ON ANALYTICALLY INVARIANT SUBSPACES AND SPECTRA

BY

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ABSTRACT. Let $T$ be a bounded linear operator from a complex Banach space $X$ into itself. Let $\mathcal{B}_T$ and $\mathcal{R}_T$ denote the weak closure of the polynomials and the rational functions (with poles outside the spectrum $\sigma(T)$ of $T$) in $T$, respectively. The lattice $\mathcal{L}(\mathcal{B}_T)$ of (closed) invariant subspaces of $\mathcal{B}_T$ is a very particular subset of the invariant subspace lattice $\mathcal{L}(T)$. It is shown that: (1) If the resolvent set of $T$ has finitely many components, then $\mathcal{L}(\mathcal{B}_T)$ is a clopen (i.e., closed and open) sublattice of $\mathcal{L}(T)$, with respect to the "gap topology" between subspaces. (2) If $M_1, M_2 \in \mathcal{L}(T)$ and $M_1 \cap M_2 \in \mathcal{L}(\mathcal{B}_T)$, then $M_1$ and $M_2$ also belong to $\mathcal{L}(\mathcal{B}_T)$. (3) If $M \in \mathcal{L}(T)$, $R$ is the restriction of $T$ to $M$, and $\bar{T}$ is the operator induced by $T$ on the quotient space $X/M$, then $\sigma(\bar{T}) \subseteq \sigma(R) \cup \sigma(T)$. Moreover, $\sigma(T) = \sigma(R) \cup \sigma(\bar{T})$ if and only if $M \in \mathcal{L}(\mathcal{B}_T)$. The results also include an analysis of the semi-Fredholm index of $R$ and $\bar{T}$ at a point $\lambda \in \sigma(R) \cup \sigma(\bar{T}) \setminus \sigma(T)$ and extensions of the results to algebras between $\mathcal{B}_T$ and $\mathcal{B}_T^*$. 

1. Properties of the lattice $\mathcal{L}(\mathcal{B}_T)$. The study of the lattice $\mathcal{L}(\mathcal{B}_T)$ (the analytically invariant subspaces of $T$) began in [5]. This article is based on and complements the results contained there. In what follows, $X$ will denote a Banach space over the complex field $C$; operator and subspace will mean bounded linear map from a Banach space into itself and closed linear manifold, respectively. We shall consider invariant subspace lattices under the topology induced by the "gap between subspaces", i.e., the metric in the family of all subspaces of $X$ defined by $d(M, N) = \text{Hausdorff distance between the closed unit ball of the subspace } M \text{ and the closed unit ball of the subspace } N$. The Banach algebra of all operators in $X$ will be denoted by $\mathcal{L}(X)$. Let $\Sigma$ be a subset of $\mathcal{L}(X)$. It is well known (see [1], [5]; [6, Chapter IV]) that $(\mathcal{L}(\Sigma), d)$ is a complete metric space; therefore, $\mathcal{L}(\mathcal{B}_T)$ is always a closed subset of $\mathcal{L}(T)$. We shall show that, under suitable restrictions on the spectrum of $T$, $\mathcal{L}(\mathcal{B}_T)$ is also open in $\mathcal{L}(T)$. 

THEOREM 1. Let $T \in \mathcal{L}(X)$. For each $\lambda$ in the resolvent set $\rho(T) = C \setminus \sigma(T)$ of $T$, there exists a constant $r(T, \lambda) > 0$ such that $d(M, N) > r(T, \lambda)$ for all $M, N \in \mathcal{L}(\mathcal{B}_T)$.
\[ \mathcal{M} \in \text{Lat} T \cap \text{Lat}(T - \lambda)^{-1} \] and all \( \mathcal{M} \in \text{Lat} T \setminus \text{Lat}(T - \lambda)^{-1} \). \( \text{In particular,} \) \( \text{Lat} T \cap \text{Lat}(T - \lambda)^{-1} \) \( \text{is a clopen sublattice of} \) \( \text{Lat} T \).

**Proof.** For each pair of subspaces \( \mathcal{X}_1, \mathcal{X}_2 \) define

\[
\delta(\mathcal{X}_1, \mathcal{X}_2) = \begin{cases}
0 & \text{if } \mathcal{X}_1 = \{0\}, \\
\sup\{\text{distance}(x, \mathcal{X}_2) : x, \in \mathcal{X}_1, \|x\| = 1\} & \text{if } \mathcal{X}_1 \neq \{0\},
\end{cases}
\]

and

\[ \delta(\mathcal{X}_1, \mathcal{X}_2) = \max\{\delta(\mathcal{X}_1, \mathcal{X}_2), \delta(\mathcal{X}_2, \mathcal{X}_1)\}. \]

Then (see [6, p. 198])

\[ \delta(\mathcal{X}_1, \mathcal{X}_2) < \hat{d}(\mathcal{X}_1, \mathcal{X}_2) < 2\delta(\mathcal{X}_1, \mathcal{X}_2); \]

if \( \mathcal{X}_1 \supset \mathcal{X}_2 \) and \( \mathcal{X}_1 \neq \mathcal{X}_2 \), then it follows from Riesz' lemma that \( \delta(\mathcal{X}_1, \mathcal{X}_2) = \hat{d}(\mathcal{X}_1, \mathcal{X}_2) = 1 \).

