A SUPERPOSITION THEOREM FOR UNBOUNDED CONTINUOUS FUNCTIONS

BY

RAOUF DOSS

Abstract. Let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space. We prove that there are $4n$ real functions $q_p$ continuous on $\mathbb{R}^n$ with the following property: Every real function $f$, not necessarily bounded, continuous on $\mathbb{R}^n$, can be written $f(x) = \sum_{q=1}^{2n+1} g(q_p(x)) + \sum_{q=2n+2}^{4n} h(q_p(x))$, $x \in \mathbb{R}^n$, where $g$, $h$ are 2 real continuous functions of one variable, depending on $f$.

Let $I = [0, 1]$ be the closed unit interval and let $C(I^n)$, $n = 1, 2, \ldots$, be the Banach space of real functions continuous on the cube $I^n$, with the usual norm. In 1957, Kolmogorov [11] proved the following theorem, giving an elegant solution to the celebrated Hilbert's Problem 13:

For every $n = 1, 2, \ldots$ there are $n(2n + 1)$ continuous increasing functions $q_{pq}$ on $I$ with the following property: Every $f \in C(I^n)$ may be written in the form

$$f(x_1 \cdots x_n) = \sum_{q=1}^{2n+1} g_q \left( \sum_{p=1}^{n} q_{pq}(x_p) \right), \quad (x_1, \ldots, x_n) \in I^n,$$

where the $g_q$ are $2n + 1$ continuous functions of one variable, depending on $f$.

New research on Hilbert's Problem 13 has been carried out in three main directions:

(a) Concerning the functions $q_{pq}$. See Fridman [4], [5], Hedberg [6], Henkin [7], Kahane [9], Kaufman [10], Sprecher [14], Vituškin [16].

(b) Concerning the functions $g_q$. See Bassalygo [1], Doss [2], [3], Kahane [8], [9], Lorentz [12], Sternfeld [15].

(c) Concerning the basic space $I^n$. See Ostrand [13]. We quote here Ostrand's theorem for we shall make use of it: Let $K$ be a compact metric space of topological dimension $n$; then there are $2n + 1$ real functions $q_{pq}$, continuous on $K$, with the following property: Every real function $f$ continuous on $K$ may be written:

$$f(x) = \sum_{q=1}^{2n+1} g(q_p(x)), \quad x \in K,$$

where $g$ is a continuous real function of one variable.
We shall be interested here in a new situation concerning the basic space, namely, instead of $I^n$, we shall consider the open unit cube $I^n_0$, or the open unit ball $B$ in $R^n$, or even $R^n$ itself, and the (possibly unbounded) continuous function on $I^n_0$ or $B$ or $R^n$.

We shall prove the following:

**Theorem.** For every fixed $n$, there are $2n - 1$ functions $\psi_i, i = 1, \ldots, 2n - 1$, and $2n + 1$ functions $\varphi_q, q = 1, \ldots, 2n + 1$, $4n$ functions in all, continuous on $R^n$, taking values in the semi-open interval $[0, 1)$, tending to 1 at infinity, with the following property: Every real $f$, continuous on $R^n$, may be written:

$$f(x) = \sum_{i=1}^{2n-1} h(\psi_i(x)) + \sum_{q=1}^{2n+1} g(\varphi_q(x)), \quad x \in R^n,$$

where $h, g$ are real functions of one variable continuous on $[0, 1)$.

The same is true if $R^n$ is replaced throughout by $I^n_0$ or by the open unit ball $B$ in $R^n$, or any space homeomorphic to these, and the various forms of the theorem are equivalent. The proof will be carried out for the open unit ball $B$.

**Lemma 1.** Let $\mathbb{R}$ be the closed region bounded by concentric spheres $S, S'$ in $R^n$, and let $\delta > 0$. Then there are $2n + 1$ sets $C_\delta = \bigcup C_\delta(i)$ satisfying the conditions:

(i) For every $q = 1, \ldots, 2n + 1$, the sets $C_\delta(i)$ form a finite family of closed disjoint sets of diameter less than $\delta$.

