue by induction on \( k \). Suppose \( N = (n_1, \ldots, n_k) \) satisfies the conditions. Choose \( E^* \in \mathcal{E}^k \) such that \( |E^* \cdot N| \) is maximal. Choose \( n_{k+1} < 0 \) with the desired parity and such that \( |n_{k+1}| > 2|E^* \cdot N| \). Let \( N' = (n_1, \ldots, n_{k+1}) \in \mathcal{Z}^{k+1} \). If \( \sum_{i=1}^{k+1} \epsilon_i n_i = \sum_{i=1}^{k+1} \epsilon'_i n_i \), \( \epsilon_i, \epsilon'_i \in \{0, 1, -1\} \), then

\[
(e'_{k+1} - e_{k+1})n_{k+1} = \sum_{i=1}^{k} \epsilon_i n_i - \sum_{i=1}^{k} \epsilon'_i n_i = E \cdot N - E' \cdot N.
\]

We consider the only possible cases:

(1) \( |e'_{k+1} - e_{k+1}| = 1 \),
(2) \( |e'_{k+1} - e_{k+1}| = 2 \),
(3) \( e'_{k+1} = e_{k+1} \) and \( E \cdot N = E' \cdot N \).

In case (1), \( |n_{k+1}| = |E \cdot N - E' \cdot N| < 2|E^* \cdot N| \) and this contradicts the assumption that \( |n_{k+1}| > 2|E^* \cdot N| \). In case (2), \( |n_{k+1}| < |E^* \cdot N| \), again a contradiction. Thus case (3) alone is possible. By the inductive hypothesis, \( E = E' \) and so \( e_i = e'_i, 1 < i < k + 1 \). Furthermore, if \( E'' = (e''_1, \ldots, e''_{k+1}) \in \mathcal{E}^{k+1} \), we claim \( |E'' \cdot N'| > R \). Indeed, if \( e''_{k+1} = 0 \), this comes from the inductive hypothesis and, if \( e''_{k+1} \neq 0 \), then

\[
|E'' \cdot N'| = \sum_{i=1}^{k} e''_i n_i \pm n_{k+1} > |n_{k+1}| - \sum_{i=1}^{k} e''_i n_i
\]

\[
> 2|E^* \cdot N| - |E^* \cdot N'| = |E^* \cdot N| > R. \quad \square
\]

We also need the following fairly well-known fact.

**Lemma 1.6.** If \( W \) is a compact connected \( n \)-manifold, \( \nu \) a vector field defined along \( \partial W \) and transverse to \( \partial W, \partial W_+ \) the part of \( \partial W \) along which \( \nu \) points out of \( W \), \( \partial W_- \) the part along which \( \nu \) points inward, then \( \nu \) extends to a nowhere zero vector field on \( W \) if and only if \( x(W) = x(\partial W_+) = x(\partial W_-) \). Equivalently, for \( n \) even, this holds precisely when \( x(W) = 0 \), and for \( n \) odd, it holds precisely when \( x(\partial W_+) = x(\partial W_-) \).

Indeed, \( \nu \) always extends to \( \partial \) with only isolated nondegenerate singularities in \( \text{int}(W) \) and, \( W \) being connected, the singularities can be eliminated if the sum of their indices \( \iota(\partial) = 0 \). The condition that \( \iota(\partial) = 0 \) is exactly the condition in (1.6), as is seen by standard relative Hopf index formulas (a very general formula is given by C. Pugh [4]).

**Proof of (1.3).** To begin with, note that adding a single handle to \( L_i \) produces \( L_i' \) with \( H^1(L_i'; R) \neq 0 \) and this property is never lost by adding
more handles. This is easily established via the Mayer-Vietoris sequence. Thus, without loss of generality, we can suppose that $H^1(L_i; \mathbb{R}) \neq 0$, $1 < i < r$, and this property is never lost during the surgical modification, so property (2) is assured.

For the rest of the argument we must distinguish the cases in which $n$ is odd or even. First suppose that $n$ is even. Let $U_1, \ldots, U_q$ be the components of $M - \bigcup_{i=1}^p L_i$. Each $\overline{U_i}$ is a manifold with boundary $\partial \overline{U_i}$ equal to the union of some of the manifolds $L_i$. If $L_{i_1}, \ldots, L_{i_k}$ are in $\text{int}(\overline{U_j})$, then $\overline{U_j} - \bigcup_{m=1}^p L_m$ is connected and we produce a compact connected manifold $W_j$ by cutting $\overline{U_j}$ open along each $L_m$. Note that $\chi(W_j) = \chi(\overline{U_j})$. Also $0 = \chi(M) = \sum_{i=1}^q \chi(U_i)$ since the boundary components of every $\overline{U_j}$ are odd dimensional, hence have vanishing Euler characteristic.

Find a sequence $U_{n_1}, \ldots, U_{n_q}$ such that each $U_i$ appears at least once in the sequence and such that $\partial \overline{U_{n_i}}$ and $\partial \overline{U_{n_{i+1}}}$ have a common component $L_{n_i}$, $1 < j < q - 1$. (Repetitions have to be allowed. For instance, $\partial \overline{U_1}$ might contain every $L_i$, which would force such a sequence as $U_1$, $U_2$, $U_3$, $U_4$, $U_5$, $U_6$, $U_7$.) If $\chi(U_{n_i}) \neq 0$, add a suitable number of handles to $L_{n_i}$ (into $\overline{U_{n_i}}$ if $\chi(\overline{U_{n_i}}) > 0$, into $\overline{U_{n_i}}$ if $\chi(\overline{U_{n_i}}) < 0$) so that the new $\chi(\overline{U_{n_i}}) = 0$. Proceed in this way through the sequence. Since some $U_{n_i}$ may equal some $U_{n_{i+1}}$, the good work already done on $U_{n_{i+1}}$ may be undone in passing from $U_{n_{i+1}}$ to $U_{n_i}$, but if $j < q$, this will be remedied in passing from $U_{n_i}$ to $U_{n_{i+1}}$. When one finally arrives at $U_{n_q}$, every $U_{n_i} \neq U_{n_q}$ satisfies $\chi(\overline{U_{n_i}}) = 0$. Assume $U_{n_i} = U_q$ and observe that