Without loss of generality, we can assume that \( \lambda = 0 \), i.e., that \( T \) is invertible. Let \( \mathcal{M} \in \text{Lat} T \cap \text{Lat} T^{-1} \) and \( \mathcal{N} \in \text{Lat} T \setminus \text{Lat} T^{-1} \); then (see [5]) \( \mathcal{M} = T^*\mathcal{M} \), while \( T^*\mathcal{M} \subset \mathcal{M} \) but \( \mathcal{M} \neq T^*\mathcal{M} \). We have

\[ 1 = \hat{d}(\mathcal{M}, T^*\mathcal{M}) < \hat{d}(\mathcal{M}, \mathcal{N}) + \hat{d}(\mathcal{N}, T^*\mathcal{M}) \]

\[ < \hat{d}(\mathcal{M}, \mathcal{N})(1 + 2\|T\|\|T^{-1}\|), \]

where the first equality follows from Riesz' lemma, the first inequality is just the triangular inequality for the metric \( \hat{d} \), and the second one follows from the relations between \( \delta \) and \( \hat{d} \) and Lemma 4.2 of [5], which implies that

\[ \delta(\mathcal{M}, T^*\mathcal{M}) = \delta(T^*\mathcal{M}, T^*\mathcal{N}) < \|T\| \|T^{-1}\| \delta(\mathcal{M}, \mathcal{N}). \]

Therefore,

\[ \hat{d}(\mathcal{M}, \mathcal{N}) > r(T, 0) = (1 + 2\|T\|\|T^{-1}\|)^{-1}. \]

The general case and the second statement follow immediately from this result. \( \square \)

**Corollary 2.** Let \( \sigma(T; \mathcal{C}_T) \) denote the spectrum of \( T \) in the Banach algebra \( \mathcal{C}_T \). If \( \sigma(T; \mathcal{C}_T) \setminus \sigma(T) \) has finitely many components, then \( \text{Lat} \mathcal{C}_T \) \( \text{is a clopen sublattice of} \) \( \text{Lat} T \). \( \text{In the general case,} \) \( \text{Lat} \mathcal{C}_T \) \( \text{is a countable intersection of clopen sublattices of} \) \( \text{Lat} T \).

**Proof.** It is enough to recall that \( \mathcal{C}_T \) is generated by \( T \) and \( \{(T - \lambda_n)^{-1}\} \), where the (possibly empty) countable set \( \{\lambda_n\} \) has exactly one point in common with each bounded component of \( \mathbb{C} \setminus \sigma(T; \mathcal{C}_T) \) [5]. Now the result follows immediately from Theorem 1. \( \square \)
**Remark.** Since $\sigma(T) \subset \sigma(T; \mathcal{A}_T)$ and $\sigma(T; \mathcal{A}_T) \setminus \sigma(T)$ is the union of a (possibly empty) subfamily of bounded components of $\rho(T)$, it easily follows that $\text{Lat } \mathcal{A}_T^*$ is clopen in $\text{Lat } T$ whenever $\rho(T)$ has finitely many components (see [5] for details).

**Example A.** Let $T$ be the bilateral shift "multiplication by $e^{ix}$" acting on $L^2(\partial D, dm)$, where $D$ denotes the unit disc of the complex plane, $\partial D$ and $D^-$ are the boundary and the closure of $D$, respectively, and $dm = dx/2\pi$ is the normalized Lebesgue measure on $\partial D$. Then (see [2], [3])

$$\text{Lat } \mathcal{A}_T^* = \{ L^2(M, dm) : M \text{ is a measurable subset of } \partial D \}$$

$L^2(M_1, dm) = L^2(M_2, dm)$ if and only if $m(M_1 \triangle M_2) = 0$, and

$$\text{Lat } T = \text{Lat } \mathcal{A}_T^* \cup (\text{Lat } T)' ,$$

where

$$(\text{Lat } T)' = \{ \{0\}, L^2(\partial D, dm) \}_{\text{uH}^2 : u \in L^\infty(\partial D, dm), |u(e^{ix})| = 1 \text{ (a.e., } dm) }$$

$(H^2$ is a subspace of $L^2$ spanned by the orthonormal set $\{e^{inx}\}_{n=0}^\infty$; $uH^2 = vH^2$ if and only if $u\overline{v}$ is constant a.e.). We have:

(i) By Theorem 1, $\text{Lat } \mathcal{A}_T^*$ and $(\text{Lat } T)'$ are clopen subsets of $\text{Lat } T$; $\text{Lat } \mathcal{A}_T^*$ is actually a boolean algebra and $\text{d}(L^2(M_1), L^2(M_2)) = 1$ whenever $L^2(M_1) \neq L^2(M_2)$. The spectrum of $T$ is equal to $\partial D$ and the constant $r(T, 0)$ can be chosen as being equal to 1.

