THE $p$-ADIC LOG GAMMA FUNCTION AND $p$-ADIC EULER CONSTANTS

BY

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ABSTRACT. We define $G_p$, a $p$-adic analog of the classical log gamma function and show it satisfies relations similar to the standard formulas for log gamma. We also define $p$-adic Euler constants and use them to obtain results on $G'_p$ and on the logarithmic derivative of Morita’s $\Gamma_p$.

1. Introduction. Leopoldt and Kubota defined $p$-adic $L$-functions by summing a function of two variables with respect to one of the variables. We present a general theorem on this technique and then use it to define $G_p$, a $p$-adic analog of the classical log $\Gamma$ function. We work with log $\Gamma$ rather than $\Gamma$ because the only continuous $p$-adic function defined on a subset of $\Omega_p$ and satisfying $f(x + 1) = xf(x)$ is the zero function. It is possible to construct an analog of $\Gamma$ by modifying the functional equation (see Morita [7]), but then we do not have close analogs of the standard formulas for $\Gamma$ or log $\Gamma$. For $G_p$, which is not the log of Morita’s gamma function, we have the functional equation, an extension theorem, the Stirling series, the Gauss multiplication theorem, a power series, certain “Laurent” series and a formula due to Gauss which is valid for $G'_p$ at rational points.

This last formula was discussed by Lehmer in [6], where he defined Euler constants for arithmetic progressions. We define $p$-adic Euler constants and present a proof of Gauss’ theorem which is valid in both the $p$-adic and complex systems. We also apply the results on Euler constants to obtain a finite expression for the logarithmic derivative of Morita’s $p$-adic gamma function at certain rational values in its domain.

2. Notation and definitions. We will use $Q$, $Q_p$, $Z$, $Z_p$, $C$ and $\Omega_p$ for, respectively, the field of rational numbers, the $p$-adic completion of $Q$, the ring of rational integers, the $p$-adic completion of $Z$ in $Q_p$, the field of complex numbers and the completion of the algebraic closure of $Q_p$. $B_n$ will be the $n$th Bernoulli number defined by $te'/(e' - 1)$. $v$ will be the $p$-adic valuation on

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The set \( \Omega_p \) with \( r(p) = 1 \) and \( |p|_p \) will be the absolute value on \( \Omega_p \) with \( |p|_p = p^{-1} \).

We will use boldface letters to indicate \( r \)-tuples.

A polydisc about \( c \in \Omega_p^r \) is a set of the form
\[
\{(x_1, \ldots, x_r) : |x_i - c_i|_p < \rho_i, i = 1, 2, \ldots, r\}
\]
where \( c = (c_1, \ldots, c_r) \) and all \( \rho_i > 0 \). \( \rho_1, \ldots, \rho_r \) is called the radius of the polydisc. \( a \) and \( M \) will denote \( (a_1, \ldots, a_r) \) and \( (M_1, \ldots, M_r) \), respectively.

We call a function defined on a subset of \( \Omega_p^r \) holomorphic if it can be represented by a single power series and locally holomorphic if at each point in the domain we can represent the function by a power series on some polydisc containing the point.

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3. \( p \)-adic sums. We begin by considering sums of the type used by Leopoldt and Kubota [5]. The first theorem is a generalization of a result in [2, p. 309].

**Theorem 1.** Suppose we have rational integers \( a_i, b_i, M_i \) with \( a_i > 0, b_i > 1, M_i > 1 \) for \( i = 1, 2, \ldots, r \). Let \( R \) be an open set in \( \Omega_p^r \) with \( a + MZ_p \subset R \). \( B \) is a Banach space over \( \Omega_p \) and \( f: R \to B \) is locally holomorphic.

We define
\[
S(k_1, \ldots, k_r, b_1, \ldots, b_r) = \frac{1}{b_1 \cdots b_r p_1^{k_1} \cdots p_r^{k_r}} \sum_{i=1}^{r} M_i b_i p_i^{k_i - 1} \sum_{n_k=0; n_k \equiv a_k \pmod{M_k}}^{b_p k_i - 1} f(n_1, \ldots, n_r).
\]

Then
(i) \( L = \lim\limits_{(n_k) \to \infty} S(k_1, \ldots, b_r) \) exists;
(ii) \( L \) is independent of the \( \{b_i\} \) used;
(iii) \( L \) may be calculated by iteration of the limit in any order.

**Proof.** We begin with \( a_1, \ldots, a_r = 0, M_1, \ldots, M_r = 1 \) and \( f \) holomorphic on \( R \) with \( Z_p^r \subset R \). We can write
\[
f(x) = \sum_{j} a_j x^j,
\]
where the right side represents a power series in \( r \) variables with \( J \) running through the \( r \)-tuples of nonnegative integers.

After we substitute the series for \( f \) in the formula for \( S \) and use the fact (see [2]) that
\[
\left| (-1)^j B_j - \frac{1}{b^k} \sum_{n=0}^{b^k-1} n^j \right|_p < p^{2-k}
\]
for \( k = 0, 1, \ldots \) and \( j = 0, 1, \ldots \), it is easy to verify that
\[ \lim_{k_i \to \infty} \ldots \lim_{k_1 \to \infty} S(k_1, \ldots, b_r) = \sum_j a_j (-1)^j B_j, \]

and each limit is uniform with respect to the remaining variables. We can now conclude that

\[ L = \lim_{(k_i) \to \infty} S(k_1, \ldots, b_r) \text{ exists.} \]

The next two parts of the theorem are obvious.