$$0 = \chi(M) = \sum_{j=1}^q \chi(\overline{U_j}) = \chi(\overline{U_q}).$$

Thus, without loss of generality, we suppose $\mathcal{E}$ is such that each $\chi(W_j) = 0$. Let $\nu$ be the unit normal field along $\bigcup_{i=1}^p L_i$. This defines a field $v_j$ along $\partial W_j$; hence (1.6) implies that $\nu$ extends to a nonsingular field on all of $M$. Property (1) follows and, since $n$ is even, property (3) is irrelevant.

Next suppose that $n$ is odd. Surgical modification reduces the Euler characteristic of $L_i$ by 2 for each handle added to $L_i$. In particular, this allows us to assume that $\chi(L_i) < 0$, $1 < i < r$. Remark also that for any compact odd-dimensional manifold $W$, $\chi(\partial W)$ is an even number since $\partial W$ is cobordant to zero. Let $\{[L_1], \ldots, [L_k]\}$ be a maximal linearly independent subset of $[\mathcal{E}]$. If $[L_i] = \sum_{j=1}^k \epsilon_j[L_j]$ then $L_i$, together with those $L_j$ for which $\epsilon_j \neq 0$, cobound in $M$, so

$$\chi(L_i) \equiv \sum_{j=1}^k \epsilon_j \chi(L_j) \mod 2.$$
Let $R > |\chi(L_i)|$, $1 < i < r$, and let $N = (n_1, \ldots, n_k)$ satisfy (1.5) with $n_j \equiv \chi(L_j) \mod 2$, $1 < j < k$. Let $\gamma$, $\delta_j$, $1 < j < k$, be circles in $M$ such that $[\gamma] \cdot [L_j] = \delta_j$, $1 < j < k$. Let $\gamma = \sum_{j=1}^{k} n_j [\sigma_j]$. Then, if $E_i = (e_i^1, \ldots, e_i^k)$ denotes the coefficient $k$-tuple of $[L_j]$, we see that $\gamma \cdot [L_i] = E_i \cdot N$. Since $E_i \not= \pm E_j$ for $i \not= j$, (1.5) implies that $|\gamma \cdot [L_i]|$, $1 < i < k$, are distinct nonzero integers.

Changing the orientation of $L_i$, if necessary, we can assume all $\gamma \cdot [L_i]$ are negative. By the above,

$$|\gamma \cdot [L_i]| > |\chi(L_i)|, \quad \gamma \cdot [L_i] \equiv \chi(L_i) \mod 2, \quad 1 < i < r.$$ 

Thus, by surgical modification, we can produce $\xi$ such that $\gamma \cdot [L_i] = \gamma \cdot [L_i] = \chi(L_i) < 0$, $1 < i < r$, and these integers are distinct.

Let $\xi \in H^{n-1}(M)$ be the Poincaré dual of $\gamma$. Then, $\xi([L_i]) = \int_{L_i} \xi = \chi(L_i)$. Let $\nu$ be the unit normal field along $\cup_{i=1}^r L'_i$ directed in the positive sense (relative to the orientation of $L'_i$). Let $U_1, \ldots, U_q$ be the components of $M - \cup_{i=1}^r L'_i$ and let $\xi_i$ be the restriction of $\xi$ to $\overline{U_i}$. On $\partial U_i$, understood as oriented inward, $\xi_i$ integrates to 0; hence on $\partial U_{i+}$, oriented outward, $\xi_i$ integrates to the same value as on $\partial U_{i-}$, oriented inward. That is,

$$\chi(\partial U_{i+}) = \chi(\partial U_{i-}), \quad 1 < i < q.$$ 

Let $W_i$ be the compact connected manifold obtained by cutting $\overline{U_i}$ open along each $L'_j \subset \text{int}(\overline{U_i})$. Then $\nu$ defines $\nu_i$ along $\partial W_i$ and transverse to $\partial W_i$. Each $L'_j \subset \text{int}(\overline{U_i})$ determines two diffeomorphic boundary components of $W_i$, one in $\partial W_{i+}$ and one in $\partial W_{i-}$. Thus $\chi(\partial W_{i+}) = \chi(\partial W_{i-})$, $1 < i < q$, so $\nu_i$ extends to a nonsingular vector field over $W_i$ (by (1.6)); hence $\nu$ extends to a nonsingular vector field over $M$. This defines the desired extension of the tangent bundle of $\cup_{i=1}^r L'_i$, hence establishes property (1). Properties (2) and (3) have already been assured, so the proof of (1.3) is complete.

The proof of (1.4) will proceed by a sequence of lemmas. We will be interested only in the separation properties of $\xi$ in $M$, so all homological arguments will be carried out with $\mathbb{Z}_2$-coefficients. This is legitimate by reason of (1.2) and the remark following that lemma. Thus, $\text{rank}(\xi) = \dim(\text{span}[\xi])_2$ and the coefficient $k$-tuple $E_i$ for $[L_j]_2$ will be an element of $(\mathbb{Z}_2)^k$.

**Definition.** If $\xi$ and $\xi'$ are admissible systems, then $\xi < \xi'$ means $\xi \subset \xi'$ and $\text{rank}(\xi) = \text{rank}(\xi')$.