(ii) $(\text{Lat } T)'$ is another sublattice of $\text{Lat } T$. The topological properties of $(\text{Lat } T)'$ are very far from those of $\text{Lat } \mathcal{A}_T^*$. Indeed, $(uH^2 : u \in L^\infty(\partial D, dm), |u(e^{ix})| = 1 \text{ (a.e., } dm))$ is an arcwise connected subset of $\text{Lat } T$.

(iii) The operator theoretical properties of these two lattices are also very different. In fact, $\text{Lat } \mathcal{A}_T^*$ is a reflexive lattice in the sense of H. Radjavi and P. Rosenthal [8]. On the contrary, if $A \in \mathcal{L}(L^2)$ leaves invariant every subspace in $(\text{Lat } T)'$, then (see [4, §3]) $A \in \mathcal{A}_T$ and, therefore, $\text{Lat } A \subset \text{Lat } T \neq (\text{Lat } T)'$; i.e., $(\text{Lat } T)'$ is not reflexive.

(iv) Let $\chi$ be the characteristic function of the upper half part of $\partial D$ and let $\mathcal{M}_1 = H^2$, $\mathcal{M}_2 = (1 - 2\chi)H^2$. Then [2], [3] $\mathcal{M}_1, \mathcal{M}_2 \in \text{Lat } T \setminus \text{Lat } \mathcal{A}_T$, $\mathcal{M}_1 \cap \mathcal{M}_2 = \{0\}$ and closure $(\mathcal{M}_1 + \mathcal{M}_2) = L^2$ (this property implies that the hypothesis "$\mathcal{M}_1 + \mathcal{M}_2$ is closed" of Theorem 4 below cannot be relaxed).

2. **Analytically invariant subspaces and the spectrum of $T$.** Our next two theorems are consequences of the results contained in [5, §§2 and 6]. Recall that $T \in \mathcal{L}(\mathcal{X})$ is a semi-Fredholm operator if it has closed range and either $\dim \ker T$ or $\text{codim ran } T = \dim \mathcal{X} / T \mathcal{X}$ is finite; in that case, the index of $T$ is defined by $\text{ind } T = \dim \ker T - \text{codim ran } T$. The reader is referred to [6, Chapter IV] for the properties of the semi-Fredholm operators.
Theorem 3. Let $T \in \mathcal{L}(\mathcal{H})$ and let $\mathcal{M} \in \text{Lat} T$. If $\pi: \mathcal{H} \to \mathcal{H}/\mathcal{M}$ is the canonical projection of $\mathcal{H}$ onto the quotient space, $\overline{T} \in \mathcal{L}(\mathcal{H}/\mathcal{M})$ is defined by $\overline{T}(\pi x) = \pi T x$ and $R = T|\mathcal{M}$ is the restriction of $T$ to $\mathcal{M}$, then

$$\sigma(T) \cup \sigma(R) = \sigma(T) \cup \sigma(\overline{T}) = \sigma(R) \cup \sigma(\overline{T}).$$

If $\lambda \in \sigma(R) \setminus \sigma(T)$ or $\lambda \in \sigma(\overline{T}) \setminus \sigma(T)$, then $\lambda \in \sigma(R) \cap \sigma(\overline{T})$, $R - \lambda$ and $\overline{T} - \lambda$ are semi-Fredholm operators and $\text{ind}(\overline{T} - \lambda) = -\text{ind}(R - \lambda) > 0$. In particular, $\sigma(T) = \sigma(R) \cup \sigma(\overline{T})$ if and only if $\mathcal{M} \in \text{Lat} \mathcal{H}^2$.

Proof. Assume that $0 \notin \sigma(R) \cup \sigma(\overline{T})$. Since $\overline{T}$ is invertible, given $x \in \mathcal{H}$ there exists $y \in \mathcal{H}$ such that $\pi x = \pi T y = \pi T y$; hence, $x = T y - x \in \mathcal{M}$. Since $R$ is invertible, $x = R w$ for some $w \in \mathcal{M}$. It follows that $x = T y - R w = T(y - w)$; therefore, $T$ maps $\mathcal{H}$ onto $\mathcal{H}$.