Now suppose \( f \) is locally holomorphic on \( R \) and \( R \supset Z_p^r \). Then there is a finite covering of \( Z_p^r \) by polydiscs whereby \( f \) is holomorphic on each polydisc, each polydisc has the same radius and the radius has the form \( (p^{-N}, \ldots, p^{-N}) \). We let \( A = \{0, \ldots, p^N - 1\} \) and for each \( W \in A^r \) we define

\[ f_W(x) = p^{-N}f(W + p^N x). \]

Each \( f_W \) is holomorphic on the disc with center \( (0, \ldots, 0) \) and radius \( (1, \ldots, 1) \).

It is convenient now to introduce an integral type notation.

We define

\[ \int_{a, M} f(x) \, dx = L, \]

where \( L \) is defined in Theorem 1.

We have

\[ \int_{0, 1} f(x) \, dx = \sum_{W \in A^r} \int_{0, 1} f_W(x) \, dx, \]

and the conclusion of the theorem follows directly.

Finally, if \( f: R \to B \) is locally holomorphic and \( a + M\mathbb{Z}_p \subset R \), we define \( g(x) = f(a' + xM) \) where \( xM = (x_1 M_1, \ldots, x_r M_r) \) and \( a' \) is the least nonnegative residue of \( a \) mod \( M \). Since \( g \) satisfies the conditions needed earlier in this proof, and \( \int_{a, M} f(x) \, dx = \int_{0, 1} g(x) \, dx \), we have established Theorem 1.

The next result is our basic device for constructing \( p \)-adic functions. We will use it to define a \( p \)-adic analog of log \( \Gamma \) and to define \( p \)-adic Euler constants. It can be used to show the existence and holomorphy of the \( p \)-adic \( L \)-functions and similarly constructed functions occurring in the works cited as references.

\textbf{Theorem 2.} Suppose \( a_i, M_i \) are rational integers with \( a_i > 0, M_i > 1 \) for \( i = 1, 2, \ldots, r \), \( \{C_1, \ldots, C_t\} \) is a set of polydiscs in \( \Omega_p^r \), \( R = \bigcup_{i=1}^t C_i \) and \( a + M\mathbb{Z}_p \subset R \). Let \( D \) be a polydisc in \( \Omega_p^r \) and suppose \( f: R \times D \to \Omega_p \) is holomorphic on each \( C_i \times D, i = 1, 2, \ldots, t \). Then
exists and is holomorphic on \( D \).

**Proof.** We let \( \Lambda(D) \) = Banach space of holomorphic functions from \( D \to \Omega_p \). For \( u \in \mathbb{R} \) we define
\[
\phi(u) = \text{the mapping } x \to f(u, x).
\]
For a fixed \( u_i \in C_i \) we have
\[
f(u, x) = \sum_j a_{i,j}(x)(u - u_i)^j
\]
for all \( u \in C_i \) and \( x \in D \).

If \( a_{i,j} \) denotes the map on \( D \), \( x \to a_{i,j}(x) \), then
\[
\phi(u) = \sum_j a_{i,j}(u - u_i)^j
\]
for \( u \in C_i \).

Each \( a_{i,j} \in \Lambda(D) \), so \( \phi: \mathbb{R} \to \Lambda(D) \) is locally holomorphic and we may apply Theorem 1. Since
\[
\int_{u,M} f(u, x) \, du = \left( \int_{u,M} \phi(u) \, du \right)(x),
\]
we conclude that \( F \in \Lambda(D) \).

The following corollary is a useful form of Theorem 2.

**Corollary.** Suppose \( a, b, M \) are rational integers with \( a > 0, b > 1, M > 1 \). Let \( f \) be locally holomorphic on a set \( A \subset \Omega_p \). Let \( x \in \Omega_p^e \) and \( T(u, x) \) be locally holomorphic on some subset of \( \Omega_p^{n+1} \). Define \( A^* = \{ x | T(a + MZ_p, x) \subset A \} \).

Then \( A^* \) is open, and if \( A^* \neq \emptyset \),
\[
F(x) = \lim_{k \to \infty} \frac{1}{bp^k} \sum_{n=0}^{Mbp^k-1} f(T(n, x))
\]
is independent of \( b \) and locally holomorphic on \( A^* \).

**Proof.** Given \( c \in A^* \) and \( u \in a + MZ_p \) there is a polydisc \( D(u, c) \) containing \( (u, c) \) on which \( f \circ T \) is holomorphic. Holding \( c \) fixed, a finite number of \( D(u, c) \) cover \( (a + MZ_p, c) \). Each \( D(u, c) \) has the form \( C_i \times D_i \) where \( C_i \) is a disc in \( \Omega_p \) containing \( u_i \) and \( D_i \) is a polydisc about \( c \). Let \( D = \cap D_i \). We know the following:
(i) \( T(C_i \times D) \subset A \) for each \( i \), so \( D \subset A^* \);
(ii) \( \cup C_i \) covers \( a + MZ_p \);
(iii) \( f \circ T \) is holomorphic on each \( C_i \times D \).
From Theorem 2 we see $F(x)$ is holomorphic on $D$ and therefore locally holomorphic on $\mathbb{A}^\ast$.

We will occasionally wish to differentiate $F(x)$. We have

**Theorem 3.** Using the definitions and conditions of Theorem 2,

$$
\frac{\partial F(x)}{\partial x_i} = \int_{a,M} \frac{\partial f(u, x)}{\partial x_i} du.
$$

**Proof.** We fix $x \in D$ and for $t \in \Omega_p$ we let $t^* = (0, \ldots, t, \ldots, 0)$, $t$ being in the $i$th position, $t^* \in \Omega_p^\ast$. We define

$$
h(u, t) = \frac{f(u, x + t^*) - f(u, x)}{t} \text{ for } t \neq 0
$$

and $h(u, 0) = \lim_{t \to 0} h(u, t)$.