**Definition.** An admissible system $\xi = \{L_1, \ldots, L_r\}$ is **polygonal** if each $L_i$ occurs as a boundary component of some $U_j$ (hence of exactly two) where $U_1, \ldots, U_r$ is the set of components of $M - \cup_{i=1}^r L'_i$.

The term “polygonal” is due to a heuristic procedure by which each $\overline{U_j}$ is represented by a diagram such as
where \( \partial \overline{U}_j = L_1 \cup L_2 \cup L_3 \cup L_4 \cup L_5 \) and \( L_6 \subset \text{int}(\overline{U}_j) \). The word “polygonal” indicates that each \( \overline{U}_j \) is represented by a simple closed polygon, hence has no \( L_i \) in the interior.

In the above example, produce \( L_i' \) and \( L_6' \subset \text{int}(U'_j) \) by displacing \( L_i \) and \( L_6 \) a small distance \( \epsilon > 0 \) along normal trajectories. A schematic representation of this situation would be

since \( L_i \) and \( L_i' \) cobound for \( i = 1, 6 \). A connected sum of \( L_i' \) and \( L_6' \) along a small tube lying entirely in \( \text{int}(\overline{U}_j) \) and missing \( L_6 \) produces \( L \subset \text{int}(\overline{U}_j) \) such that \( L \cap L_6 = \emptyset \) and \( L \cup L_6 \) disconnects \( \overline{U}_j \). The schematic representation of the final state of affairs would be

Finally, if \( \{L_1, \ldots, L_k\} = \mathcal{L} \) for every \( L_i \in \mathcal{L} \), this construction produces \( \mathcal{L}' = \{L_1, \ldots, L_r, L\} \) with \( \mathcal{L} < \mathcal{L}' \). This idea is exploited in the following lemma.

**Lemma 1.7.** If \( \mathcal{L} \) is an admissible system of rank at least 2, then \( \mathcal{L} < \mathcal{L}' \) for some admissible polygonal system \( \mathcal{L}' \).

**Proof.** As usual, arrange that \( \{[L_1], \ldots, [L_k]\} \) is a maximal linearly independent subset of \( [\mathcal{L}]_2 \). If \( L_i \subset \text{int}(U_j) \), then \( L_1 \cup \cdots \cup L_k \cup L_i \) cannot
separate $M$, hence $i < k$. Arrange that $\{L_1, \ldots, L_p\}, p < k$, is the subset of those $L_i$ each of which is contained in the interior of some $U_j$. Thus every $E_j = (e_i^1, \ldots, e_i^k)$ has $e_i^j = 0$ for $1 < i < p$. The system $\mathcal{L}$ is polygonal if and only if $p = 0$.

If $p = k$, then $k = r > 2$ and by (1.1) the class $x = [L_1]_2 + \cdots + [L_r]_2$ can be represented by connected orientable $L_{r+1} \subset M$ disjoint from every $L_i$, $1 < i < r$, $[L_{r+1}]_2$ being linearly independent of $[L_i]_2$, $1 < i < r$. $\mathcal{L}' = \{L_1, \ldots, L_r, L_{r+1}\}$ is admissible of rank $k = r$ and is also polygonal.

If $0 < p < k$, let $U_i$ be the component such that $L_p \subset \text{int}(U_i)$. Let $L_s \subset \partial U_i$. As in the remarks preceding the lemma, displace both $L_s$ and $L_p$ slightly and form a connected sum so as to produce $L_{r+1} \subset \text{int}(U_i)$ with

$$L_{r+1} \cap L_j = \emptyset, \quad 1 < j < r,$$

$$[L_{r+1}]_2 = [L_p]_2 + [L_s]_2.$$

This gives an admissible system $\{L_1, \ldots, L_{r+1}\}$ of rank $k$ in which $\{L_1, \ldots, L_{p-1}\}$ is the subset of those $L_i$ contained in some $\text{int}(U_i)$. This process, repeated finitely often, produces the desired polygonal system $\mathcal{L}'$.

Remark that polygonal systems must have rank at least 2, hence the necessity of the restriction on rank($\mathcal{L}$) in (1.7). This in turn is the reason why $\varphi(k)$ must be defined by a different formula when $k = 0, 1$.

DEFINITION. An admissible system is triangular if it is polygonal and if each $\partial U_j$ has exactly three components.

**Lemma 1.8.** Let $\mathcal{L}$ be an admissible polygonal system. Then $\mathcal{L} \subset \mathcal{L}'$ for some admissible triangular system.

**Proof.** If some $\partial(U_j)$ has more than three components (fewer are impossible by the definition of an admissible system), let $L_1, L_2, L_3, L_4$ be distinct components of $\partial U_j$. If both $\{[L_1]_2, [L_2]_2, [L_3]_2\}$ and $\{[L_2]_2, [L_3]_2, [L_4]_2\}$ are linearly dependent sets, then $[L_1]_2 = [L_2]_2 + [L_3]_2$ and $[L_4]_2 = [L_2]_2 + [L_3]_2$, and this would contradict $[L_1]_2 \neq [L_4]_2$. Thus we can assume that $\{[L_1]_2, [L_2]_2, [L_3]_2\}$ is a linearly independent set. We claim that either $[L_1]_2 + [L_2]_2$ or $[L_1]_2 + [L_3]_2$ is not in $[\mathcal{L}]_2$. Indeed, if one can find $L_p, L_q \in \mathcal{L}$ such that $[L_p]_2 = [L_1]_2 + [L_2]_2$, $[L_q]_2 = [L_1]_2 + [L_3]_2$, then $p, q \neq 1, 2, 3$ and $p \neq q$. Thus $\{L_1, L_2, L_3, L_p, L_q\}$ is an admissible system of rank 3. The complement in $M$ of $L_1 \cup L_2 \cup L_p$ has two components. Neither $L_3$ nor $L_q$ individually disconnects the component in which it lies, but $L_1 \cup L_3 \cup L_q$ disconnects $M$, so $L_3$ and $L_q$ must lie in the same component of $M - L_1 - L_2 - L_p$ and together disconnect it into two components, one of which must have boundary $L_1 \cup L_3 \cup L_q$ (because all other possibilities lead to a linear dependence between $[L_1]_2$ and $[L_2]_2$). Thus $M - (L_1 \cup L_2 \cup L_3 \cup L_p \cup L_q)$
has components $W_1$, $W_2$, $W_3$ with