On the other hand, if $T x = 0$, then $\pi T x = \pi T y = 0$, and the invertibility of $\overline{T}$ implies that $\pi x = 0$, i.e., $x \in \mathcal{M}$. Finally, since $R$ is also invertible, $R x = T x = 0$ implies that $x = 0$. We conclude that $T$ is invertible. Replacing $T$ by $T - \lambda$ for each $\lambda \in \sigma(R) \cup \sigma(\overline{T})$, it follows that $\sigma(T) \subset \sigma(R) \cup \sigma(\overline{T})$.

A fortiori $\sigma(T) \cup \sigma(R)$ and $\sigma(T) \cup \sigma(\overline{T})$ are also contained in $\sigma(R) \cup \sigma(\overline{T})$.

Assume that $\lambda \in \sigma(R) \cup \sigma(\overline{T}) \setminus \sigma(T)$; then either $\lambda \in \sigma(R) \setminus \sigma(T)$ or $\lambda \in \sigma(\overline{T}) \setminus \sigma(T)$. In both cases the conclusion is the same: $\mathcal{M} \notin \text{Lat} \mathcal{H}^2$ and $\lambda \in \sigma(R) \cap \sigma(\overline{T})$ (see [5, Lemmas 2.2 and 6.3]). Therefore $\sigma(T) \cup \sigma(R) \supset \sigma(R) \cup \sigma(\overline{T})$ and $\sigma(T) \cup \sigma(\overline{T}) \supset \sigma(R) \cup \sigma(\overline{T})$, whence we obtain the equalities of the first statement; moreover, the same arguments show that $\sigma(T) = \sigma(R) \cup \sigma(\overline{T})$ if and only if $\mathcal{M} \in \text{Lat} \mathcal{H}^2$.

Since $\lambda \notin \sigma(T)$, $(T - \lambda)\mathcal{M}$ is closed; in fact, it is a proper subspace of $\mathcal{M}$, and therefore $(R - \lambda)$ is a semi-Fredholm operator of negative index because $\ker(R - \lambda) \subset \ker(T - \lambda) = \{0\}$. On the other hand, $(T - \lambda)$ maps $\mathcal{H}$ onto $\mathcal{H}$ and, therefore, $(\overline{T} - \lambda)$ maps $\mathcal{H}/\mathcal{M}$ onto $\mathcal{H}/\mathcal{M}$, i.e., $(\overline{T} - \lambda)$ is a semi-Fredholm operator of positive index. Finally, observe that $\ker(\overline{T} - \lambda) = (T - \lambda)^{-1}\mathcal{M}/\mathcal{M}$ is isomorphic to $\mathcal{M}/(T - \lambda)\mathcal{M} = \mathcal{M}/(R - \lambda)\mathcal{M}$ and, therefore, $\text{ind}(\overline{T} - \lambda) = -\text{ind}(R - \lambda) > 0$. □

Theorem 4. Let $T \in \mathcal{L}(\mathcal{H})$ and let $\mathcal{M}_1, \mathcal{M}_2 \in \text{Lat} T$. Assume that $\mathcal{M}_3 = \mathcal{M}_1 + \mathcal{M}_2$ is closed in $\mathcal{H}$ and let $\mathcal{M}_0 = \mathcal{M}_1 \cap \mathcal{M}_2$. Then

$$\sigma(T_0) \cup \sigma(T_3) = \sigma(T_0) \cup \sigma(T_1) \cup \sigma(T_2),$$

where $T_j = T|\mathcal{M}_j$, $j = 0, 1, 2, 3$. Moreover, if $\lambda \in \sigma(T_0) \cup \sigma(T_3) \setminus \sigma(T_1) \cup \sigma(T_2)$, then $\lambda \in \sigma(T_0) \cap \sigma(T_3)$, $T_0 - \lambda$ and $T_3 - \lambda$ are semi-Fredholm operators and $\text{ind}(T_3 - \lambda) = -\text{ind}(T_0 - \lambda) > 0$.

Proof. Let $R = T|\mathcal{M}_3$, let $\overline{T}$ be the operator induced by $T$ on $\mathcal{H}/\mathcal{M}_0$ (as in Theorem 3 above) and let $R = \overline{T}|\mathcal{M}_3/\mathcal{M}_0$. Then the fact that $\mathcal{M}_3$ is
closed implies that $M_2/M_0 = M_1/M_0 \oplus M_2/M_0$ (algebraic direct sum).