We observe that $h(u, 0) = \frac{\partial f(u, x)}{\partial x_i}$ and that there is a neighborhood $D_0$, of zero, so $h$ is holomorphic on each $C_i \times D_0$.

From the definition of derivative we have

$$
\frac{\partial F(x)}{\partial x_i} = \lim_{t \to 0} \int_{a,M} h(u, t) du.
$$

Now, if we let $H(t) = \int_{a,M} h(u, t) du$ we can use Theorem 2 to see that $H$ is holomorphic on $D_0$ and, in particular, continuous at 0.

Thus we have

$$
\frac{\partial F(x)}{\partial x_i} = \lim_{t \to 0} H(t) = H(0) = \int_{a,M} \frac{\partial f(u, x)}{\partial x_i} du.
$$

The next result shows how certain sums can be used to solve difference equations.

**Theorem 4.** If $a, M$ are rational integers where $M > a > 0$, $f'(x + a)$ exists and $F(x) = \int_{a,M} f(x + u) du$, then $F(x + M)$ exists and $F(x + M) = F(x) + Mf'(x + a)$.

**Proof.** This follows directly from the definition of the right side.

4. The $p$-adic log $\Gamma$ function. We now consider the problem of constructing a $p$-adic analog of log $\Gamma(x)$.

In looking for a $p$-adic analog of log $\Gamma(x)$ we want a function $G_p$ which sends a subset of $\Omega_p$ into $\Omega_p$ and satisfies the functional equation $G_p(x + 1) = G_p(x) + \log(x)$. $\log(x)$ is defined by the usual power series when $|x - 1|_p < 1$, and by setting $\log(p) = 0$ and using the functional equations for $\log(x)$ when $|x - 1|_p > 1$ and $x \neq 0$. There is a complete discussion of this idea in [3]. Just as in the complex case, this functional equation forces $G_p$ to be discontinuous on either the positive integers or the negative integers. This is somewhat unfortunate in the $p$-adic case because if we want a locally
holomorphic function we must exclude $\mathbb{Z}_p$ from the domain of $G_p$. However, this is all we need exclude because on the domain $\Omega_p - \mathbb{Z}_p$ we have a locally holomorphic function which satisfies $G_p(x + 1) = G_p(x) + \log(x)$ and several other relations similar to those of the complex log $\Gamma(x)$. We will use the construction given in §3 to define $G_p(x)$ and demonstrate its properties.

An alternative approach is to slightly modify the functional equation to obtain a functional locally holomorphic on all of $\Omega_p$. After considering $G_p(x)$ we will exhibit a sequence of such functions. We have the relation that the sequence of functions locally holomorphic on $\Omega_p$ converges pointwise to $G_p$.

The technique of changing the functional equation has been used by Morita [7] to define $\Gamma_p$, a function on $\mathbb{Z}_p$, which is an analog of $\Gamma$. Our $G_p$ is clearly not log $\Gamma_p$.

**Definition of $G_p$.** We use the corollary of Theorem 2 with $T(u, x) = u + x$ and $f(x) = x \log(x) - x. f$ is locally holomorphic on $\Omega_p - \{0\}$. We then have

$$G_p(x) = \lim_{k \to \infty} \frac{1}{p^k} \sum_{n=0}^{p^k-1} (x + n)\log(x + n) - (x + n).$$

$G_p$ is locally holomorphic on $\Omega_p - \mathbb{Z}_p$, and at each $c \in \Omega_p - \mathbb{Z}_p$ the disc of holomorphy is the largest (open) disc $D(c)$ such that $D(c) \cap \mathbb{Z}_p = \emptyset$.

An immediate consequence of Theorem 4 is the functional equation:

**Theorem 5.** $G_p(x + 1) = G_p(x) + \log x$.

Stirling's Theorem, which is an asymptotic formula for log $\Gamma(x)$, is simpler in $\mathbb{Z}_p$. We have

**Theorem 6.** When $|x|_p > 1$,

$$G_p(x) = \left(x - \frac{1}{2}\right) \log(x) - x + \sum_{r=1}^{\infty} \frac{B_{r+1}}{r(r+1)x^r}.$$

**Proof.**

$$G_p(x) = \frac{1}{2} - x + \lim_{k \to \infty} \frac{1}{p^k} \sum_{n=0}^{p^k-1} (x + n)(\log(x) + \log(1 + n/x)).$$

Using the power series for log$(1 + n/x)$ will lead to the result.

If we match Theorem 6 and the next result with the corresponding classical formulas we see that it is more accurate to speak of $G_p(x)$ as the analog of $-\frac{1}{2}\log(2\pi) + \log \Gamma(x)$. However, for simplicity we will continue to refer to $G_p$ as the analog of log $\Gamma$.

The following relation is the $p$-adic version of Gauss' Multiplication Theorem.

**Theorem 7.** Given any $m \in \mathbb{Z}^+$ we have

$$G_p(x) = \left(x - \frac{1}{2}\right) \log(x) - x + \sum_{r=1}^{\infty} \frac{B_{r+1}}{r(r+1)x^r}.$$
\[ G_p(x) = \left(x - \frac{1}{2}\right) \log(m) + \sum_{a=0}^{m-1} G_p\left(\frac{x + a}{m}\right) \]

provided the right side is defined.