\[
\partial \bar{W}_1 = L_1 \cup L_3 \cup L_q,
\]

\[
\partial \bar{W}_2 = L_3 \cup L_q \cup L_2 \cup L_p,
\]

\[
\partial \bar{W}_3 = L_1 \cup L_2 \cup L_p.
\]

Since $\mathcal{L}$ contains this admissible system, $U_j$ must be contained in one of $W_1$, $W_2$, $W_3$. But $\bar{U}_j \subset \bar{W}_1$ contradicts $L_2 \not\subset \bar{W}_1$, $\bar{U}_j \subset \bar{W}_2$ contradicts $L_1 \not\subset \bar{W}_2$, and $\bar{U}_j \subset \bar{W}_3$ contradicts $L_3 \not\subset \bar{W}_3$. Thus we can assume that no $L_p \in \mathcal{L}$ has $[L_p]_2 = [L_1]_2 + [L_2]_2$. Thus, displace both $L_1$ and $L_2$ to the interior of $U_j$ and form their connected sum, producing $L_{r+1} \subset U_j$ which separates $U_j$ into two components, one with boundary $L_{r+1} \cup L_1 \cup L_2$, the other with one fewer boundary component than $U_j$ had. By the above, $\{L_1, \ldots, L_r, L_{r+1}\}$ is admissible, polygonal, and has the same rank as $\mathcal{L}$. Finite iteration of this process will finally yield the desired triangular system. □

Thus every admissible $\mathcal{L}$ can be extended to an admissible triangular system $\mathcal{L}'$ of the same rank. The next project is to produce a formula for the number of elements in $\mathcal{L}'$ in terms of the integer rank($\mathcal{L}'$).

**Lemma 1.9.** Let $W$ be a compact, orientable, connected manifold and suppose that $\partial W$ has $h$ components, $h \geq 3$. Let $L_1, \ldots, L_p \subset \text{int}(W)$ be disjoint, closed, connected, orientable submanifolds of $\text{int}(W)$ of codimension one. Suppose that $\{L_1, \ldots, L_q\}$ is a maximal subset such that $W - \bigcup_{i=1}^q L_i$ is connected. Suppose that, for each component $U$ of $W - \bigcup_{i=1}^q L_i$, the manifold $\partial U$ has three components and that every $L_i$ occurs as some boundary component for some $U$. Then $p = h + 3s - 3$.

**Proof.** We do a double induction on $h \geq 3$ and $s \geq 0$. If $h = 3$ and $s = 0$, we claim that $p = 0$. Otherwise, $L_1$ separates $W$ into two components one of which, say $W'$, must be bounded by $L_1$ and a single component of $\partial W$. If there is any $L_i \subset W'$, then, since $s = 0$, we again get a component of $W - (L_1 \cup L_i)$ with two boundary components. Proceeding in this way, we finally see that some component of $W - \bigcup_{i=1}^p L_i$ has just two boundary components, contrary to our hypothesis.

Suppose the assertion is true for $s = 0$ and for all $h$ such that $3 < h < m$. If $\partial W$ has $m$ components and $s = 0$, then $L_1$ divides $W$ into two components each of which has at least three boundary components (for the same reason as above). If the number of boundary components in each is $h_1$ and $h_2$, respectively, then $3 < h_i < m, i = 1, 2$, and $h_1 + h_2 = m + 2$. But each of $L_2, \ldots, L_p$ belongs to one or another component of $W - L_1$ and these components are divided into triangular components. The inductive hypothesis implies that
\[ p - 1 = h_1 - 3 + h_2 - 3 = h_1 + h_2 - 6 = m - 4 \]

and so \( p = m - 3 \) as desired.

Finally, assume the assertion for \( 0 < s < n \) and for all \( h \). Suppose \( \{L_1, \ldots, L_n\} \) is a maximal subset not separating \( W \). Necessarily \( n < p \) since, otherwise, \( h = 3, p = 0 \), and so \( n = 0 \). Thus, the set \( \{L_1, \ldots, L_n, L_{n+1}\} \) separates \( W \) into two components \( W_1 \) and \( W_2 \). We can suppose that the common boundary is \( \partial W_1 \cap \partial W_2 = L_1 \cup \cdots \cup L_q \cup L_{n+1} \) where \( 0 < q < n \). We can even suppose \( q > 0 \) since, if this were impossible to arrange, none of \( L_1, \ldots, L_n \) could ever occur as a boundary component of any component of \( W - \bigcup_{i=1}^q L_i \), contradicting the assumption that \( n > 0 \). Suppose that \( s_1 \) of \( \{L_{q+1}, \ldots, L_n\} \) fall into \( W_1 \) and that \( s_2 \) of them fall into \( W_2 \). Evidently these \( s_i \) manifolds are maximal nonseparating in \( W_i \) and \( s_i < n, i = 1, 2 \). Let \( h_i \) be the number of boundary components of \( W_i \) and \( p_i \) the number of \( L_j \subset \text{int}(W_i) \), \( i = 1, 2 \). By the inductive hypothesis \( p_i = h_i + 3s_i - 3, i = 1, 2 \). But \( h_1 + h_2 = h + 2q + 2, s_1 + s_2 + q = n, p_1 + p_2 = p - q - 1 \), and so

\[
\begin{align*}
p &= p_1 + p_2 + q + 1 = (h_1 + 3s_1 - 3) + (h_2 + 3s_2 - 3) + q + 1 \\
&= (h + 2q + 2) + 3s_1 + 3s_2 + q - 5 \\
&= h + 3(s_1 + s_2 + q) - 3 = h + 3n - 3.
\end{align*}
\]

**Lemma 1.10.** Let \( \mathcal{L} = \{L_1, \ldots, L_r\} \) be an admissible triangular system of rank \( k \). Then \( r = 3k - 3 \).