(ii) Every $x \in \mathbb{R}$ belongs to at least $n + 1$ of the sets $C_\delta, q = 1, \ldots, 2n + 1$.

These sets $C_\delta(i)$ are the intersection with $\mathbb{R}$ of the well-known cubes considered by Kolmogorov in his classical paper [11]. There is great freedom in the choice of these sets; in the sequel of this paper we assume that they have been chosen once and for all.

**Lemma 2.** Let $\mathbb{R}$ be the closed region bounded by the concentric spheres $S, S'$ in $R^n$, of radius $\alpha, \alpha', \alpha < \alpha'$. There is a decreasing sequence $\epsilon_m \to 0$ and there are $2n + 1$ functions $\varphi_q$, continuous on $\mathbb{R}$, such that:

1. $\alpha \leq \varphi_q \leq \alpha'$ in the region $\mathbb{R}, q = 1, \ldots, 2n + 1$,
2. $\varphi_q(x) = \alpha$ if and only if $x \in S$,
3. $\varphi_q(x) = \alpha'$ if and only if $x \in S'$,
4. for all $m$, and for all sets $C_{\epsilon_{m+1}}(i)$, not meeting the $\epsilon_m$-neighborhood of $S \cup S'$, we have

$$\varphi_q(C_{\epsilon_{m+1}}(i)) \cap \varphi_q(C_{\epsilon_{m+1}}(i')) = \emptyset$$

if either $q \neq q'$ or $q = q', i \neq i'$.

**Proof.** Let $\delta_m$ be a decreasing sequence tending to 0. Choose two fixed real
functions $\phi_1, \phi_2$, continuous on $\mathcal{R}$ such that $\phi_1 = \phi_2 = \alpha$ on $S$, $\phi_1 = \phi_2 = \alpha'$ on $S'$, $\alpha < \phi_1 < \phi_2 < \alpha'$ between $S$ and $S'$.

Let $A$ be the set of all $(2n + 1)$-tuples $(\varphi_q)$ of functions $\varphi_q$, continuous on $\mathcal{R}$, and satisfying the conditions

(5) \[ \varphi_1 < \varphi_q < \varphi_2. \]

Such functions $\varphi_q$ satisfy necessarily conditions (1), (2), and (3) of the lemma. With the usual definition of the uniform norm $\| \cdot \|$, $A$ is a complete metric space.

For an integer $m$, define the subset $B_m$ of $A$ as follows: The element $(\varphi_q)$ of $A$ belongs to $B_m$ if there exists an integer $l > m$ with the property that for any sets $C_\delta^q(i), C_\delta^q(i')$ not meeting the $\delta_m$-neighborhood of $S \cap S'$ we have

$$\varphi_q(C_\delta^q(i)) \cap \varphi_q(C_\delta^q(i')) = \emptyset$$

if either $q \neq q'$ or $q = q', i \neq i'$.

We see easily that $B_m$ is open in $A$.

We shall prove that $B_m$ is dense in $A$. So let $(\varphi_q^0) \in A$ and let $\varepsilon > 0$ be given. We must show that we can find $(\varphi_q) \in B_m$ such that

(6) \[ \| \varphi_q - \varphi_q^0 \| < \varepsilon, \quad q = 1, \ldots, 2n + 1. \]

On the closed set $\mathcal{R}_m = \mathcal{R} \setminus \delta_m$-nbhd of $S \cup S'$ we have $\varphi_1 < \varphi_2$. Hence, there is $\gamma > 0$ such that

$$\varphi_1(x) < \varphi_2(x) - \gamma, \quad x \in \mathcal{R}_m.$$

Choose $l$ so large that the variation of $\varphi_1, \varphi_2$ and every $\varphi_q^0$ on any set of diameter $< \delta_l$ is less than $\varepsilon/2$ and also less than $\gamma/3$. For any set $C_\delta^q(i)$ lying in $\mathcal{R}_m$ put $\varphi_q(C_\delta^q(i)) = k^q(i)$ where the constants $k^q(i)$ are all in the open interval $(\alpha, \alpha')$, are all different, and