**Proof.** We can write

\[ G_p(x) = \lim_{k \to \infty} \frac{1}{mp^k} \sum_{n=0}^{mp^k-1} (x + n) \log(x + n) - (x + n) \]

\[ = \lim_{k \to \infty} \frac{1}{mp^k} \sum_{n=0}^{p^k-1} \sum_{a=1}^{m-1} (x + a + mn) \log(x + a + mn) - (x + a + mn). \]

With a little rearranging, Theorem 7 is easily obtained.

**Corollary.**

\[ G_p(x) = \sum_{a=0}^{p^r-1} G_p\left(\frac{x + a}{p^r}\right) \quad \text{for } r = 0, 1, 2, \ldots. \]

This last corollary provides us with a means for transferring results about \( G_p(x) \) when \( |x|_p > 1 \) to \( G_p(x) \) with \( |x|_p < 1 \).

For the extension theorem we have

**Theorem 8.** \( G_p(x) + G_p(1 - x) = 0. \)

**Proof.** We can see immediately from Theorem 6 that \( G_p(x) + G_p(-x) = -\log(x) \) when \( |x|_p > 1 \). Combining this with \( G_p(x + 1) = G_p(x) + \log(x) \) we have \( G_p(x) + G_p(1 - x) = 0 \) when \( |x|_p > 1 \).

Given any \( x \in \mathbb{Q}_p - \mathbb{Z}_p \) with \( |x|_p < 1 \) we can choose an \( r \in \mathbb{Z}^+ \) so \( |(x + a)/p^r|_p > 1 \) for all \( a \in \mathbb{Z} \). Then

\[ G_p(x) + G_p(1 - x) = \sum_{a=0}^{p^r-1} G_p\left(\frac{x + a}{p^r}\right) + G_p\left(\frac{1 - x + a}{p^r}\right) \]

\[ = \sum_{a=0}^{p^r-1} G_p\left(\frac{x + a}{p^r}\right) - G_p\left(\frac{1 - x + a}{p^r}\right) \]

\[ = \sum_{a=0}^{p^r-1} G_p\left(\frac{x + a}{p^r}\right) - G_p\left(\frac{x + p^r - a - 1}{p^r}\right), \]

and, since as \( a \) goes from 0 to \( p^r - 1 \), \( p^r - a - 1 \) goes from \( p^r - 1 \) to 0, Theorem 8 is proven.

The complex log \( \Gamma(x) \) has a simple power series about 1, with values of the Riemann \( \zeta \)-function appearing in the coefficients. We will now find the power series for \( G_p(x) \) about \( 1/p \).

We use Theorem 4 to obtain
\[ D^{(1)}G_p(x) = \lim_{k \to \infty} \frac{1}{p^k} \sum_{n=0}^{p^k-1} \log(x + n) \]

and, in particular,

\[ D^{(1)}G_p(1/p) = p \lim_{k \to \infty} \frac{1}{p^k} \sum_{m=0}^{p^k-1} \log(m). \]

We write this as

\[ D^{(1)}G_p(1/p) = -p\gamma_p(1,p), \]

\[ D^{(r)}G_p(x) = (-1)^r (r - 2)! \lim_{k \to \infty} \frac{1}{p^k} \sum_{n=0}^{p^k-1} \frac{1}{(x + n)^{r-1}} \quad \text{for } r > 2, \]

\[ \frac{D^{(r)}G_p(1/p)}{r!} = \frac{(-1)^r}{r(r-1)} \lim_{k \to \infty} \frac{1}{p^k} \sum_{n=0}^{p^k-1} \frac{1}{(n + 1/p)^{r-1}} \]

\[ = \frac{(-1)^r p^r}{r} \cdot \frac{1}{r-1} \lim_{k \to \infty} \frac{1}{p^k} \sum_{m=0}^{p^k-1} \frac{1}{m^{r-1}}. \]

We will write this last expression as \((-1)^r p \, \zeta_p(r)/r\). Using the notation introduced above we have

**Theorem 9.**

\[ G_p(x) = G_p\left(\frac{1}{p}\right) - p\gamma_p(1,p)\left(x - \frac{1}{p}\right) + \sum_{r=2}^{\infty} \frac{(-1)^r \zeta_p(r) p^r}{r} \left(x - \frac{1}{p}\right)^r. \]

This series converges for \(|x - 1/p|_p < p\).

It is interesting to compare this with the classical formula for \(\log \Gamma(x)\):

\[ \log \Gamma(x) = -\gamma(x - 1) + \sum_{r=2}^{\infty} \frac{(-1)^r \zeta(r)}{r} (x - 1)^r \quad \text{for } |x - 1| < 1. \]

The idea of having \(p = 1\) give us classical results from a \(p\)-adic formula, while only formal here, is valid in certain formulas for \(\zeta(n)\) and \(\zeta_p(n)\) discussed in [1].

Our next result for \(G_p(x)\) is a set of formulas for \(G_p(x)\) valid on the annular regions \(A_n = \{ x : n - 1 < v(x) < n \}\) for \(n = 1, 2, 3, \ldots \). Since these regions have no points of \(\mathbb{Z}_p\) we are able to find series which are almost Laurent series. To simplify the discussion we introduce a function \(G^*\) defined by

\[ G^*(x) = \lim_{k \to \infty} \frac{1}{p^k} \sum_{n=0}^{p^k-1} f(x + n) \]

where \(f(x) = x \log(x) - x\).
We can write
\[ G^*(x) = \sum_{a=1}^{p-1} \lim_{k \to \infty} \frac{1}{p^k} \sum_{n=0}^{p^k-1} f(x + n). \]
For each value of \( a \) the inner lim is locally analytic for \( x \) with \( x + a \not\in p\mathbb{Z}_p \). Therefore \( G^* \) is locally analytic for \( x \in \Omega_p - V_p \), where \( V_p \) is the set of units in \( \mathbb{Z}_p \).
For \( |x|_p < 1 \), \( G^* \) coincides with a function defined by Morita [7] in the study of the function he calls \( \Gamma_p(x) \).
To obtain our formulas for \( G_p \) we need the power series for \( G^*(x) \) at \( x = 0 \).