According to [5, Theorems 6.1 and 6.2], $M_1, M_2 \in \text{Lat } R$ and $M_1/M_0, M_2/M_0 \in \text{Lat } R$; moreover, it is clear that $R$ commutes with the projection of $M_3/M_0$ onto $M_1/M_0$ along $M_2/M_0$ and, therefore, $M_1/M_0, M_2/M_0 \in \text{Lat } \mathcal{E}_r$ (where $\mathcal{E}_r = \{ S \in \mathcal{L}(X) : SV = VS \text{ for all } V \in \mathcal{L}(X) \}$ commuting with $L$) denotes the double commutant of $L \in \mathcal{L}(X)$; see [5, Lemma 2.3]). Hence

$$\sigma(R) = \sigma(R|M_1/M_0) \cup \sigma(R|M_2/M_0).$$

By applying Theorem 3 to $T_1, T_2$ and $T_3$, we obtain

$$\sigma(T_0) \cup \sigma(T_3) = \sigma(T_0) \cup \sigma(T_3),$$

$$\sigma(T) = \bigcup_{j=0}^2 \sigma(T_j) = \bigcup_{j=0}^2 \sigma(T_j).$$

Let $\lambda \in [\sigma(T_0) \cup \sigma(T_3)] \setminus [\sigma(T_1) \cup \sigma(T_2)]$. Since $\bigcup_{j=0}^2 \sigma(T_j) = \bigcup_{j=0}^3 \sigma(T_j)$, it follows that $\lambda \in \sigma(T_0)$. On the other hand, $(T - \lambda)M_0$ is closed and $\ker(T_0 - \lambda) = \{0\}$, because $T_1 - \lambda$ is invertible and $M_0$ is a subspace of $M_1$; therefore $T_0 - \lambda$ is a semi-Fredholm operator of negative index.

Consider the map $W : M_1 \oplus M_2 \to M_3$ defined by $W(x_1, x_2) = x_1 - x_2$. Clearly, $\lambda \in \sigma(T_1 \oplus T_2) = \sigma(T_1) \cup \sigma(T_2)$, $\ker W = \{(x_0, x_0) : x_0 \in M_0\}$ is an “isometrically isomorphic copy” of $M_0$ and $\text{ind}(T_1 \oplus T_2|\ker W - \lambda) = \text{ind}(T_0 - \lambda)$. By using the canonical isomorphism between $M_3 = \text{ran } W$ and $M_1 \oplus M_2/\ker W$ and applying Theorem 3, we conclude that $\lambda \in \sigma(T_3)$ and, moreover, that $T_3 - \lambda$ is a semi-Fredholm operator with index $\text{ind}(T_3 - \lambda) = -\text{ind}(T_0 - \lambda) > 0$. This proves, in particular, that $\bigcup_{j=0}^2 \sigma(T_j) = \bigcup_{j=0}^3 \sigma(T_j)$.

**Corollary 5.** Let $M_1$ and $M_2$ be two invariant subspaces of $T$ satisfying the hypotheses of Theorem 4 and assume, moreover, that $M_0, M_3 \in \text{Lat } \mathcal{E}_r$. Then $M_1, M_2 \in \text{Lat } \mathcal{E}_r$.

**Proof.** Our hypotheses on $M_0, M_3$, Theorem 4 and Lemma 2.2 of [5] imply that $\sigma(T_1) \cup \sigma(T_2) \subset \sigma(T_0) \cup \sigma(T_3) \subset \sigma(T)$. Hence $\sigma(T_j) \subset \sigma(T), j = 1, 2, 3$ and (according to [5, Lemma 2.2]) this is equivalent to $M_1, M_2 \in \text{Lat } \mathcal{E}_r$. □

The arguments of the proof of Theorem 4 can be applied to other situations; namely, we have

**Theorem 6.** Let $\{M_\nu, \nu \in \Phi\}$ be an arbitrary family of invariant subspaces of $T \in \mathcal{L}(X)$ and assume that $X = \Sigma_{\nu \in \Phi} M_\nu$, the algebraic sum of the $M_\nu$'s. Then every $\lambda \in \sigma(T) \setminus \bigcup_{\nu} \sigma(T|M_\nu)$ is an interior point of $\sigma(T)$ such that
$T - \lambda$ is a semi-Fredholm operator of positive index.

