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GENERAL POSITION OF EQUIVARIANT MAPS

BY

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ABSTRACT. A natural generic notion of general position for smooth maps which are equivariant with respect to the action of a compact Lie group is introduced. If G is a compact Lie group, and M, N are smooth G-manifolds, then the set of smooth equivariant maps $F: M \to N$ which are in general position with respect to a closed invariant submanifold P of N, is open and dense in the Whitney topology. The inverse image of P, by an equivariant map in general position, is Whitney stratified. The inverse images, by nearby equivariant maps in general position, are topologically ambient isotopic.

In the local context, let V, W be linear G-spaces, and $F: V \to W$ a smooth equivariant map. Let $F_1, \ldots, F_k$ be a finite set of homogeneous polynomial generators for the module of smooth equivariant maps, over the ring of smooth invariant functions on V. There are invariant functions $h_1, \ldots, h_k$ such that $F = U \circ \operatorname{graph} h$, where $\operatorname{graph} h$ is the graph of $h(x) = (h_1(x), \ldots, h_k(x))$, and $U(x, h) = \sum_{i=1}^k h_i F_i(x)$. The isomorphism class of the real affine algebraic subvariety $(U = 0)$ of $V \times \mathbf{R}^k$ is uniquely determined (up to product with an affine space) by V, W. F is said to be in general position with respect to $0 \in W$ at $0 \in V$ if $\operatorname{graph} h: V \to V \times \mathbf{R}^k$ is transverse to the minimum Whitney stratification of $(U = 0)$, at $x \in V$.

1. **Introduction.** Let G be a compact Lie group and M, N smooth ($\mathcal{C}^\infty$) G-manifolds. In this paper we introduce and study a natural generic notion of general position of a smooth equivariant map $F: M \to N$ with respect to a smooth invariant submanifold P of N. The problem of defining general position locally amounts to defining general position of a smooth equivariant map $F: V \to W$, where V, W are linear G-spaces, with respect to the origin of W, at the origin of V. Let $F_1, \ldots, F_k$ be a finite set of polynomial generators for the module $\mathcal{C}_G^\infty(V, W)$ of smooth equivariant maps, over the ring $\mathcal{C}_G^\infty(V)$ of smooth invariant functions on the source. There are invariant functions $h_1, \ldots, h_k$ such that

$$F(x) = \sum_{i=1}^k h_i(x) F_i(x) = U \circ \operatorname{graph} h(x),$$

where $\operatorname{graph} h$ is the graph of $h(x) = (h_1(x), \ldots, h_k(x))$, and $U(x, h) =$

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$\Sigma_{i=1}^{k} h_i F_i(x)$ is the "universal" part of the equivariant map. The analytic isomorphism class of the germ at $(x = 0)$ of the real affine algebraic subvariety $(U(x, h) = 0)$ of $V \times \mathbf{R}^k$ is uniquely determined (up to product with an affine space) by V, W (§4).

DEFINITION 1.1. F is *in general position with respect to* $0 \in W$ *at* $0 \in V$ if graph h: $V \to V \times \mathbf{R}^k$ is transverse to (the minimum Whitney stratification of) the affine algebraic variety $(U(x, h) = 0)$, at $0 \in V$.

The most naive approach to equivariant general position, of course, is to try to deform a given equivariant map F: $M \to N$ into an equivariant map which is transverse to P (Ted Petrie [18] has recently announced an obstruction theory for deforming a proper equivariant map by a proper equivariant homotopy; the obstructions are of a global nature, though transversality is a local property). This approach has been moderately successful in certain problems in equivariant algebraic topology [25], [17]. But in simple examples there are no transverse equivariant maps. In studying the local structure of smooth equivariant maps, one seeks a natural local notion of general position, which is satisfied by almost all equivariant maps, and for which there are natural equivariant analogues of the basic results of Thom's transversality theory [23], [1].