**Theorem 10.** If \( |x|_p < 1 \), then
\[ G^*(x) = M_\chi(\log)(x) + \sum_{r=3}^{\infty} \frac{L_p(r, \overline{\chi}^r)(-1)^r}{r} x^r. \]
\( L_p(r, \chi) \) is the Leopoldt \( L \)-function for the character \( \chi \), \( \overline{\chi} \) is the principal character mod \( p \), \( M_\chi(f) \) is the Leopoldt \( \chi \)-mean [5], and \( \overline{\chi} \) is the character mod \( p \) defined by
\[ \overline{\chi}(n) = \begin{cases} \lim_{n \to \infty} n^{-p^k} & \text{for } (n, p) = 1, \text{if } p > 2, \\ 1 & \text{if } n \equiv 1 \pmod{4}, \\ -1 & \text{if } n \equiv 3 \pmod{4} \text{ for } p = 2. \end{cases} \]

**Proof.** Apply Theorem 3.
This result has also been found by Morita [7].
We are now prepared to find series for \( G_p(x) \) on the annular domains \( A_n = \{ x: n - 1 < v(x) < n \} \), \( n \in \mathbb{Z}_+ \).
With \( x \in A_n \) we write the equations
\[ G_p(x/p^i) - G_p(x/p^{i+1}) = G^*(x/p^i) \quad \text{for } i = 0, 1, \ldots, n - 1; \]
adding these equations we obtain
\[ G_p(x) = G_p(\frac{x}{p^n}) + \sum_{i=0}^{n-1} G^*(\frac{x}{p^i}). \]
We now use Theorems 6 and 10 to obtain

**Theorem 11.** On the annulus \( A_n \) we have the formula
\[ G_p(x) = \left( \frac{x}{p^n} - \frac{1}{2} \right) \log(x) - \frac{x}{p^n} M_\chi(\log) \left( \frac{p^n - 1}{p^n - p^{n-1}} \right) x 
+ \sum_{r=3}^{\infty} \frac{L_p(r, \overline{\chi}^r)(-1)^r}{r} \left( \frac{p^{rn} - 1}{p^{rn} - p^{rn-1}} \right) x^r + \sum_{r=1}^{\infty} \frac{B_{r+1} p^{nr} x^{-r}}{r(r + 1)}. \]
If we define $A_0$ as $\{x: |x|_p > 1\}$, then the above formula is valid for $n = 0, 1, 2, \ldots$.

5. Analyticity. The function $G_p$ is not an analytic function in the sense of Krasner [4], but its second derivative $G''_p$ is an analytic function on $\Omega_p - Z_p$.

**Theorem 12.** $G''_p$ is an analytic function on $\Omega_p - Z_p$.

**Proof.** First for $a, m \in Z$ we define $D(a, m) = \{x: x \in \Omega_p, v(x - a) > m\}$. For each $m \in Z^+$ the set $A_m = \Omega_p - \bigcup_{a=0}^{p^{m-1}} D(a, m)$ is a quasi-connected set. $(A_m: m \in Z^+)$ is nested and $\bigcup_{m=1}^{\infty} A_m = \Omega_p - Z_p$. Therefore if we can prove $G''_p$ is an analytic element on each $A_m$, i.e. the uniform limit of a sequence of rational functions having no poles in $A_m$, then we will know $G''_p$ is an analytic function on $\Omega_p - Z_p$.

If we apply Theorem 7 we can write

$$G''_p(x) = \frac{1}{p^{2m+2}} \sum_{a=0}^{p^{m+1}-1} G''_p\left(\frac{x + a}{p^m+1}\right)$$

for each $m \in Z^+$ and $x \in \Omega_p - Z_p$.

If we consider just $x \in A_m$, then $|(x + a)/p^{m+1}|_p > p > 1$ for all $a \in Z$. Therefore we may use Theorem 6 and obtain

$$G''_p\left(\frac{x + a}{p^m+1}\right) = \sum_{r=0}^{\infty} \frac{B_r}{[(x + a)/p^{m+1}]^{r+1}}.$$ 

Since $|(x + a)/p^{m+1}|_p > p$ for all $x \in A_m$, this last series converges uniformly on $A_m$.

Thus $G''_p$ is an analytic element on $A_m$ and $G''_p$ is an analytic function on $\Omega_p - Z_p$.

6. An alternative approach. Earlier we mentioned another approach to the idea of $p$-adic log $\Gamma(x)$: to change the functional equation. Of course, it must only be a slight change so we can associate it with log $\Gamma(x)$. We will construct a sequence of such functions, which will be locally holomorphic on $\Omega_p$ and have $G_p(x)$ as their pointwise limit.

**Definition.** Let

$$H_N(x) = \lim_{k \to \infty} \frac{1}{p^k} \sum_{n=0}^{p^k-1} f_N(x + n) \quad \text{for } N = 1, 2, \ldots,$$

where

$$f_N(x) = \begin{cases} x \log(x) - x & \text{if } v(x) < N, \\ 0 & \text{if } v(x) \geq N. \end{cases}$$

Each $f_N$ is locally analytic on $\Omega_p$, so each $H_N$ is also locally analytic on $\Omega_p$. 