(6') \[ |\varphi_q(x) - \varphi_q^0(x)| < \varepsilon, \quad x \in C_\delta^q(i) \subset \mathcal{R}_m. \]

Moreover, since $\sup_{x \in C} \varphi_1(x) < \inf_{x \in C} \varphi_2(x) - \gamma/3$, where $C$ stands for $C_\delta^q(i)$, we may choose the constants $k^q(i)$ such that

(5') \[ \sup_{x \in C} \varphi_1(x) < k^q(i) < \inf_{x \in C} \varphi_2(x) \]

for $C_\delta^q(i) \subset \mathcal{R}_m$. Next, put $\varphi_q = \alpha$ on $S$, $\varphi_q = \alpha'$ on $S'$, and then extend these $\varphi_q$, so far defined, to functions $\varphi_q$, continuous on $\mathcal{R}$, and satisfying conditions (5) and (6). This proves that $B_m$ is dense in $A$.

Now the intersection $B = \cap_m B_m$ is dense by Baire's theorem and, hence, is nonempty. Choose a fixed $(\varphi_q) \in B$. Then, inductively, there is a subsequence $\varepsilon_m$ of $\delta_m$ such that for any sets $C_\varepsilon^{q,i}, C_\varepsilon^{q,i'}$ not meeting the $\varepsilon_m$-neighborhood of $S \cup S'$, we have

$$\varphi_q(C_\varepsilon^{q,i}) \cap \varphi_q(C_\varepsilon^{q,i'}) = \emptyset$$

if either $q \neq q'$ or $q = q', i \neq i'$. 

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This completes the proof of Lemma 2.

**Lemma 3.** Let $R$ be the closed ring bounded by the two concentric spheres $S, S'$ in $R^n$, of radius $\alpha, \alpha', \alpha < \alpha'$. Then there are $2n + 1$ functions $\varphi_q, q = 1, \ldots, 2n + 1$, continuous on $R$, taking values in the interval $[\alpha, \alpha']$, such that $\varphi_q(x) = \alpha$ iff $x \in S$, $\varphi_q(x) = \alpha'$ iff $x \in S'$, $q = 1, \ldots, 2n + 1$, with the following property:

To every function $F$, continuous on $R$, vanishing on $S \cup S'$, such that $|F(x)| < M$ for $x \in R$, there corresponds a $g$, continuous on $[\alpha, \alpha']$, such that

$$g(\alpha) = g(\alpha') = 0, \quad |g| < \frac{1}{2n + 3} M,$$

and

$$|F(x) - \sum_{q=1}^{2n+1} g(\varphi_q(x))| < \frac{2n+2}{2n+3} M \quad \text{for } x \in R. \quad (1)$$

**Proof.** We may assume $M = 1$. Since $\epsilon_m \to 0$ (cf. Lemma 2), we may choose $m$ so large that the oscillation of $F$, on any set of diameter $< \epsilon_{m-1}$, is less than $\frac{1}{2} \cdot (2n + 3)^{-1}$. Define $g$ as follows:

If $F(x) > 0$ throughout a set $C_q(i)$ not meeting the $\epsilon_{m-1}$-nbhd of $S \cup S'$ put $g(\varphi_q(C_q(i))) = (2n + 3)^{-1}$. If $F(x) < 0$ throughout such a set, put $g(\varphi_q(C_q(i))) = -(2n + 3)^{-1}$. Because the closed sets $\varphi_q(C_q(i))$ are disjoint, these constructions are consistent. Also, if for some $C_q(i)$, the image $g(\varphi_q(C_q(i))) = \pm (2n + 3)^{-1}$ has been defined, then $\alpha, \alpha' \notin \varphi_q(C_q(i))$, since $\varphi_q = \alpha, \alpha'$ only on $S \cup S'$, while $C_q(i)$ does not meet $S \cup S'$. Therefore, it is consistent with the above construction to put $g(\alpha) = g(\alpha') = 0$. Finally, extend $g$ to a continuous function on $[\alpha, \alpha']$, still bounded by $(2n + 3)^{-1}$.