As a first approach in this direction, one might consider "stratumwise transversality" of a smooth equivariant map F: $M \to N$ to an invariant submanifold P of N. M is stratified by the bundles $M_{(H)}$ of orbits of type (H), for each isotropy subgroup H of the action of G on M ((H) denotes the conjugacy class of H). Let M^H be the fixed point submanifold of H in M, and $M_H = M_{(H)} \cap M^H$, the associated principal bundle of $M_{(H)}$. It is not difficult to see that any smooth equivariant map can be deformed by an arbitrarily small amount (in either the $\mathcal{C}^\infty$ or Whitney topology) to a smooth equivariant map F: $M \to N$ which is stratumwise transverse to P in the only way that makes sense: for each isotropy subgroup H of M, the restriction of F to M_H, considered as a map $M_H \to N^H$, is transverse to the submanifold P^H of N^H. Unfortunately, this stratumwise transversality is not a *generic* condition. Though the subspace of smooth equivariant maps which are stratumwise transverse to P is dense, it is generally not open, even if M is compact and P closed in N. Our generic notion of equivariant general position, defined locally by 1.1, has the property that an equivariant map F: $M \to N$ which is in general position with respect to an invariant submanifold P of N is automatically stratumwise transverse to P (§6).

§2 contains some motivating examples, and §3 some basic results on spaces of smooth equivariant maps. In §5 we recall the definition of the minimum Whitney stratification of a semialgebraic set, and show that general position of a smooth equivariant map with respect to an invariant submanifold of the

target is well defined. Definition 1.1 is shown there to be independent of the choice of F_i and h_i, and invariant under equivariant coordinate changes in the source and target.

DEFINITION 1.1′. If $W = W_1 \oplus W_2$ is a G-direct sum, then $F = (F_1, F_2)$: $V \to W_1 \oplus W_2$ is *in general position with respect to* $W_1 \subset W$ *at* $0 \in V$ if F_2 is in general position with respect to $0 \in W_2$ at $0 \in V$.

By invariance under equivariant coordinate changes, this definition also makes sense on G-manifolds, at a fixed point of the source. In general:

DEFINITION 1.1″. If F: $M \to N$ is a smooth equivariant map between G-manifolds, and P a G-submanifold of N, then F is *in general position with respect to* P *at* $x \in M$ if either $F(x) \notin P$, or $F(x) \in P$ and for any slice S for the orbit Gx at x, the G_x-equivariant map $F|S$: $S \to N$ is in general position with respect to P at $x \in S$ (G_x denotes the isotropy subgroup of x).

This definition is shown to be independent of the choice of slice.

DEFINITION 1.2. A smooth equivariant map F: $M \to N$ is *in general position with respect to* an invariant submanifold P of N if it is in general position with respect to P at each point of M.

Some elementary local properties of equivariant general position are established in §6. If $G = 1$, then general position = transversality. If F: $M \to N$ is in general position with respect to $P \subset N$ at $x \in M$, then it is also in general position at nearby points. If F is in general position with respect to P, then $F^{-1}(P)$ is "strongly stratified" (in the sense of Thom [21]) by invariant submanifolds. In fact, in the local situation of Definition 1.1, general position of $F(x) = \sum_{i=1}^{k} h_i(x)F_i(x)$ with respect to $0 \in W$, in a neighbourhood of $0 \in V$, is equivalent to transversality of graph h to $(U(x, h) = 0)$ in this neighbourhood; $F^{-1}(0)$ is the transversal intersection of the affine algebraic subset $(U(x, h) = 0)$ of $V \times \mathbf{R}^k$ by the diffeomorphism graph h of V into $V \times \mathbf{R}^k$.

§§7, 8 and 9 are devoted to equivariant analogues of the transversal openness, density and isotopy theorems. In the above local context, we pass from the equivariant map $F(x) = \sum_{i=1}^{k} h_i(x)F_i(x)$ in general position with respect to $0 \in W$, to the map graph h transverse to the affine algebraic variety $(U(x, h) = 0)$, and apply Thom's theorems on transversality to a stratified set, and his First Isotopy Lemma [20], [16]. The equivariant results are:

THEOREM 1.3. *If P is a closed G-submanifold of N, then the set of smooth equivariant maps F: $M \to N$ which are in general position with respect to P (respectively in general position at each point of a compact subset K of M) is open in the Whitney (respectively $\mathcal{C}^\infty$) topology.*

THEOREM 1.4. *If P is an invariant submanifold of N, then the set of smooth*

equivariant maps $F: M \to N$ which are in general position with respect to P is a countable intersection of open dense sets (in the Whitney of $\mathcal{C}^\infty$ topology).