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We have the following equation

\[ H_N(x + 1) = \begin{cases} 
H_N(x) + \log(x) & \text{if } \nu(x) < N, \\
H_N(x) & \text{if } \nu(x) > N.
\end{cases} \]

It can be shown that \( H_N(0) = 0 \) for each \( N \), so we have for \( n \in \mathbb{Z}^+ \),

\[ H_N(n + 1) = \log \prod_{i=1}^{n} t_i^{p^{\nu_{1/n}}}, \]

in particular, \( H_N(n + 1) = \log(n!) \) if \( n < p^N \). The following theorem shows the relation between \( G_p(x) \) and \( H_N(x) \).

**Theorem 13.** If \( x \) is such that \( \nu(x) < p^{-N} \) for all \( a \in \mathbb{Z}_p \) then \( H_N(x) = G_p(x) \).

**Proof.** Inspection of the definitions of \( G_p(x) \) and \( H_N(x) \).

Theorem 13 shows us that the sequence \( H_N(x) \), \( N = 1, 2, \ldots \), and \( x \) fixed with \( x \in \mathbb{Z}_p \), eventually becomes constant with the value \( G_p(x) \).

For \( x \) with \( \nu(x) > 1 \) the functions \( H_1 \) and \( G^* \) coincide. However, for other \( x \) they are not the same and it is \( H_1 \) which is the log of the function on \( \mathbb{Z}_p \) which Morita has called \( \Gamma_p(x) \) [7].

**7. \( p \)-adic Euler constants.** In a recent paper [6], D. H. Lehmer proves a theorem of Gauss by defining a generalization of Euler's constant. Gauss’ theorem is a formula for the logarithmic derivative of the Gamma function at rational points \( r/k \) with \( 0 < r < k \). The formula is notable because it is a constant plus a linear combination of logarithms of integers in \( Q(\sqrt[k]{\chi}) \).

We shall define \( p \)-adic Euler constants, give the basic results for them and then prove the \( p \)-adic version of Gauss’ theorem: a formula for \( G_p(r/f) \) with \( 0 < r < f \) and \( \nu(r/f) < 0 \).

We show how Gauss’ theorem follows from the classical formula for \( L(1, \chi) \), and since we have the same expression for \( L_p(1, \chi) \) [3], we have a proof valid in both \( C \) and \( \Omega_p \).

Lehmer defines the (generalized) Euler constants by

\[ \gamma(r, k) = \lim_{\chi \to \infty} \left\{ \sum_{0 < n < x \atop n \equiv r \pmod{k}} \frac{1}{n} - \frac{1}{k} \log x \right\} \]

and then, using the relation

\[ \psi(r/k) = D^{(1)} \log \Gamma(r/k) = \log k - k \gamma(r, k) \] for \( r, k \in \mathbb{Z}^+, r < k, \)

he proves: If \( r, k \in \mathbb{Z}^+, r < k, \) then
\( \psi \left( \frac{r}{k} \right) = -\gamma - \log \left( \frac{k}{2} \right) - \frac{\pi}{2} \cot \left( \frac{\pi r}{k} \right) \\
+ 2 \sum_{0 < j < k/2} \cos \left( \frac{2\pi j}{k} \right) \log \sin \left( \frac{\pi j}{k} \right) \)

Gauss’ theorem, combined with the functional equation, enables us to calculate \( \psi(x) \) in closed form at every rational value of \( x \) for which the function is defined.

Working in \( \Omega_p \) we can define \( \gamma_p(r,f) \) when \( r, f \in \mathbb{Z}, f > 1 \), and find a similar formula for \( G'_p(r/f) \).

When \( \nu(r/f) < 0 \) we define

\[
\gamma_p(r,f) = \lim_{k \to \infty} \frac{1}{f^p k} \sum_{m=0}^{f^p-1} \log(m).
\]

When \( \nu(r/f) > 0 \) we write \( f = p^m f^* \) with \( (p,f^*) = 1 \) and let \( \phi = \phi(f^*) \) (the Euler function). We then define

\[
\gamma_p(r,f) = \frac{p^*}{p^* - 1} \sum_{n \in N(r,f)} \gamma_p(r + nf, p^*f)
\]

where

\[
N(r,f) = \{ n: 0 < n < p^*, nf + r \equiv 0 \pmod{p^{k+k}} \}.
\]

Theorem 1 applies to show \( \gamma_p(r,f) \) exists.

To obtain Gauss’ theorem in \( \Omega_p \), we need several results which are mostly the same as Lehmer has given for \( C \). The proofs follow from the definition of \( \gamma_p(r,f) \), previous results for \( G'_p(x) \), and Theorem 18. We will write \( \psi_p(x) = G'_p(x) \).

**Theorem 14.**

(i) If \( d|(r,f) \), then \( f \gamma_p(r,f) = (f/d) \gamma_p(r/d, f/d) - \log d \).

(ii) If \( \nu(r/f) < 0 \) and \( 0 < r < f \), then \( \psi_p(r,f) = -\log f - f \gamma_p(r,f) \).

(iii) \( \gamma_p(r,f) = \gamma_p(f - r, f) \).

(iv) If \( b \in \mathbb{Z}^+ \), then

\[
\gamma_p(r,f) = \sum_{n=0}^{b-1} \gamma_p(r + nf, bf).
\]

(v) If \( p^* \equiv 1 \pmod{f^*} \) and \( \nu(r/f) > 0 \), then

\[
\gamma_p(r,f) = \frac{p^*}{p^* - 1} \sum_{n=0}^{p^*-1} \gamma_p(r + nf, p^*f).
\]

We are going to need a \( p \)-adic analog of Euler’s constant. The value
\[ \gamma_p = \gamma_p(0, 1) = -\frac{p}{p - 1} \lim_{k \to \infty} \frac{1}{p^k} \sum_{m=1 \atop (m,p)=1}^{p^k-1} \log(m) \]

fits our formulas precisely as we need. Morita has also realized the connection between Euler's constant and \( M_\mathbb{p} \log \) and he gives a slightly different value [7].