**Proof.** Let \( U \) be any component of \( M - \bigcup_{i=1}^r L_i \) and let \( \partial \overline{U} = L_1 \cup L_2 \cup L_r \). These are compact manifolds each with three boundary components. Also, \( W_2 \) is connected since \( \{[L_1]_2, [L_2]_2, [L_r]_2\} \) is pairwise linearly independent. The set \( \{[L_1]_2, [L_2]_2\} \) extends to a maximal independent set \( \{[L_1]_2, \ldots, [L_k]_2\} \). If \( i \neq 1, 2, r \), then \( L_i \subset \text{int}(W_2) \) and \( \{L_3, \ldots, L_k\} \) is a maximal nonseparating subset in \( W_2 \). Thus, by (1.9), \( r - 3 = 3 + 3(k - 2) - 3 = 3k - 6 \) and so \( r = 3k - 3 \). \( \square \)

We now prove (1.4). If \( \mathcal{L} = \{L_1, \ldots, L_r\} \) is an admissible system of rank \( k \), let \( \{[L_1], \ldots, [L_k]\} \) be a maximal independent subset. We emphasize the return to integral homology. By (1.1), this set spans a direct summand of \( H_{n-1}(M) \). If \( x_i \) denotes the Poincaré dual of \( [L_i] \), \( 1 < i < k \), then \( \{x_1, \ldots, x_k\} \) extends to a basis of \( H^1(M) \). But \( x_i \cup x_j \) is the Poincaré dual of \( [L_i] \ast [L_j] = 0 \); hence \( k < \alpha(M) \). If \( k = 0 \), then \( r = 0 \) and there is nothing to prove. If \( k = 1 \), then \( r = 1 \) and \( 1 < \alpha(M) \), so \( r = \varphi(1) - 1 < \varphi(\alpha(M)) - 1 \). If \( k > 2 \), then (1.7), (1.8) and (1.10) together imply

\[
r < 3k - 3 = \varphi(k) - 1 < \varphi(\alpha(M)) - 1. \quad \square
\]

2. **Applications to foliated \( n \)-manifolds.** It is easy to prove Theorem A' as a corollary to (1.4). Let \( (M, F) \) be a foliated \( n \)-manifold with \( h(F) = r \). Let
\[ \{L_1, \ldots, L_r\} \] be a set of closed leaves of \( F \) such that \( \{[L_1], \ldots, [L_r]\} \) are distinct modulo sign. If no \( [L_i] = 0 \), then \( \{L_1, \ldots, L_r\} \) is an admissible system. If, say, \( [L_r] = 0 \), then \( \{L_1, \ldots, L_{r-1}\} \) is admissible. In the first case, \( r < \varphi(\alpha(M)) - 1 \), and in the second, \( r - 1 < \varphi(\alpha(M)) - 1 \). In any case, \( h(F) < \varphi(\alpha(M)) \). Also, by (1.2), \( h_2(F) = h(F) \).

If \( L \) and \( L' \) are closed leaves of \( F \) with \( |\chi(L)| \neq |\chi(L')| \), then \( [L] \neq \pm [L'] \). Indeed, let \( c \in H^{n-1}(M) \) be the Euler class of the tangent bundle to \( F \). Then, if \( [L] = \pm [L'] \), we would have
\[ \pm \chi(L) = c[L] = \pm c[L'] = \pm \chi(L'), \]
a contradiction. It follows that \( e(F) < h(F) \) and the proof of Theorem A' is complete. As remarked in the introduction, Theorem A is a trivial consequence when \( \dim(M) = 3 \). \( \Box \)

**Definition.** An admissible system \( \mathcal{L} \) is integrable if there is a foliation \( F \) of \( M \) with each \( L_i \in \mathcal{L} \) as a leaf. (Note that the orientation of \( L_i \) as a leaf of \( F \) may not be the same as that of \( L_i \in \mathcal{L} \).)

**Theorem 2.1.** If \( \chi(M) = 0 \) and \( \mathcal{L} \) is admissible, then a suitable surgical modification on \( \mathcal{L} \) produces an integrable \( \mathcal{L}' \). If, in addition, \( \dim(M) \) is odd, it can be arranged that the Euler characteristics of the elements of \( \mathcal{L}' \) form a set of distinct negative integers.

**Proof.** We allow \( \mathcal{L} = \emptyset \), in which case \( \text{rank}(\mathcal{L}) = 0 \). Produce \( \mathcal{L}' \) satisfying (1.3). Then, by Thurston ([6], relative version if \( \text{rank}(\mathcal{L}) \neq 0 \)), there is a foliation of \( M \) having each \( L_i \in \mathcal{L} \) as a leaf. (Note that the orientation of \( L_i \) as a leaf of \( F \) may not be the same as that of \( L_i \in \mathcal{L} \).)