THEOREM 1.5. *Let S, M, N be smooth G-manifolds, with G acting trivially on S, and M compact, and let P be a closed G-submanifold of N. Let F: $S \times M \to N$ be a smooth equivariant map, and $F_s(x) = F(s, x)$, where $(s, x) \in S \times M$. If F_{s_0} is in general position with respect to P, then there is an open neighbourhood W of s_0 in S, and an equivariant homeomorphism A: $W \times M \to W \times M$ covering the identity on W, such that $A|\{s_0\} \times M =$* id (id *denotes the identity map) and*

$$A\big((F|W \times M)^{-1}(P)\big) = W \times F_{s_0}^{-1}(P).$$

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A theory of "G-transversality" has also recently been announced in an independent work of Mike Field [9]. It may turn out that our definition and that of Field are equivalent, though an openness theorem has not yet been obtained for the latter.[1]

2. Examples.

EXAMPLE 2.1. Let $V = W = \mathbf{R}$, with Z_2 acting by reflection in the origin. We show that the set of smooth equivalent maps $V \to W$ which are stratumwise transverse to $0 \in W$ is not open in the Whitney topology on the space of smooth equivariant maps. Let f: $\mathbf{R} \to \mathbf{R}$ be a smooth function with $f(x) = 0$ for $x \leqslant 0$, $f(x) = 1$ for $x \geqslant 1$, and $0 < f(x) < 1$ for $0 < x < 1$. For any $a \geqslant 0$, define a smooth equivariant map F_a: $V \to W$ by $F_a(x) = f(x - a)$ for $x \geqslant 0$ and $F_a(x) = -F_a(-x)$ for $x < 0$. The sequence $\{F_{1/n}\}$ converges to F_0 in the Whitney topology as $n \to \infty$, but F_a is stratumwise transverse to $0 \in W$ if and only if $a = 0$.

EXAMPLE 2.2. Let $V = W = \mathbf{R}^2$, with Z_2 acting by reflection in the vertical axis. There certainly are equivariant maps $V \to W$ transverse to the fixed point set W^{Z_2} of W, but the set of such maps is not dense in the space of smooth equivariant maps. The map $(x_1, x_2) \mapsto (x_1^3 - x_1x_2, x_2)$, for instance, is equivariantly stable [5]; any sufficiently close smooth equivariant map has a cusp point whose image lies in W^{Z_2}. This map is in equivariant general position with respect to W^{Z_2}.

EXAMPLE 2.3. Let V, W be 2-dimensional linear Z_4-spaces, the actions of Z_4 on V, W generated respectively by 90°, 180° rotation about the origin. Any smooth equivariant map F: $V \to W$ can be written

[1](Added in proof.) M. Field has shown that the definitions are equivalent (*Stratifications of equivariant varieties*, preprint).

$$F(x_1, x_2) = \begin{pmatrix} h_{11}(x_1, x_2) & h_{12}(x_1, x_2) \\ h_{21}(x_1, x_2) & h_{22}(x_1, x_2) \end{pmatrix} \begin{pmatrix} x_1^2 - x_2^2 \\ 2x_1x_2 \end{pmatrix},$$

where the h_{ij} are smooth invariant functions on V. F is in general position with respect to $0 \in W$ at $0 \in V$ if and only if the matrix $(h_{ij}(0))$ is nonsingular. F is in general position with respect to the first coordinate axis of W at $0 \in V$ if and only if either $h_{21}(0)$ or $h_{22}(0)$ is nonzero. In this case the inverse image of the first coordinate axis (in a neighbourhood of 0) is a pair of smooth curves intersecting transversely at $0 \in V$.

EXAMPLE 2.4. REGULAR $O(n)$-MANIFOLDS. Let $O(n)$ act on the space $M(k, n)$ of k by n matrices by the standard action on each row vector (i.e. by matrix multiplication $(g, y) \mapsto yg^{-1}$, where $g \in O(n), y \in M(k, n)$). We consider the category of $O(n)$-manifolds whose slice representations are of the form $\mathbf{R}^a \times M(k, n)$, where a and k are nonnegative integers, and $O(n)$ acts trivially on $\mathbf{R}^a$ (see [6], [8]). The case $n = 1$ covers all Z_2-manifolds. Examining general position of equivariant maps with respect to an invariant submanifold of the target amounts to examining, in the local situation, general position of an equivariant map F: $\mathbf{R}^a \times M(k, n) \to \mathbf{R}^b \times M(l, n)$ with respect to the origin of the target, at the origin of the source. Denote points in $\mathbf{R}^a \times M(k, n)$ by pairs (x, y), where $x = (x_1, \ldots, x_a)$, and

$$y = \begin{bmatrix} y_1 \\ \vdots \\ y_k \end{bmatrix} = \begin{bmatrix} y_{11} & \cdots & y_{1n} \\ \vdots & & \vdots \\ y_{k1} & \cdots & y_{kn} \end{bmatrix}.$$