**Proof.** Let $\lambda \in \sigma(T) \setminus \bigcup \sigma(T|\mathcal{M}_\nu)$. Then $(T - \lambda)\mathcal{M}_\nu = \mathcal{M}_\nu$ for all $\nu \in \Phi$ and, therefore,

$$(T - \lambda)\mathcal{K} = (T - \lambda) \sum \mathcal{M}_\nu = \sum (T - \lambda)\mathcal{M}_\nu = \sum \mathcal{M}_\nu = \mathcal{K},$$

i.e. $T - \lambda$ maps $\mathcal{K}$ onto $\mathcal{K}$. Hence, $T - \lambda$ is a semi-Fredholm operator of positive index and, therefore (see [6, Chapter IV]), $\lambda$ is an interior point of $\sigma(T)$. □

**Example B.** Let $T$ be as in Example A, let $S = T|H^2$ (the unilateral shift) and set $L = T^* \oplus S^*$ (where $L^*$ denotes the adjoint of the operator $L$) acting in the usual fashion on the orthogonal direct sum $\mathcal{K} = L^2 \oplus H^2$. Then we can decompose

$$\mathcal{K} = \left[ (H^2)^\perp \oplus H^2 \right] \oplus H^2 = \left( (H^2)^\perp \oplus \{(f, f) : f \in H^2\} \right) \oplus \{(f, -f) : f \in H^2\} = \mathcal{M}_1 + \mathcal{M}_2,$$

where $\mathcal{M}_1 = (H^2)^\perp \oplus \{(f, f) : f \in H^2\}$ and $\mathcal{M}_2 = (H^2)^\perp \oplus \{(f, -f) : f \in H^2\}$. Straightforward computations show that $\mathcal{M}_1$ and $\mathcal{M}_2$ are invariant under $L$; $L|\mathcal{M}_1$ and $L|\mathcal{M}_2$ are similar to $T$ and, therefore, $\sigma(T|\mathcal{M}_1) = \sigma(T|\mathcal{M}_2) = \partial D$. However, $\mathcal{M}_0 = \mathcal{M}_1 \cap \mathcal{M}_2 = (H^2)^\perp \oplus \{0\}$ and $L|\mathcal{M}_0$ is unitarily equivalent to $S$; hence $\sigma(L|\mathcal{M}_0) = \sigma(L|\mathcal{M}_3) = \partial D$ (where $\mathcal{M}_3 = \mathcal{K}$).

**Example C.** Now set $B = T \oplus T$ acting on $L^2 \oplus L^2$. Then $\sigma(B) = \partial D$; $\mathcal{M}_1 = L^2 \oplus H^2$ and $\mathcal{M}_2 = H^2 \oplus L^2$ belong to $\mathcal{L} B \setminus \mathcal{L} \mathcal{B}^*$, because $\sigma(B|\mathcal{M}_1) = \sigma(B|\mathcal{M}_2) = \partial D$ is not included in $\partial D$. This example shows that, in general, from $\mathcal{K} = \mathcal{M}_1 + \mathcal{M}_2$, $\mathcal{M}_1 \neq \mathcal{K} \neq \mathcal{M}_2$, we cannot conclude that $\sigma(B) \subseteq \sigma(B|\mathcal{M}_1)$.

3. Algebras between $\mathcal{B}_T$ and $\mathcal{B}_{T}^*$. Let $T \in \mathcal{L}(\mathcal{K})$, let $\Lambda$ be a subset of $C$ containing at most one point of each bounded component of $\rho(T)$, and let $\mathcal{B}_T(\Lambda)$ denote the weakly closed algebra generated by $T$ and $\{(T - \lambda)^{-1} : \lambda \in \Lambda\}$ (for instance, $\mathcal{B}_T = \mathcal{B}_T(\emptyset)$). Then part of the results of [5, §6] and the above theorems can be extended to the algebras $\mathcal{B}_T(\Lambda)$ by using the same kind of arguments. Thus, we shall establish without proof the following:

**Theorem 7.** (i) If $\mathcal{K} \in \mathcal{L} T$, $R = T|\mathcal{K}$ and $\overline{T}$ is the operator induced by $T$ on $\mathcal{K}/\mathcal{K}$, then the following are equivalent: (a) $\mathcal{K} \in \mathcal{L} \mathcal{B}_T(\Lambda)$; (b) $\Lambda \subseteq \rho(R)$; (c) $\Lambda \subseteq \rho(\overline{T})$; (d) $\Lambda \subseteq \rho(R) \cup \rho(\overline{T})$; (e) $\sigma(T; \mathcal{B}_T(\Lambda)) = \sigma(R) \cup \sigma(T)$.

(ii) If $\mathcal{K} \in \mathcal{L} \mathcal{B}_T(\Lambda)$ and $\mathcal{K}$ is a subspace of $\mathcal{M}$, then $\mathcal{K} \in \mathcal{L} \mathcal{B}_R(\Lambda)$ implies that $\mathcal{K} \in \mathcal{L} \mathcal{B}_T(\Lambda)$.
(iii) If $\mathcal{M} \in \text{Lat}_{\mathcal{T}}(\Lambda)$ and $\overline{\mathcal{M}}$ is a subspace of $\mathcal{K}/\mathcal{M}$, then $\overline{\mathcal{M}} \in \text{Lat}_{\mathcal{T}}(\Lambda)$ implies that $\mathcal{M} = \pi^{-1}(\overline{\mathcal{M}}) \in \text{Lat}_{\mathcal{T}}(\Lambda)$.