To prove that relation (1) holds we assume first that $F(x) > (2n + 3)^{-1}$. By Lemma 1, there are at least $n + 1$ sets $C_q(i)$ containing $x$. By our choice of $m$, no such set can meet the $\epsilon_{m-1}$-nbhd of $S \cup S'$ $[F = 0$ on $S \cup S'$ while $F > \frac{1}{2} \cdot (2n + 3)^{-1}$ on such a set]. Therefore

$$F(x) - \sum_{q=1}^{2n+1} g(\varphi_q(x)) < 1 - \frac{n+1}{2n+3} + \frac{n}{2n+3} = \frac{2n+2}{2n+3} \quad (2)$$

since at least $n + 1$ of the terms $g(\varphi_q(x))$ are equal to $(2n + 3)^{-1}$. Also, the left side of (2) is larger than $(2n + 3)^{-1} - (2n + 1)/(2n + 3)$, hence larger than $-(2n + 2)/(2n + 3)$, so that (1) is verified in this case.

The case $F(x) < -(2n + 3)^{-1}$ is treated similarly.

Finally, if $|F(x)| < (2n + 3)^{-1}$, the expression on the left side of (2) has absolute value not exceeding $(2n + 2)/(2n + 3)$ so that (1) holds also in this case.

The proof of Lemma 3 is now complete.
Lemma 4. Let $\mathcal{R}$ be the closed ring bounded by the two concentric spheres $S, S'$, in $\mathbb{R}^n$, of radius $\alpha, \alpha'$, $\alpha < \alpha'$; then there are $2n + 1$ functions $\varphi_q$, $q = 1, \ldots, 2n + 1$, continuous on $\mathcal{R}$, taking values in the interval $[\alpha, \alpha']$, such that $\varphi_q(x) = \alpha$ for $x \in S$, $\varphi_q(x) = \alpha'$ for $x \in S'$ with the following property:

To every function $F$, continuous on $\mathcal{R}$ and vanishing on $S \cup S'$, there corresponds a real function $g$, defined and continuous on the interval $[\alpha, \alpha']$, such that

$$F(x) = \sum_{q=1}^{2n+1} g(\varphi_q(x)), \quad x \in \mathcal{R}.$$  

Observe that for such a $g$ we necessarily have $g(\alpha) = g(\alpha') = 0$

This is deduced from Lemma 3, using the very familiar Kolmogorov technique; see [11].

Lemma 5. Let $\alpha_m$ be an increasing sequence of positive numbers tending to 1, and let $S_m$ be the spheres in $\mathbb{R}^n$ of center 0, and radius $\alpha_m$. Then there are $2n - 1$ functions $\psi_i$, $i = 1, \ldots, 2n - 1$, continuous on the open unit ball $\mathbb{B}$ in $\mathbb{R}^n$ of center 0, taking values in $[0, 1)$, with the following property: To every $f$ continuous on $\mathbb{B}$ there corresponds a function $h$, continuous on $[0, 1)$, such that

$$f(x) = \sum_{i=1}^{2n-1} h(\psi_i(x)), \quad x \in S_{2m}, m = 1, 2, \ldots.$$  

Proof. By Ostrand's theorem [13], since $\dim S_{2m} = n - 1$, there are $2n - 1$ functions $\psi^m_i$, $i = 1, \ldots, 2n - 1$, continuous on $S_{2m}$, taking values in the interval $[\alpha_{2m}, \alpha_{2m+1}]$ with the property that every $f$ continuous on $S_{2m}$ may be written:

$$f(x) = \sum_{i=1}^{2n-1} h_m(\psi^m_i(x)) \quad \text{for } x \in S_{2m},$$

where $h_m$ is continuous on $[\alpha_{2m}, \alpha_{2m+1}]$.

Let $\psi_i$ be a continuous function on the open unit ball $\mathbb{B}$ such that $\psi_i(x) = \psi^m_i(x)$ for $x \in S_{2m}$, and taking values in $[0, 1)$.