Lehmer has defined the formula \( \Phi(f) \) by

\[ \Phi(f) = \sum_{r=1 \atop (r,f)=1}^{f} \gamma(r,f). \]

He then proves

\[ \Phi(f) = \frac{\phi(f)}{f} \gamma + \frac{\phi(f)}{f} \sum_{q|f} \frac{\log q}{q - 1}. \]

In this formula, \( q \) is prime, \( \phi \) is the Euler \( \phi \)-function and \( \gamma \) is Euler's constant.

We define

\[ \Phi_p(f) = \sum_{r=1 \atop (r,f)=1}^{f} \gamma_p(r,f) \text{ when } \nu(f) > 0. \]

We then have

**Theorem 15.**

\[ \Phi_p(f) = \frac{\phi(f)}{f} \gamma_p + \frac{\phi(f)}{f} \sum_{q|f} \frac{\log q}{q - 1}. \]

**Proof.** We can use Theorem 14(iv) with \( b = f_1/f \) to show that if \( f \) has the same distinct prime factors as \( f_1 \) and \( f|f_1 \) then \( \Phi_p(f) = \Phi_p(f_1) \).

It is then sufficient to consider square free \( f \). This is accomplished by induction on the number of prime factors of \( f \).

We will need the following algebraic identity.

**Theorem 16.** If \( \xi \) is a primitive \( f \)-th root of unity, \( f > 1 \), \( \epsilon_f \) the principal character mod \( f \) and \( \tau_a(\epsilon_f) \) the Gauss sum,

\[ \tau_a(\epsilon_f) = \sum_{n=1}^{f} \epsilon_f(n) \xi^{an}, \]

then

\[ \prod_{a=1}^{f-1} (1 - \xi^{-a})^{\tau_a(\epsilon_f)} = \prod_{q|f} q^{-\phi(f)/(q-1)}. \]

The product on the right side is over the distinct prime divisors of \( f \).
PROOF. Let \( \omega_r \) be a primitive \( r \)th root of unity and \( Q_r = Q(\omega_r) \) for \( r = 2, 3, \ldots \).

We observe that

\[
\tau_a(e_j) = \sum_{n=1}^{f-1} \omega_r^{an} = \text{tr}_{Q_r/Q}(\omega_r).
\]

When we group together conjugate elements we have

\[
(1) \prod_{a=1}^{f-1} (1 - \omega_r^{-a}) = \prod_{r \mid f} \left( N_{Q_r/Q}(1 - \omega_r) \right)^{\text{tr}_{Q_r/Q}(\omega_r)}.
\]

Examination of the minimal polynomial of \( \omega_r \) shows that:

(i) if \( r \) is not square free, then \( \text{tr}_{Q_r/Q}(\omega_r) = 0 \);
(ii) if \( r \) is square free, but not prime, then \( N_{Q_r/Q}(1 - \omega_r) = 1 \);
(iii) if \( r \) is prime, \( N_{Q_r/Q}(1 - \omega_r) = r \);
(iv) if \( r \) is prime

\[
\text{tr}_{Q_r/Q}(\omega_r) = \frac{\phi(f)}{r-1} \text{tr}_{Q_r/Q}(\omega_r) = \frac{-\phi(f)}{r-1}.
\]

Placing these four values into (1) establishes the theorem.

Next we state Gauss' theorem in \( \Omega_p \).

THEOREM 17. If \( r, f \in \mathbb{Z}^+, r < f \) and \( \nu(r/f) < 0 \), then

\[
\psi_p(r/f) = -\log f - \gamma_p + \sum_{a=1}^{f-1} \omega_r^{-a} \log(1 - \omega_r^{-a}).
\]

If \( \psi_p \) is replaced by \( \psi \) and \( \gamma_p \) by \( \gamma \) we have Gauss' theorem in \( \mathbb{C} \). Of course \( \log \) is either \( p \)-adic or complex as required.

PROOF. Since we have shown (Theorem 14(ii)) that \( \psi_p(r/f) = -\log f - f\gamma_p(r, f) \), it will be sufficient to prove

THEOREM 18. If \( f > 1 \) and \( \omega_r \) is a primitive \( f \)th root of unity, then

\[
f\gamma_p(r, f) = \gamma_p - \sum_{a=1}^{f-1} \omega_r^{-a} \log(1 - \omega_r^{-a}).
\]

Notice that we do not need the restriction \( \nu(r/f) < 0 \) for this result.

PROOF. We begin by assuming \( (r, f) = 1 \) and \( \nu(r/f) < 0 \) and proceed to evaluate \( \sum_{\chi \neq \chi} \chi(r)L_p(1, \chi) \) in two different ways. The sum is over all non-principal characters mod \( f \).

For \( \chi \) not principal and if \( p \mid f \), we have the formula [3]

\[
L_p(1, \chi) = -\frac{1}{f} \sum_{a=1}^{f-1} \tau_a(\chi) \log(1 - \omega_r^{-a}).
\]
(Note: Iwasawa gives this formula in a form valid only for primitive characters, but if \( \overline{\chi}(\alpha)\tau(\chi) \) is replaced by \( \gamma_\alpha(\chi) \), then his proof can be modified to be valid for all nonprincipal characters.)