F can be written $F(x, y) = (f(x, y), H(x, y)y)$, where $f = (f_1, \ldots, f_b)$,

$$H = \begin{bmatrix} h_{11} & \cdots & h_{1k} \\ \vdots & & \vdots \\ h_{l1} & \cdots & h_{lk} \end{bmatrix},$$

and the f_i, h_{ij} are $O(n)$-invariant functions on the source (i.e. functions of x and the inner products $\langle y_\mu, y_\nu \rangle$). It is not hard to see that F is in general position with respect to $0 \in \mathbf{R}^b \times M(l, n)$ at $0 \in \mathbf{R}^a \times M(k, n)$ if and only if the map $x \mapsto (f(x, 0), H(x, 0))$ of $\mathbf{R}^a$ into $\mathbf{R}^b \times M(l, k)$ is transverse to the natural stratification of $\{0\} \times M(l, k)$ by matrix rank.

EXAMPLE 2.5. Consider S^1 acting on $\mathbf{C} = \mathbf{R}^2$ by the standard action, and on $\mathbf{C}^2$ by $e^{i\theta} \cdot (y_1, y_2) = (e^{i\theta}y_1, e^{2i\theta}y_2)$. Any smooth equivariant map F: $\mathbf{C} \to \mathbf{C}^2$ can be written $F(x) = (\alpha(\bar{x}x)x, \beta(\bar{x}x)x^2)$, where α, β are smooth $\mathbf{C}$-valued functions. F is in general position with respect to $0 \in \mathbf{C}^2$ at $0 \in \mathbf{C}$ if the map $x \mapsto (x, \alpha(\bar{x}x), \beta(\bar{x}x))$ is transverse at $0 \in \mathbf{C}$ to the affine algebraic variety $\{(x, \alpha, \beta) \in \mathbf{C}^3 | \alpha x = 0, \beta x^2 = 0\}$ (i.e. the union of the complex line ($\alpha = \beta$

$= 0$) and the complex plane ($x = 0$)). The map $x \mapsto (x, \alpha(\bar{x}x), \beta(\bar{x}x))$ is transverse to this variety at $x = 0$ as long as $\alpha(0)$, $\beta(0)$ are not both zero. In a subsequent paper we hope to refine the notion of equivariant general position (using a jet bundle version of the results in this paper), so that the map $x \mapsto (x, 0)$ of $\mathbf{C}$ into $\mathbf{C}^2$ will be in general position with respect to $0 \in \mathbf{C}^2$, whereas the map $x \mapsto (0, x^2)$ will not.[2]

EXAMPLE 2.6. Consider S^1 acting on $\mathbf{C}^2$ by $e^{i\theta} \cdot (x_1, x_2) = (e^{i\theta}x_1, e^{i\theta}x_2)$, and on $\mathbf{C}$ by $e^{i\theta} \cdot y = e^{4i\theta}y$. Any smooth equivariant map F: $\mathbf{C}^2 \to \mathbf{C}$ can be written $F(x_1, x_2) = \Sigma_{i=0}^4 h_i(\bar{x}_1x_1, \bar{x}_2x_2, \bar{x}_1x_2)x_1^{4-i}x_2^i$, where the h_i are smooth $\mathbf{C}$-valued functions. Let $F_t(x_1, x_2) = F(t, x_1, x_2) = x_1x_2(x_2 - x_1)(x_2 - (3 + t)x_1)$, $t \in (-1, 1)$. Then $F_0(x_1, x_2) = 3x_1^3x_2 - 4x_1^2x_2^2 + x_1x_2^3$ is in general position with respect to $0 \in \mathbf{C}$ at $0 \in \mathbf{C}^2$, since in a neighbourhood U of $((0, 0), (0, 3, -4, 1, 0)) \in \mathbf{C}^2 \times \mathbf{C}^5$, $U \cap \mathbf{C}^5$ is the stratum of $(\Sigma_{i=0}^4 h_i x_1^{4-i}x_2^i = 0)$ containing this point. For $t \in (-1, 1)$, $F_t^{-1}(0)$ is the union of 4 complex lines intersecting at $0 \in \mathbf{C}^2$. These lines have a cross ratio which provides an obstruction to the $\mathcal{C}^1$ local triviality of the stratification of $F^{-1}(0) \subset (-1, 1) \times \mathbf{C}^2$ [26, §13], [16, §8]. Hence the ambient isotopy of Theorem 1.5 cannot, in general, be smooth (the author is grateful to Mike Field for pointing this out to him).