If now $h$ is a continuous function on $[0, 1)$ such that $h(y) = h_m(y)$ for $y \in [\alpha_{2m}, \alpha_{2m+1}]$, then

$$f(x) = \sum_{i=1}^{2n-1} h(\psi_i(x)) \quad \text{for } x \in S_{2m}, m = 1, 2, \ldots.$$  

This proves Lemma 5.

Theorem. There are $4n$ functions $\psi_i, \varphi_q, i = 1, \ldots, 2n - 1, q = 1, \ldots, 2n + 1$, continuous on the open unit ball $\mathbb{B}$ in $\mathbb{R}^n$, taking values in the semi-open interval $[0, 1)$, with the following property:
To every real function $f$, continuous on the open ball $B$, there correspond two real functions $h$, $g$, continuous on $[0, 1)$ such that

$$f(x) = \sum_{i=1}^{2n-1} h(\psi_i(x)) + \sum_{q=1}^{2n+1} g(\varphi_q(x)), \quad x \in B.$$ 

**Proof.** By Lemma 5, starting with any increasing sequence $\alpha_0 = 0, \alpha_1, \ldots, \alpha_m \ldots$ of real numbers tending to 1 and with the spheres $S_m$ of center 0 and radius $\alpha_m$, we have $2n - 1$ functions $\psi_i, i = 1, \ldots, 2n - 1$, continuous on $B$, taking values in $[0, 1)$ such that to every $f$ continuous on $B$ there corresponds a real function $h$, continuous on $[0, 1)$, such that

$$f(x) = \sum_{i=1}^{2n-1} h(\psi_i(x)), \quad x \in S_{2m}, m = 0, 1, 2, \ldots.$$ 

Next, by Lemma 4, if $R_m$ is the closed ring bounded by the two spheres $S_{2m}$, $S_{2m+2}$, then there are $2n + 1$ functions $\varphi_q^m, q = 1, \ldots, 2n + 1$, continuous on $R_m$, taking values in the interval $[\alpha_{2n}, \alpha_{2n+2}]$ such that $\varphi_q^m(x) = \alpha_{2m}$ for $x \in S_{2m}$, $\varphi_q^m(x) = \alpha_{2m+2}$ for $x \in S_{2m+2}$ with the property that to every $F$ continuous on $R_m$ and vanishing on $S_{2m} \cup S_{2m+2}$, there corresponds a real function $g_m$, defined and continuous on $[\alpha_{2m}, \alpha_{2m+2}]$, such that $g_m(\alpha_{2m}) = 0$ and such that

$$F(x) = \sum_{q=1}^{2n+1} g_m(\varphi_q^m(x)), \quad x \in R_m.$$ 

Let $\varphi_q(x) = \varphi_q^m(x)$ for $x \in R_m, m = 0, 1, \ldots, q = 1, \ldots, 2n + 1$. These functions $\varphi_q$ are continuous on the open unit ball $B$ and take values in the semi-open interval $[0, 1)$.

Put now

$$F(x) = f(x) - \sum_{i=1}^{2n-1} h(\psi_i(x)), \quad x \in B.$$ 

Then $F$ is continuous on $B$, and, by (1), $F(x) = 0$ for $x \in S_{2m}, m = 0, 1, \ldots$. Put $g(x) = g_m(x)$ for $x \in [\alpha_{2m}, \alpha_{2m+2}]$. Since $g(\alpha_{2m}) = g(\alpha_{2m+2}) = 0$, this function is unambiguously defined and is continuous on $[0, 1)$. We have, by (2),

$$F(x) = \sum_{q=1}^{2n+1} g(\varphi_q(x)), \quad x \in R_m, m = 0, 1, 2, \ldots,$$

that is $F(x) = \sum_{q=1}^{2n+1} g(\varphi_q(x)), x \in B$. Finally, by (3),

$$f(x) = \sum_{i=1}^{2n-1} h(\psi_i(x)) + \sum_{q=1}^{2n+1} g(\varphi_q(x)), \quad x \in B,$$

and the theorem is proved.
REFERENCES


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