**Corollary 2.2.** If \( \chi(M) = 0 \) and \( M \) supports an admissible system of rank \( k \), then \( M \) admits a foliation \( F \) with \( h(F) > \varphi(k) \). (Here we allow an empty admissible system; hence \( k = 0 \).) If \( \dim(M) \) is odd, such \( F \) can also be chosen so that \( e(F) > \varphi(k) \).

**Proof.** If \( k > 2 \), then by (1.7), (1.8), and (1.10) there is an admissible system \( \mathcal{L} \) of cardinality \( 3k - 3 \). In any case, there is such a system \( \mathcal{L} \) of cardinality \( \varphi(k) - 1 \). By (2.1) we can assume the existence of a foliation \( F \) having each \( L \in \mathcal{L} \) as a leaf. Also, on odd-dimensional manifolds it can be arranged that the Euler characteristics \( \chi(L), L \in \mathcal{L} \), form a set of distinct negative integers. By the Reeb stability theorem we can assume that some leaf is noncompact; hence standard methods show that there is a closed transversal to \( F \) missing every leaf \( L \in \mathcal{L} \). Modifying \( F \) along this transversal [7] produces a foliation \( F' \) in which each element of \( \mathcal{L} \) is a leaf, but which also has a homologically trivial closed leaf with Euler characteristic zero. Thus \( h(F') > \varphi(k) \), and on odd dimensional manifolds \( M \), one also has \( e(F') > \varphi(k) \). \( \Box \)

In order to prove Theorem B', we will now construct a suitable \( M_k^a \) and
apply the above corollary together with Theorem A'.

First suppose that \( n \) is odd and at least equal to 3. Let \( M^n_\circ = S^n \) and \( M^1_\circ = T^s \). Evidently \( \alpha(S^n) = 0 \) and \( \alpha(T^n) = 1 \). For \( k \geq 2 \), let \( W^k \) be the complement in \( D^n \) of \( k \) disjoint open balls with closures in the interior of \( D^n \) and let \( M^n_k \) be the double of \( W^n_k \). By the above corollary and the obvious fact that \( M^n_k \) supports an admissible system \( \{ L_1, \ldots, L_k \} \) of rank \( k \), we see that \( M^n_k \) supports a foliation \( F_k \) with \( h(F_k) > e(F_k) > \varphi(k) \).

If \( n \) is even and at least equal to 4, and if \( k > 0 \), set \( M^n_k = M^n_{k-1} \times S^1 \). If \( \{ L_1, \ldots, L_k \} \) is an admissible system of rank \( k \) on \( M^n_{k-1} \), then the manifolds \( L'_i = L_i \times S^1 \subset M^n_k \), \( 1 \leq i \leq k \), form an admissible system of rank \( k \). For the case \( k = 0 \), set \( M^6_0 = S^3 \times S^3 \) and \( M^6_n = S^3 \times S^3 \times S^{n-6} \), \( n > 6 \). In all cases, \( \chi(M^n_k) = 0 \) and the above corollary guarantees the existence of a foliation \( F_k \) with \( h(F_k) > \varphi(k) \).

The following lemma together with Theorem A' clearly completes the proof of Theorem B'. As remarked in the introduction, when \( n = 3 \) we get Theorem B as a special case.

**Lemma 2.3.** \( \alpha(M^n_k) = k \) for all \( n > 3 \) and all \( k > 0 \) (excepting, of course, the nonexistent \( M^3_0 \)).

**Proof.** For \( k = 0, 1 \) this is obvious. For \( n \) odd and \( k > 2 \), \( \pi_i(W^n_k) = 0 \), \( 0 < i < n - 2 \), so the Hurewicz theorem gives

\[
\pi_{n-1}(W^n_k) = \pi_{n-1}(W^n_k) = \mathbb{Z}^k.
\]

There is an exact Mayer-Vietoris sequence

\[
\begin{align*}
H_n(W^n_k) \oplus H_n(W^n_k) &\to H_n(M^n_k) \to H_{n-1}(W^n_k) \to H_{n-1}(W^n_k) \\
\oplus H_{n-1}(W^n_k) &\to H_{n-1}(M^n_k) \to H_{n-2}(W^n_k).
\end{align*}
\]

But \( H_n(W^n_k) = 0 \), \( H_n(M^n_k) = \mathbb{Z} \), \( H_{n-1}(W^n_k) = \mathbb{Z}^{k+1} \), \( H_{n-1}(W^n_k) = \mathbb{Z}^k \) and \( H_{n-2}(W^n_k) = 0 \), so we get

\[
0 \to \mathbb{Z} \to \mathbb{Z}^{k+1} \to \mathbb{Z}^{2k} \to H_{n-1}(M^n_k) \to 0;
\]

hence \( 0 \to \mathbb{Z}^{k} \to \mathbb{Z}^{2k} \to H_{n-1}(M^n_k) \to 0 \). As earlier remarked, \( H_{n-1}(M^n_k) \) is free abelian, so the sequence splits and \( \mathbb{Z}^{2k} \cong \mathbb{Z}^k \oplus H_{n-1}(M^n_k) \). Therefore \( H_{n-1}(M^n_k) \cong \mathbb{Z}^k \) for \( n \) odd. The admissible system \( \{ L_1, \ldots, L_k \} \) provides a free basis of \( H_{n-1}(M^n_k) \) with all intersection products zero. By Poincaré duality, we get a basis \( x_1, \ldots, x_k \) of \( H^1(M^n_k) \) with all \( x_i \cup x_j = 0 \), so \( \alpha(M^n_k) = k \).

If \( n \) is even, then

\[
H_{n-1}(M^n_k) \cong H_{n-1}(M^n_{k-1}) \oplus H_{n-1}(M^n_{k-1}) \cong \mathbb{Z}^{k+1}.
\]

Again we get classes \( x_1, \ldots, x_k \in H^1(M^n_k) \) with all \( x_i \cup x_j = 0 \). These extend to a basis by adjunction of the class \( x_{k+1} \) defined by the projection map.
of $M_k^n$ onto the factor $S^1$, and $x_i \cup x_{k+1} \neq 0$, $1 < i < k$. It is again clear that $\alpha(M_k^n) = k$. □

3. A class of examples. Let $T_g$ denote the closed oriented surface of genus $g$. Let $E_{g,n}$ denote the total space of the oriented $S^1$-bundle over $T_g$ having Euler class $n \in H^2(T_g) = \mathbb{Z}$. Since $E_{g,n} \cong E_{g,-n}$, we will always take $n > 0$. Remark, in particular, that $E_{0,0} = S^1 \times S^2$, $E_{0,1} = S^3$ and $E_{1,0} = T^3$.

We will compute the following.

\[
\alpha(E_{g,n}) = \begin{cases} 
1, & n = g = 0, \\
2g, & n = 1, \\
g, & \text{otherwise.}
\end{cases}
\]

Indeed, $H^1(E_{0,0}) = \mathbb{Z}$, so the first equality is immediate.

Since the bundle $S^1 \to E_{g,n} \to T_g$ is orientable, the cohomology spectral sequence has ordinary coefficients. Since $\gamma_x(T_g) = \mathbb{Z}$, the $E_2$ stage takes the following form.

\[
\begin{array}{cccc}
\mathbb{Z} & \mathbb{Z} & 0 & \cdots \\
0 & 0 & 0 & \cdots \\
\mathbb{Z} & \mathbb{Z}^2 & \mathbb{Z} & 0 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\end{array}
\]

If $1 \in \mathbb{Z} = E_{2,1} = H^1(S^1)$ corresponds to the orientation of the fiber, then $d_2(1) = n \in \mathbb{Z} = E_2^{2,0} = H^2(T_g)$ is the Euler class of the bundle. It is also clear that $E_3 = E_{\infty}$.

**Lemma 3.2.** $H^1(E_{g,1}) = \mathbb{Z}^2g$ and the cup products of elements in this group are all zero. Thus $\alpha(E_{g,1}) = 2g$.

**Proof.** In the spectral sequence, $d_2(1) = 1$ and so $d_2: E_2^{0,1} \to E_2^{2,0}$ is an isomorphism. Thus, $E_3^{0,1} = 0$ and $E_3^{1,0} = \mathbb{Z}^{2g}$. Since $E_3 = E_{\infty}$, we have $H^1(E_{g,1}) = \mathbb{Z}^{2g}$. Indeed, by the edge homomorphism, $\pi^*: H^1(T_g) \to H^1(E_{g,1})$ is an isomorphism. Since $E_3^{2,0} = 0$, the edge homomorphism also shows that $\pi^*: H^2(T_g) \to H^2(E_{g,1})$ is trivial. It follows that all cup products of elements in $H^1(E_{g,1})$ are trivial. □

**Lemma 3.3.** If $n > 1$, then $H^1(E_{g,n}) = \mathbb{Z}^{2g}$, $H^2(E_{g,n}) = \mathbb{Z}^{2g} \oplus \mathbb{Z}$, and there is a basis $x_1, \ldots, x_{2g}$ of $H^1(E_{g,n})$ such that
(1) $x_i \cup x_{g+1} = x_j \cup x_{g+j}$ is a generator of $\mathbb{Z}_n \subset H^2(E_{g,n})$, $1 \leq i, j \leq g$.
(2) $x_i \cup x_j = 0$ if $|i - j| \neq g$.

**Proof.** In the spectral sequence, $d_2(1) = n$, so $E_3^{0,1} = 0$, $E_3^{2,0} = \mathbb{Z}_n$, $E_3^{1,0} = E_3^{1,1} = \mathbb{Z}^{2g}$. By the edge homomorphism, $\pi^*: H^1(T_g) \to H^1(E_{g,n})$ is an isomorphism, the group being $\mathbb{Z}^{2g}$, and $\pi^*: H^2(T_g) \to H^2(E_{g,n})$ has image $\mathbb{Z}_n$. Also, there is an exact sequence

$$0 \to \mathbb{Z}_n \to H^2(E_{g,n}) \to \mathbb{Z}^{2g} \to 0$$

so $H^2(E_{g,n}) = \mathbb{Z}_n \oplus \mathbb{Z}^{2g}$. Finally, $H^1(T_g)$ has a basis $y_1, \ldots, y_{2g}$ such that, for a generator $z \in H^2(T_g)$,

$$y_i \cup y_{g+i} = z, \quad 1 \leq i \leq g$$
$$y_i \cup y_j = 0, \quad |i - j| \neq g.$$

Setting $x_i = \pi^*(y_i)$ gives the desired basis of $H^1(E_{g,n})$. □

**Corollary 3.4.** If $n > 1$, $\alpha(E_{g,n}) = g$.

**Proof.** Let $p > 1$ be a prime which divides $n$. The bilinear map $\mathbb{Z}^{2g} \times \mathbb{Z}^{2g} \to \mathbb{Z}_n$ given by

$$H^1(E_{g,n}) \times H^1(E_{g,n}) \to \mathbb{Z}_n \subset H^2(E_{g,n}),$$

when tensored with $\mathbb{Z}_p$, gives a bilinear map $\mathbb{Z}_p^{2g} \times \mathbb{Z}_p^{2g} \to \mathbb{Z}_p$. This is a bilinear form on the $\mathbb{Z}_p$-vector space $\mathbb{Z}_p^{2g}$ whose matrix, relative to the basis $x_1, \ldots, x_{2g}$, is

$$J = \begin{bmatrix} 0 & I_g \\ -I_g & 0 \end{bmatrix}.$$

If $z_1, \ldots, z_{2g}$ is another basis of $H^1(E_{g,n})$ and if $z_i \cup z_j = 0, 1 \leq i, j \leq g + 1$, then the matrix of the form becomes

$$P^*J = \begin{bmatrix} 0 & \ast \\ \ast & \ast \end{bmatrix}$$

where $P$ is the nonsingular matrix corresponding to the change of basis. But $J$ is nonsingular while the above matrix is clearly singular, so we have reached a contradiction. □

The manifolds $E_{g,0} = S^1 \times T_g$ must be handled a bit differently. We need the following lemma which is easily checked via the Künneth theorem.