(iv) If $\Lambda$ is finite, there exists a constant $s(T, \Lambda) > 0$, such that $d(\mathcal{M}, \mathcal{N}) > s(T, \Lambda)$ for all $\mathcal{M} \in \text{Lat}_{\mathcal{T}}(\Lambda)$ and all $\mathcal{N} \in \text{Lat} \setminus \text{Lat}_{\mathcal{T}}(\Lambda)$. In particular, $\text{Lat}_{\mathcal{T}}(\Lambda)$ is a clopen sublattice of $\text{Lat} T$. Moreover, both results remain true under the weaker assumption: $\Lambda$ only intersects finitely many components of $C \setminus \sigma(T; \mathcal{T}_T(\Lambda))$.

(v) If $\mathcal{M}_1, \mathcal{M}_2 \in \text{Lat} T$, $\mathcal{M}_0 = \mathcal{M}_1 \cap \mathcal{M}_2 \in \text{Lat}_{\mathcal{T}}(\Lambda)$ and $\mathcal{M}_3 = \mathcal{M}_1 + \mathcal{M}_2$ is closed and belongs to $\text{Lat}_{\mathcal{T}}(\Lambda)$, then $\mathcal{M}_1, \mathcal{M}_2 \in \text{Lat}_{\mathcal{T}}(\Lambda)$.

Our last result says that each of the lattices $\text{Lat}_{\mathcal{T}}(\Lambda)$ is invariant under "small perturbations of the dimension". In fact, we have

**Theorem 8.** If $\mathcal{M}, \mathcal{N} \in \text{Lat} T$, $\mathcal{M} \subset \mathcal{N}$ and $\dim \mathcal{N}/\mathcal{M} < \infty$, then $\mathcal{N} \in \text{Lat}_{\mathcal{T}}(\Lambda)$ if and only if $\mathcal{M} \in \text{Lat}_{\mathcal{T}}(\Lambda)$. Moreover, $\sigma(T|\mathcal{N}) = \sigma(T|\mathcal{M}) \cup \{\lambda_1, \ldots, \lambda_n\}$, where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of the finite dimensional operator induced by $T|\mathcal{N}$ on $\mathcal{K}/\mathcal{M}$ and $\sigma(T|\mathcal{N}) = \sigma(T|\mathcal{M}) \cup \{\lambda_1, \ldots, \lambda_n\}$, where $\overline{T|\mathcal{N}}$ and $\overline{T|\mathcal{M}}$ denote the operators induced by $T$ on $\mathcal{K}/\mathcal{M}$ and $\mathcal{K}/\mathcal{N}$, respectively. In particular, if $\mathcal{M} \in \text{Lat} T$ and $\dim \mathcal{M} < \infty$ or codim $\mathcal{M} < \infty$, then $\mathcal{M} \in \text{Lat}_{\mathcal{T}}(\Lambda)$.

**Proof.** By applying Theorem 3 to $T|\mathcal{N}$, we obtain

$$\sigma(T|\mathcal{N}) \cup \sigma(T|\mathcal{M}) = \sigma(T|\mathcal{M}) \cup \{\lambda_1, \ldots, \lambda_n\} = \sigma(T|\mathcal{M}) \cup \{\lambda_1, \ldots, \lambda_n\}$$

Since $\{\lambda_1, \ldots, \lambda_n\}$, the spectrum of the operator $(T|\mathcal{N})^-$ induced by $T|\mathcal{N}$ on $\mathcal{K}/\mathcal{M}$, is a finite set, we have that $\sigma((T|\mathcal{N})^-) = \sigma((T|\mathcal{M})^-) \subset \sigma(T|\mathcal{M})$ (see [5, §2]), we conclude that $\sigma(T|\mathcal{N}) = \sigma(T|\mathcal{M}) \cup \{\lambda_1, \ldots, \lambda_n\}$.

Similarly, by applying Theorem 3 to $\overline{T|\mathcal{N}}$, we obtain the equality $\sigma(\overline{T|\mathcal{N}}) = \sigma(\overline{T|\mathcal{M}}) \cup \{\lambda_1, \ldots, \lambda_n\}$ (to see this, observe that $(T|\mathcal{N})^-$ coincides with $\overline{T|\mathcal{N}}$). We have proved that the symmetric difference $\sigma(T|\mathcal{N}) \Delta \sigma(T|\mathcal{M})$ is contained in the finite set $\{\lambda_1, \ldots, \lambda_n\}$; thus, since $\sigma(T|\mathcal{N}) \Delta \sigma(T|\mathcal{M})$ cannot contain a component of $\rho(T)$, $\mathcal{M}$ and $\mathcal{N}$ belong to exactly the same lattices $\text{Lat}_{\mathcal{T}}(\Lambda)$ (where $\Lambda$ runs over all possible sets of the above described type).

If $\dim \mathcal{M} < \infty$ (codim $\mathcal{M} < \infty$, resp.), then $\mathcal{N} \in \text{Lat}_{\mathcal{T}}$ because $\dim \mathcal{N}/\{0\} < \infty$ and $\{0\} \in \text{Lat}_{\mathcal{T}}$ (dim $\mathcal{N}/\mathcal{M} < \infty$ and $\mathcal{M} \in \text{Lat}_{\mathcal{T}}$, resp.)

It is interesting to observe that not every finite dimensional (or finite codimensional) invariant subspace of $T$ is bi-invariant, i.e., invariant under the double commutant $\mathcal{T}_T$ of $T$. Indeed, we have the following counterexample, inspired in a paper of A. L. Lambert and T. R. Turner:
Example D. Let $S_a$ and $S_\beta$ be injective unilateral weighted shifts in $l^2$ such that the operator $L = S_a^* \otimes S_\beta \in \mathcal{L}(l^2 \otimes l^2)$ satisfies the relations $\mathcal{L}_L = \mathcal{L}_L^{(a)} \neq \mathcal{L}_L^{(\beta)} = \mathcal{L}_L$ (for a concrete numerical example, see [7]). Assume that $S_a$ ($S_\beta$, resp.) is defined in the usual way with respect to the orthonormal basis $(e_n)_{n=0}^\infty$ of $l^2$ (resp. $(f_n)_{n=0}^\infty$ of $(0) \oplus l^2$, resp.). Then $\ker L = \{\lambda e_0 + \mu f_0 : \lambda, \mu \in \mathbb{C}\} \subseteq \mathcal{L}_L^{(a)} = \mathcal{L}_L^{(\beta)}$ (i.e., ker $L$ is actually a hyperinvariant subspace), and the orthogonal projection $P$ of $l^2 \oplus l^2$ onto $l^2 \oplus (0)$ belongs to $\mathcal{L}_L$. Clearly, every one-dimensional subspace of ker $L$ belongs to $\mathcal{L}_L$. On the other hand, since $P \in \mathcal{L}_L$, $\mathcal{L}_L$ splits with respect to the above direct sum decomposition, i.e., if $\mathcal{M} \in \mathcal{L}_L$, then $\mathcal{M} = P \mathcal{M} \oplus (I - P)\mathcal{M}$ (see [5, §5]). It is not hard to conclude that $L$ has exactly two one-dimensional bi-invariant subspaces: the ones generated by $e_0$ and by $f_0$.

Furthermore, this example answers strongly in the negative a question raised in [5, §6]:

(i) $\ker L \in \mathcal{L}_L^{(a)}$ and $(\lambda(e_0 + f_0) : \lambda \in \mathbb{C}) \in \mathcal{L}_L^{(a,\ker L)}$. However, $(\lambda(e_0 + f_0) : \lambda \in \mathbb{C}) \not\subseteq \mathcal{L}_L^{(a)}$.

(ii) Let $S_a$ and $S_\beta$ be as above and set $Q = S_a \oplus S_\beta \in \mathcal{L}(l^2 \oplus l^2)$. Then the subspace spanned by $(e_n, f_n)_{n=1}^\infty$ is equal to $\mathcal{M} = \text{closure}(\text{ran} Q) \subseteq \mathcal{L}_Q$ and $\mathcal{M} = \mathcal{M} \oplus \{\lambda(e_0 + f_0) : \lambda \in \mathbb{C}\} = \mathcal{M} \subseteq \mathcal{L}_Q$, where $Q = 0$ is the operator induced by $Q$ on $l^2 \oplus l^2$. However, $\mathcal{M} = \mathcal{M} = \mathcal{M} \oplus \{\lambda(e_0 + f_0) : \lambda \in \mathbb{C}\} = \mathcal{L}_Q$.

Let $T \in \mathcal{L}(\mathcal{M})$ and set $\mathcal{M} = \mathcal{M} \subseteq \mathcal{L}_Q$, $\mathcal{M} \subseteq \mathcal{M}$. Whether or not $\mathcal{M} \subseteq \mathcal{L}_Q$ (and, similarly, $\mathcal{M} \subseteq \mathcal{L}_Q$, where $T$ is the operator induced by $T$ on $\mathcal{M} / \mathcal{M}$) is an open problem, even under the stronger assumption $\mathcal{M} \subseteq \mathcal{L}_Q$.

References


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