This is the same as the formula for \( L(1, \chi) \) in \( C \). Using this result and Theorems 15 and 16 we have

\[
(*) \quad \sum_{\chi \neq \eta} \overline{\chi}(r)L_p(1, \chi) = \frac{\phi(f)\gamma_p}{f} - \Phi_p(f) - \frac{\phi(f)}{f} \sum_{a=1}^{f-1} \zeta^{ar}\log(1 - \zeta^{-a}).
\]

This result is also valid, correctly interpreted, in \( C \).

If we use the expression [5] that

\[
L_p(1, \chi) = -\lim_{k \to \infty} \frac{1}{f_p^k} \sum_{n=0}^{f_p^{k-1}} \chi(n)\log n \quad \text{where} \quad (n, p) = 1,
\]

we obtain

\[
(**) \quad \sum_{\chi \neq \eta} \overline{\chi}(r)L_p(1, \chi) = \phi(f)\gamma_p(r, f) - \Phi_p(f).
\]

(**) is obtained in \( C \) by using

\[
L(1, \chi) = \lim_{x \to \infty} \sum_{0 < n < x} \frac{\chi(n)}{n}.
\]

Equating (*) and (**) yields Theorem 18 in the case where \( (r, f) = 1 \) and \( \nu(r/f) < 0 \).

Now suppose \( (r, f) = d > 1 \) and \( \nu(r/f) < 0 \). Then we can use 14(i) to obtain

\[
f_p(r, f) = -\log d + \gamma_p - \sum_{a=1}^{f/d - 1} \zeta^{-ar}\log(1 - \zeta^{-ad}).
\]

We can factor \( 1 - \zeta^{-ad} \) and obtain

\[
f_p(r, f) = -\log d + \gamma_p - \sum_{a=1}^{f/d - 1} \sum_{b=0}^{d-1} (\zeta^a \lambda^b)^{-r}\log(1 - \zeta^a \lambda^b)
\]

where \( \lambda \) is a primitive \( d \)th root of unity.

Since

\[
\{\zeta^a \lambda^b: 0 < a < f/d, 0 < b < d - 1\} = \{\zeta^a: 0 < a < f\},
\]

\[
f_p(r, f) = -\log d + \gamma_p - \sum_{a=1}^{f-1} \zeta^{-ar}\log(1 - \zeta^a) + \sum_{b=1}^{d-1} \log(1 - \lambda^b)
\]

\[
= \gamma_p - \sum_{a=1}^{f-1} \zeta^{-ar}\log(1 - \zeta^a).
\]

This completes the proof of Gauss' theorem, but we have not yet completed the proof of Theorem 18.
If \( \nu(r,f) > 0 \) we can use the definition of \( \gamma_p(r,f) \) and the case of Theorem 18 already proven to show

\[
\frac{f_r}{r}(r,f) = \gamma_p - \frac{1}{p^\mu - 1} \sum_{\alpha=1}^{p^\mu-1} \eta^{-ar}\log(1 - \eta^a) \sum_{n \in \mathbb{N}(r,f)} \eta^{-anr},
\]

where \( \eta \) is a primitive \( p^\mu \) root of unity.

The last sum on the right is \( p^\mu - 1 \) if \( p^\mu | a \) and \(-\eta^{ar-ar^p\phi} \) if \( p^\mu \not| a \). Thus

\[
\frac{f_r}{r}(r,f) = \gamma_p - \sum_{\alpha=1}^{f-1} \xi^{-ar}\log(1 - \xi^a) + \frac{1}{p^\mu - 1} \sum_{\alpha=1}^{p^\mu-1} \xi^{-ar}\log(1 - \xi^a).
\]

The last sum on the right is 0. □

We have seen that \( \psi_p \) is locally holomorphic on \( \Omega_p - \mathbb{Z}_p \) and \( \psi_p \) is Krasner-analytic on this domain. We have also shown that the formula

\[
(*) - \log f - f_r(r,f)
\]

depends only on the ratio \( r/f \) and that for \( \nu(r/f) < 0 \), and \( 0 < r < f \),

\[
\psi_p(r/f) = (\ast) = -\log f - \gamma_p + \sum_{\alpha=1}^{f-1} \xi^{-ar}\log(1 - \xi^a).
\]

Since \( \ast \) is defined for \( r,f \) with \( \nu(r/f) > 0 \), it is tempting to use \( \ast \) to extend the definition of \( \psi_p \) onto the rational numbers in \( \mathbb{Z}_p \). However, this "continuation" would not retain the other properties of \( \psi_p \). The values of \( \ast \), though, are related to functions similar to \( \psi_p \) when \( \nu(r,f) > 0 \) and we have

**Theorem 19.** Given \( \nu(r/f) > 0 \), then for any \( \mu \) such that \( p^\mu \equiv 1 \pmod{f^\mu} \) we have

\[
(*) \quad \frac{p^\mu}{p^\mu - 1} H^\mu_{\nu}( \frac{r}{f} ) = -\log f - \gamma_p + \sum_{\alpha=1}^{f-1} \xi^{-ar}\log(1 - \xi^a).
\]

\( H_N \) is discussed at the end of §4.

**Proof.** This follows directly from previous results.

Since \( H_1 \) (on \( \mathbb{Z}_p \)) is the logarithm of Morita's \( \Gamma_p \) [7], we have a

**Corollary.** If \( 0 < r < f, \nu(r/f) > 0 \) and \( f^\mu | (p - 1) \), then

\[
\frac{\Gamma_p}{\Gamma_{p^\mu}} \left( \frac{r}{f} \right) = (1 - 1/p) \left( -\log f - \gamma_p + \sum_{\alpha=1}^{f-1} \xi^{-ar}\log(1 - \xi^a) \right).
\]

**References**


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