**Lemma 3.5.** $H^1(E_{g,0}) = \mathbb{Z}^{2g+1} = H^2(E_{g,0})$ and there is a basis $x_1, \ldots, x_{2g}, y$ of $H^1(E_{g,0})$ such that

(1) $x_i \cup x_{g+i} = x_j \cup x_{g+j} = z, 1 \leq i, j \leq g$,
(2) $x_1 \cup y, x_2 \cup y, \ldots, x_{2g} \cup y, z$ form a basis of $H^2(E_{g,0})$,
(3) $x_i \cup x_j = 0, |i - j| \neq g$. 

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In particular, of course, \( \alpha(E_{g,0}) > g \). We must prove the reverse inequality for all \( g > 1 \).

Let \( V \subset H^1(E_{g,0}; Q) \) be the vector subspace spanned by \( \{ x_1, \ldots, x_{2g} \} \). Define the bilinear form \( \varphi: V \times V \to Q \) by

\[
\varphi(v, w) = v \cup w.
\]

Then, (3.5) implies that the matrix of \( \varphi \) relative to the basis \( \{ x_1, \ldots, x_{2g} \} \) is

\[
\Phi = \begin{bmatrix} 0 & I_g \\ -I_g & 0 \end{bmatrix}.
\]

This matrix is nonsingular. As in the proof of (3.4), the following will be enough to prove \( \alpha(E_{g,0}) < g \).

**Lemma 3.6.** If \( \alpha(E_{g,0}) > g \), there is a basis \( \beta_1, \ldots, \beta_{2g} \) of \( V \) such that \( \varphi(\beta_i, \beta_j) = 0 \), \( 1 \leq i, j \leq g + 1 \).

**Proof.** There will be linearly independent elements \( z_1, \ldots, z_{g+1} \) of \( H^1(E_{g,0}; Q) \) with \( z_i \cup z_j = 0 \), \( 1 \leq i, j \leq g + 1 \). Express these elements in terms of the basis given in (3.5) as follows:

\[
z_i = \sum_{j=1}^{2g} a_j^i x_j + a_{2g+1}^i y.
\]

Then

\[
0 = z_i \cup z_k = \left( \sum_{j=1}^{g} a_j^i a_{g+j}^k - a_{g+j}^i a_j^k \right) z + \sum_{j=1}^{2g} \left( a_j^k a_{2g+1}^i - a_j^i a_{2g+1}^k \right) x_j \cup y
\]

and from (3.5) we can conclude \( 0 = \sum_{j=1}^{g} (a_j^i a_{g+j}^k - a_{g+j}^i a_j^k) \) and so, if we set

\[
\beta_i = z_i - a_{2g+1}^i y = \sum_{j=1}^{2g} a_j^i x_j, \quad 1 \leq i \leq g + 1,
\]

we obtain \( \beta_i \cup \beta_k = 0 \), \( 1 \leq i, k \leq g + 1 \). In order to show that \( \{ \beta_1, \ldots, \beta_{g+1} \} \) is linearly independent, it will be enough to show that \( y \) is linearly independent of \( \{ z_1, \ldots, z_{g+1} \} \). This will complete the proof.

Suppose \( y = \sum_{j=1}^{g+1} b_j z_j \). Then

\[
0 = \left( \sum_{j=1}^{g+1} b_j z_j \right) \cup z_k = y \cup z_k = \sum_{j=1}^{2g} a_j^k y \cup x_j, \quad 1 \leq k \leq g + 1.
\]

By (3.5) we conclude that all \( a_j^k = 0 \) for \( j < 2g \), so \( z_k = a_{2g+1}^k y \), \( 1 \leq k \leq g + 1 \). Since we are assuming \( g > 1 \), this contradicts the linear independence of \( \{ z_1, \ldots, z_{g+1} \} \).

Formula (3.1) is now completely proven. It is natural to ask whether Theorem A gives the best upper bound for \( \gamma(F) \) on \( E_{g,n} \). For \( n \neq 1 \), it is easy to show that this is true.
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Theorem 3.7. If \( n \neq 1 \) and \( (n, g) \neq (0, 0) \), then there is a foliation \( F \) of \( E_{g,n} \) with \( \gamma(F) = \varphi(g) \).

Proof. The surface \( T_g \) supports a set of disjointly imbedded circles \( \sigma_1, \ldots, \sigma_g \) which together do not disconnect \( T_g \). Thus \( \{ \pi^{-1}(\sigma_1), \ldots, \pi^{-1}(\sigma_g) \} \) is an admissible system in \( E_{g,n} \) of rank \( g \). By (2.2), there is a foliation \( F \) with \( \gamma(F) > \varphi(g) \). If \( n \neq 1 \) and \( (n, g) \neq (0, 0) \), (3.1) says that \( \alpha(E_{g,n}) = g \); hence Theorem A says that \( \gamma(F) < \varphi(g) \). □

We do not know whether \( E_{g,1} \) admits a foliation \( F \) with \( \gamma(F) = \varphi(2g) \), but we doubt it.

References