3. **Spaces of equivariant maps.** Throughout the paper, G denotes a compact Lie group, and M, N smooth G-manifolds. We consider the space $\mathcal{C}_G^\infty(M, N)$ of smooth equivariant maps from M to N, with the $\mathcal{C}^\infty$, or with the Whitney topology (the latter is defined in [14]). In the $\mathcal{C}^\infty$ topology, $\mathcal{C}_G^\infty(M, N)$ is a complete metric space, hence Baire.

PROPOSITION 3.1. *$\mathcal{C}_G^\infty(M, N)$ is a Baire space in the Whitney topology.*

Mather's proof [15, Proposition 3.1] for $G = 1$, goes through in the equivariant case.

Let V, W be linear G-spaces, and $\mathcal{O}$ an invariant open neighbourhood of $0 \in V$. The module $\mathcal{C}_G^\infty(\mathcal{O}, W)$ is finitely generated by polynomial maps over the ring $\mathcal{C}_G^\infty(\mathcal{O})$ of smooth invariant functions on $\mathcal{O}$ ($\mathcal{C}_G^\infty(\mathcal{O}) = \mathcal{C}_G^\infty(\mathcal{O}, \mathbf{R})$, G acting trivially on $\mathbf{R}$); the proof, due to Malgrange, using G. W. Schwarz's smooth invariant theorem [19], is contained in that of Proposition 3.2 below. Let $F_1, \ldots, F_k$ be a finite set of polynomial generators. *In the $\mathcal{C}^\infty$ topology*, the map $(h_1, \ldots, h_k) \mapsto \Sigma_{i=1}^k h_iF_i$ of $\mathcal{C}_G^\infty(\mathcal{O})^k$ onto $\mathcal{C}_G^\infty(\mathcal{O}, W)$ is a continuous linear surjection of Fréchet spaces, hence *open* by the Open Mapping Theorem. Though this fact will suffice for our purposes, for completeness we

[2](Added in proof.) We have given this refinement in *Generic equivariant maps*, Singularities of Real/Complex Maps, Oslo 1976, Noordhoff, Leyden, The Netherlands, 1977. Example 1.2 in that paper is a counterexample to the theorem announced by M. Field in C. R. Acad. Sci. Paris **282** (1976), A379–A380. MR **53** #4124.

give the following more general result (depending on a recent theorem of Mather [12]):

PROPOSITION 3.2. *In the $\mathcal{C}^\infty$ topology, the surjection $(h_1, \ldots, h_k) \mapsto \Sigma_{i=1}^k h_i F_i$ of $\mathcal{C}_G^\infty(\mathcal{O})^k$ onto $\mathcal{C}_G^\infty(\mathcal{O}, W)$ has a continuous section. In fact given any $F = \Sigma_{i=1}^k h_i F_i \in \mathcal{C}_G^\infty(\mathcal{O}, W)$, where $h_i \in \mathcal{C}_G^\infty(\mathcal{O})$, there is a continuous section S with $S(F) = (h_1, \ldots, h_k)$.*

PROOF. The second statement follows trivially from the first. It suffices to prove the first statement for any given set of generators $F_1, \ldots, F_k$. For let $F_1', \ldots, F_{k'}'$ be another set of generators, and $S = (S_1, \ldots, S_k)$: $\mathcal{C}_G^\infty(\mathcal{O}, W) \to \mathcal{C}_G^\infty(\mathcal{O})^k$ a continuous section for the surjection $(h_1, \ldots, h_k) \mapsto \Sigma_{i=1}^k h_i F_i$. We can write $F_i = \Sigma_{j=1}^{k'} h_{ji} F_j'$, $i = 1, \ldots, k$, where $h_{ji} \in \mathcal{C}_G^\infty(\mathcal{O})$, so that for any $F \in \mathcal{C}_G^\infty(\mathcal{O}, W)$,

$$F = \sum_{i=1}^{k} S_i(F) F_i = \sum_{i=1}^{k'} \sum_{j=1}^{k} h_{ij} S_j(F) F_i'.$$

The map $T = (T_1, \ldots, T_{k'})$: $\mathcal{C}_G^\infty(\mathcal{O}, W) \to \mathcal{C}_G^\infty(\mathcal{O})^{k'}$, defined by $T_i(F) = \Sigma_{j=1}^k h_{ij} S_j(F)$, $i = 1, \ldots, k'$, is a continuous section for the surjection $(h_1, \ldots, h_{k'}) \mapsto \Sigma_{i=1}^{k'} h_i F_i'$.

We demonstrate the existence of a continuous section using the set of generators constructed by Malgrange. The action of G on W induces an action on the dual space W^*, given by $(g\lambda)(y) = \lambda(g^{-1}y)$, where $g \in G, y \in W, \lambda \in W^*$. Let $P = (p_1, \ldots, p_k)$: $V \times W^* \to \mathbf{R}^k$ be the orbit map, given by a finite set of generators $p_1, \ldots, p_k$ for the algebra of invariant polynomials on $V \times W^*$. Let $\mathcal{U}$ be an open subset of $\mathbf{R}^k$, with $P(\mathcal{O} \times W^*) = \mathcal{U} \cap P(V \times W^*)$. The map P^*: $\mathcal{C}^\infty(\mathcal{U}) \to \mathcal{C}_G^\infty(\mathcal{O} \times W^*)$, given by composition with P, is surjective by [19], and has a continuous section s: $\mathcal{C}_G^\infty(\mathcal{O} \times W^*) \to \mathcal{C}^\infty(\mathcal{U})$ by [12]. The map $F \mapsto f$ of $\mathcal{C}_G^\infty(\mathcal{O}, W)$ into $\mathcal{C}_G^\infty(\mathcal{O} \times W^*)$, defined by $f(x, \lambda) = \lambda(F(x))$, where $(x, \lambda) \in \mathcal{O} \times W^*$, is continuous in the $\mathcal{C}^\infty$ topology, as is the map D'': $\mathcal{C}_G^\infty(\mathcal{O} \times W^*) \to \mathcal{C}_G^\infty(\mathcal{O} \times W^*, W)$, given by differentiation with respect to the second variable (we have identified W with the G-space of linear maps $L(W^*, \mathbf{R})$). Writing $f(x, \lambda) = s(f) \circ P(x, \lambda)$, and differentiating with respect to λ, at $\lambda = 0$, we have

$$F(x) = \sum_{i=1}^{k} D_i s(f)(P(x, 0)) D'' p_i(x, 0),$$

where D_i denotes the ith partial derivative. Then $F_i(x) = D'' p_i(x, 0)$, $i = 1, \ldots, k$, generate $\mathcal{C}_G^\infty(\mathcal{O}, W)$ over $\mathcal{C}_G^\infty(\mathcal{O})$, and $S = (S_1, \ldots, S_k)$: $\mathcal{C}_G^\infty(\mathcal{O}, W) \to \mathcal{C}_G^\infty(\mathcal{O})^k$, defined by $S_i(F)(x) = D_i s(f)(P(x, 0))$, $i = 1, \ldots, k$, is a continuous section for the surjection $(h_1, \ldots, h_k) \mapsto \Sigma_{i=1}^k h_i F_i$ of $\mathcal{C}_G^\infty(\mathcal{O})^k$ onto $\mathcal{C}_G^\infty(\mathcal{O}, W)$.

We will need the following result on parametrized families of equivariant maps:

PROPOSITION 3.3. *Let V, W be linear G-spaces, $\mathcal{O}$ an invariant open neighbourhood of $0 \in V$, and $F_1, \ldots, F_k$ a set of generators for $\mathcal{C}_G^\infty(\mathcal{O}, W)$ over $\mathcal{C}_G^\infty(\mathcal{O})$. Then $F_1, \ldots, F_k$ also generate $\mathcal{C}_G^\infty(\mathbf{R}^q \times \mathcal{O}, W)$ over $\mathcal{C}_G^\infty(\mathbf{R}^q \times \mathcal{O})$, where G acts trivially on $\mathbf{R}^q$.*

PROOF. Observe that it again suffices to prove the result for any given set of generators. The proposition when follows from Malgrange's construction, as in the proof of Proposition 3.2, since the invariants of $\mathbf{R}^q \times V \times W^*$ are generated by the invariants of $V \times W^*$ together with the coordinate functions on $\mathbf{R}^q$.

Let V, W be linear G-spaces. Let $\mathcal{A}$ be the ring of germs at $0 \in V$ of smooth invariant functions, and $\mathfrak{M}$ the finitely generated $\mathcal{A}$-module of germs at $0 \in V$ of smooth equivariant maps $V \to W$. $\mathcal{A}$ is a local ring, its unique maximal ideal $\mathfrak{m}$ comprising germs of invariant functions vanishing at the origin. Then $\mathcal{A}/\mathfrak{m} \cong \mathbf{R}$, and $\mathfrak{M}/\mathfrak{m} \cdot \mathfrak{M}$ is a finite dimensional vector space over the field $\mathcal{A}/\mathfrak{m}$, of dimension d say. The following proposition allows the selection of a minimal subset of any set of generators of $\mathfrak{M}$ over $\mathcal{A}$:

PROPOSITION 3.4. *Let $\mathcal{O}$ be an invariant open neighbourhood of $0 \in V$, and $F_1, \ldots, F_k$ a set of generators for $\mathcal{C}_G^\infty(\mathcal{O}, W)$ over $\mathcal{C}_G^\infty(\mathcal{O})$. Then $k \geqslant d$, and (after reordering) $F_1, \ldots, F_d$ generate $\mathfrak{M}$ over $\mathcal{A}$. If $F = \sum_{i=1}^d h_i F_i \in \mathfrak{M}$, where $h_i \in \mathcal{A}$, then the $h_i(0)$ are uniquely determined by the F_i and F.*

PROOF. The images of $F_1, \ldots, F_k$ in $\mathfrak{M}/\mathfrak{m} \cdot \mathfrak{M}$ span $\mathfrak{M}/\mathfrak{m} \cdot \mathfrak{M}$ over $\mathcal{A}/\mathfrak{m}$, so that $k \geqslant d$, and (after reordering) the images of $F_1, \ldots, F_d$ form a basis. The proposition follows immediately from Nakayama's lemma (see [24, Chapter I, Corollary 1.2]).

4. The variety associated with a space of equivariant maps. Let V, W be linear G-spaces, and $\mathcal{O}$ an invariant open neighbourhood of $0 \in V$. Let $F_1, \ldots, F_k$ be a set of polynomial generators for $\mathcal{C}_G^\infty(\mathcal{O}, W)$ over $\mathcal{C}_G^\infty(\mathcal{O})$. Any smooth equivariant map $F \in \mathcal{C}_G^\infty(\mathcal{O}, W)$ can be written

$$F(x) = \sum_{i=1}^{k} h_i(x) F_i(x) = U \circ \operatorname{graph} h(x),$$

where $h = (h_1, \ldots, h_k) \in \mathcal{C}_G^\infty(\mathcal{O})^k$, graph $h = (\mathrm{id}, h)\colon \mathcal{O} \to \mathcal{O} \times \mathbf{R}^k$, and $U\colon \mathcal{O} \times \mathbf{R}^k \to W$ is defined by $U(x, h) = \sum_{i=1}^k h_i F_i(x)$.

PROPOSITION 4.1. *The analytic isomorphism class of the germ at $(x = 0)$ of the G-invariant real affine algebraic subvariety $(U(x, h) = 0)$ of $\mathcal{O} \times \mathbf{R}^k$ is uniquely determined (up to product with an affine space) by V, W.*

PROOF. First, in the notation preceding Proposition 3.4, let $F_1, \ldots, F_d$ and $G_1, \ldots, G_d$ be 2 minimal sets of polynomial generators for $\mathfrak{M}$ over $\mathfrak{A}$. There are G-invariant convergent power series $\alpha_{ij}(x)$, $i, j = 1, \ldots, d$, such that $F_i(x) = \sum_{j=1}^d \alpha_{ji}(x) G_j(x)$, $i = 1, \ldots, d$ (that the α_{ij} can be chosen analytic, and, in fact, algebraic over the ring of polynomials, follows from implicit function theorems of M. Artin [2], [3]). The matrix $(\alpha_{ij}(0))$ is invertible by Proposition 3.4. We have

$$(*) \qquad \sum_{i=1}^d h_i F_i(x) = \sum_{i=1}^d \sum_{j=1}^d \alpha_{ij}(x) h_j G_i(x) = \sum_{i=1}^d k_i(x, h) G_i(x),$$

where $h = (h_1, \ldots, h_d)$ and $k_i(x, h) = \sum_{j=1}^d \alpha_{ij}(x) h_j$. