SOME PROPERTIES OF FAMILIES OF CONVEX CONES(*)

BY

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Abstract. The purpose of this paper is to study properties of finite families of convex cones in n-dimensional Euclidean space $\mathbb{R}^n$, whose members all have the origin as a common apex.

Of special interest are such families of convex cones in $\mathbb{R}^n$ which have the following property: Each member of the family is of dimension $n$, the intersection of any two members is at least $(n - 1)$-dimensional, ..., the intersection of any $n$ members is at least 1-dimensional and the intersection of all the members is the origin.

1. Introduction. The purpose of this paper is to study properties of families of convex cones in $n$-dimensional Euclidean space in $\mathbb{R}^n$, whose members have the origin as a common apex.

For a set $A$ in $\mathbb{R}^n$, $\dim A$ denotes the dimension of the minimal flat containing $A$. For a family $A_T = \{A_i; i \in T\}$ of sets in $\mathbb{R}^n$, $A(S)$ denotes $\bigcap \{A_i; i \in S\}$ and $\overline{A}(S) = A(T \setminus S)$. We use the convention that $A(\emptyset) = A(T) = \mathbb{R}^n$.

Unless stated otherwise a family is a finite family and a cone is a convex cone with apex 0.

Of special interest are nonempty families of cones in $\mathbb{R}^n$ which are nondegenerate in the following sense:

Each member of the family is of dimension $n$, the intersection of any two members of the family is of dimension $n - 1$ at least, ..., the intersection of any $n$ members of the family is of dimension 1 at least and the intersection of all members of the family is the origin. Such families are called nondegenerate families or N.D.F.s.

Equivalently: A family $A_T$ of cones in $\mathbb{R}^n$ is an N.D.F. if $A(T) = \{0\}$ and $\dim A(S) > n - |S| + 1$ for any nonempty $S \subset T$. Three properties of N.D.F.s are given in Theorems 1-3:

**Theorem 1 (Perles).** If $A_T$ is an N.D.F. in $\mathbb{R}^n$ then $A_T$ covers $\mathbb{R}^n$ (i.e., $\bigcup \{A_i; i \in T\} = \mathbb{R}^n$).

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Theorem 2. If \( A_T \) is an N.D.F. in \( \mathbb{R}^n \) then for each \( j \in T \) the intersection of all members of \( A_T \) excluding \( A_j \) is contained in \( -A_j \) (i.e., \( \bar{A}(\{j\}) \subset -A_j \)).

Theorem 3. If \( A_T \) is an N.D.F., \( M \) is a subset of \( T \) and \( \bar{A}(M) = A(T \setminus M) \) contains an \( M \)-dimensional subspace then \( M \) is the empty set.

Theorem 1 was originally proved by M. A. Perles (private communication). Perles' proof is algebraic in nature.

A geometric proof of Theorem 2 is given in §2. It is possible to prove Theorems 1 and 3 using the same methods described in §2.

Let \( A_T \) be a family of cones in \( \mathbb{R}^n \). Subsets \( B \) of \( T \) for which \( \bar{A}(B) \) is a subspace will be called faces of \( A_T \). \( B \) will be called a \( k \)-face of \( A_T \) if \( B \) is a face of \( A_T \) and \( \dim \bar{A}(B) = |B| - k \).

It is natural to ask the following question:

Given a family \( A_T \) of convex cones in \( \mathbb{R}^n \) and given \( \dim A(S) \) for each \( S \subset T \), can we determine the subsets \( B \) of \( T \) which are faces of \( A_T \)?

In general, as easy examples can show, the answer is negative. However, Theorem 1 yields a sufficient condition:

By Theorem 1, if \( B \neq \emptyset \) and \( \{\bar{A}(B) \cap A_i : i \in B\} \) is an N.D.F. in \( \text{span} \bar{A}(B) \), then \( \bigcup \{\bar{A}(B) \cap A_i : i \in B\} = \text{span} \bar{A}(B) \) and therefore \( \bar{A}(B) \) is a subspace. Thus we have

**Theorem 4.** If \( B = \emptyset \) and \( A(T) = \{0\} \) or if the family \( \{\bar{A}(B) \cap A_i : i \in B\} \) is an N.D.F. in \( \text{span} \bar{A}(B) \), then \( B \) is a face of \( A_T \).

A subset \( B \) of \( T \) will be called a nondegenerate subset of \( T \) relative to \( A_T \) or an N.D.S. of \( A_T \) if \( B \neq \emptyset \) and \( A(T) = \{0\} \) or if the family \( \{\bar{A}(B) \cap A_i : i \in B\} \) is an N.D.F. in \( \text{span} \bar{A}(B) \).

Equivalently: a subset \( B \) of \( T \) will be called an N.D.S. of \( A_T \) if \( B = \emptyset \) and \( A(T) = \{0\} \) or if \( \bar{A}(\emptyset) = A(T) = 0 \) and \( |S| - \dim \bar{A}(S) < |B| - \dim \bar{A}(B) \) for any proper subset \( S \) of \( B \).

\( B \) will be called a \( k \)-N.D.S. of \( A_T \) if it is an N.D.S. of \( A_T \) and if \( \dim \bar{A}(B) = |B| - k \).

If \( A_T \) is an N.D.F. then the sufficient condition in Theorem 4 is also a necessary condition:

**Theorem 5.** If \( A_T \) is an N.D.F. then a subset \( B \) of \( T \) is a \([k-]\)face of \( A_T \) iff it is a \([k-]\)N.D.S. of \( A_T \).

The proof of Theorem 5 is established in §3. Theorem 5 yields an algorithm for finding all the faces of an N.D.F. when \( \dim A(S) \) is known for all \( S \subset T \).

**Algorithm.** If \( A_T \) is an N.D.F. then a subset \( B \) of \( T \) is a face of \( A_T \) iff
$|S| - \dim \overline{A}(S) < |B| - \dim \overline{A}(B)$ for any proper subset $S$ of $B$.

In the last section we attempt to 'justify' nondegenerate families of convex cones by discussing some results which may be obtained by using N.D.F.s.

2. Proof of Theorem 2. The essence of the proof is the use of a suitable separating hyperplane and induction on $n$.

The following lemma will be used:

**Lemma 1.** If $A$ and $B$ are polyhedral cones in $\mathbb{R}^n$, $A$ is pointed and $A \cap B = 0$, then there is a hyperplane $H$ which separates $A$ and $B$ and strictly supports $A$ (i.e., $A$ and $B$ are on different sides of $H$ and $H \cap A = \{0\}$).

Let $A_T = \{A_i: i \in T\}$ be an N.D.F. in $\mathbb{R}^n$. It is enough to prove that $\bigcup \{A_i: i \in T\} = \mathbb{R}^n$ under the additional assumption that $A_T$ is a family of polyhedral cones.

The proof is by induction on $n$ and for fixed $n$ by induction on $t = |T| - n$ (since $A_T$ is an N.D.F., $t \geq 1$).

If $\overline{A}(j) = A(T \setminus \{j\}) = 0$, the proof is trivial (this includes the case $n = 0$).

Otherwise, suppose $0 \neq x \in \overline{A}(j)$. We have to show that $-x \in A_j$. There are two cases to consider.

1. For each $i \in T \setminus \{j\}$, $\overline{A}(i) \neq \{0\}$. (This includes the case $t = 1$.)

If $\overline{A}(i)$ contains a line for each $i \in T \setminus \{j\}$ we would have $A_j = \mathbb{R}^n$ (since $A_j$ contains the sum of these lines, the lines are linearly independent ($A(T) = \{0\}$), and there are at least $n$ lines).

This leads to $x \in A_j \cap \overline{A}(j) = A(T) = \{0\}$, a contradiction.

Suppose $i \in T \setminus \{j\}$ and $A(i)$ is a pointed cone. Using Lemma 1, let $H$ be a hyperplane which separates $\overline{A}(i)$ and $A_i$ (both of them are polyhedral by our assumptions).

Define $A_T^{\setminus \{i\}}$, an N.D.F. in $\mathbb{R}^{n-1} = H$ by

$$A'_l = A_l \cap H \quad \text{for } l \in T \setminus \{i\}.$$ 

(It is not difficult to prove that $A_T^{\setminus \{i\}}$ is an N.D.F.)

Let $H^+$ and $H^-$ be the two closed halfspaces determined by $H$, and suppose that $\overline{A}(i) \subset H^+$. Since $x \in \overline{A}(j) \subset A_j \subset H^-$ and $\overline{A}(i) \cap \text{int } H^+ \neq \emptyset$, there is a $y \in \overline{A}(i)$ such that $x + y \in H$. Therefore $x + y \in \overline{A}(i \cup j)$.

By the induction assumption (on $n$), $-x - y \in A'_i \subset A_i$. Consequently, $-x = (-x - y) + y \in A_j (y \in \overline{A}(i) \subset A_j)$.

2. Suppose $i \in T \setminus \{j\}$ and $\overline{A}(i) = \{0\}$. Define $A_T^{\setminus \{i\}} = \{A'_i = A_i: f \in T \setminus \{i\}\}$. Then $A_T^{\setminus \{i\}}$ is an N.D.F. of $n + (t - 1)$ cones in $\mathbb{R}^n$ and $x \in$
\( \bar{A}((i, j)) = \bar{A}'(j) \). By the induction assumption (on \( t = |T| - n \), \( -x \in A'_j = A_j \). 

The proof is now complete.

3. N.D.S.s and faces. The main object of this section is a proof of Theorem 5.

The proof of Theorem 5 relies on Lemmas 2, 3 and 4. Lemma 5 states that distinct 1-N.D.S.s are disjoint; it is presented on its own merit.

In the following, let \( A_T \) be a family of \( |T| = n + t \) cones in \( R^n \) such that \( A(T) = \{0\} \).

**Lemma 2.** If \( \dim \bar{A}(B) < |B| - k \) and \( 0 < k' < k \) then \( B \) contains a \( k' \)-N.D.S. of \( A_T \) and \( B \) contains a \( k' \)-face of \( A_T \).

**Proof of Lemma 2.** By the conditions of Lemma 2, \( \dim \bar{A}(B) < |B| - k' \). Let \( B' \) be a minimal subset of \( B \) which satisfies \( \dim \bar{A}(B) < |B'| - k' \). It is easily verified that \( B' \) is a \( k' \)-N.D.S. of \( A_T \) and, by Theorem 4, \( B' \) is a \( k' \)-face of \( A_T \).

**Lemma 3.** A. If \( B \) is an N.D.S. of \( A_T \), \( M \subset B \) and \( \bar{A}(M) \) contains an \( |M| \)-dimensional subspace, then \( M = \emptyset \).

B. If \( A_T \) is an N.D.F., \( B \subset M \subset T \), \( B \) is a \( k \)-face of \( A_T \) and \( \bar{A}(M) \) contains an \( (|M| - k) \)-dimensional subspace, then \( M = B \).

C. If \( A_T \) is an N.D.F., \( B_1 \subset B_2 \subset T \) and \( B_i \) is a \( k_i \)-face of \( A_T \), then \( k_1 \leq k_2 \).

**Proof of Lemma 3.** Part A follows from Theorem 3 applied to the family \( \{ \bar{A}(B) \cap A_i : i \in B \} \) in \( \text{span} \bar{A}(B) \). Proof of B: The case \( B = T \) is trivial, assume therefore that \( B \) is a proper subset of \( T \). Let \( Y \) be a subspace complementary to \( \bar{A}(B) \) relative to \( R^n \).

Define \( C_{T \setminus B} = \{ C_i : A_i \cap Y : i \in T \setminus B \} \), a family of cones in \( Y \).

For each \( S \subset T \setminus B \), \( C(S) = A(S) \cap Y \), \( \bar{C}(S) = \bar{A}(S \cup B) \cap Y \) and \( \dim C(S) = \dim A(S) - \dim \bar{A}(B) \).

It follows that \( C_{T \setminus B} \) is an N.D.F. in \( Y \). Since \( \bar{C}(M \setminus B) = \bar{A}(M) \cap Y \) and \( \bar{A}(M) \) contains an \( (|M| - k) \)-dimensional subspace it follows that \( \bar{C}(M \setminus B) \) contains an \( (|M| - k) - \dim \bar{A}(B) = |M| - k - (|B| - k) = |M - B| \)-dimensional subspace. By Theorem 3 applied to \( C_{T \setminus B} \), \( M \setminus B = \emptyset \), proving part B.

Part C is an immediate result of B.

**Lemma 4.** If \( A_T \) is an N.D.F. and \( B \) is a \( k \)-face of \( A_T \), then \( 0 < k < t \), \( k = 0 \) iff \( B = \emptyset \), \( k = t \) iff \( B = T \).

Lemma 4 is easily derived from the definitions of a \( k \)-face, the definition of an N.D.F. and Theorem 3.
Lemma 5. If $B_1$ and $B_2$ are distinct 1-N.D.S.s of $A_T$ then $B_1$ and $B_2$ are disjoint.

Proof. By Theorem 4, $B_1$ and $B_2$ are 1-faces of $A_T$. $B_1 \cap B_2$ is a face of $A_T$ since $\overline{A}(B_1 \cap B_2) = \overline{A}(B_1) \cap \overline{A}(B_2)$ is a subspace.

Since $B_1$ and $B_2$ are distinct, $B_1 \cap B_2$ is either a proper subset of $B_1$ or a proper subset of $B_2$. Suppose that $B_1 \cap B_2$ is a proper subset of $B_1$. Since $B_1$ is a 1-N.D.S.,

$$|B_1 \cap B_2| - \dim \overline{A}(B_1 \cap B_2) < |B_1| - \dim \overline{A}(B_1) = 1.$$  

Therefore $\dim \overline{A}(B_1 \cap B_2) > |B_1 \cap B_2|$ and, by Lemma 3.A, $B_1 \cap B_2 = \emptyset$, completing the proof.

Proof of Theorem 5. Suppose that $A_T$ is an N.D.F. in $R^n$. We have to prove that for any subset $B$ of $T$, $B$ is an N.D.S. of $A_T$ iff $B$ is a face of $A_T$.

If $B$ is a $k$-N.D.S. then $B$ is a $k$-face by Theorem 4.

We assume that $B$ is a $k$-face and not a $k$-N.D.S. and derive a contradiction.

If $B$ is not a $k$-N.D.S. of $A_T$ then there exists a proper subset $B'$ of $B$ such that $|B'| - \dim \overline{A}(B') > |B| - \dim \overline{A}(B) = k$. Therefore $\dim \overline{A}(B') < |B'| - k$ and $k > 0$ by Lemma 4. By Lemma 2, $B'$ contains a $k$-face $C$. Since $C$ is a proper subset of $B$ and $A_T$ is an N.D.F., we have by Lemma 3.C that $k < k$, a contradiction. The proof of Theorem 5 is now complete.

4. Remarks. We will briefly discuss some results which may be obtained using properties of N.D.F.s:

1. Reconstructing dimensions of intersections of convex sets.

We can prove that if $K_T$ and $K'_T$ are two finite families of convex sets in $R^n$ and $\dim \cap \{K_i : i \in S\} = \dim \cap \{K'_i : i \in S\}$ for each $S \subseteq T$ with $|S| < n + 1$, then $\dim \cap \{K_i : i \in T\} = \dim \cap \{K'_i : i \in T\}$ (see [3]).

2. In [4] we use properties of N.D.F.s and Gale diagrams (see [2, Chapter 5, §4], for Gale diagrams) to establish connections between N.D.F.s and convex polytopes:

For each N.D.F., $A_T = \{A_i : i \in T\}$ in $R^n$ there is a $|T| - n - 1$-polytope $P$ such that the lattice of faces of $A_T$ ordered by the inclusion relation is isomorphic to the lattice of faces of $P$ ordered by the inclusion relation.

This result enables one to obtain properties of families of cones by using well-known theorems on convex polytopes. An illustration is [5]:

3. Using N.D.F.s, [4], and properties of neighborly polytopes we can generalize a result of M. J. C. Baker [1] and prove a theorem which is equivalent to the following:

Let $F$ be a finite family of at least $n + 1 + t$ "convex" sets on $S_n$ ($t > 0$). If
Every \( n + 1 \) members of \( F \) have nonempty intersection then there are \( n + 1 + \lceil t/2 \rceil \) members of \( F \) whose intersection is nonempty. (\( S^n \) is the \( n \)-dimensional unit sphere and a set is 'convex' if it is the intersection of a convex cone with apex 0 in \( R^{n+1} \) with \( S^n \).)

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**References**


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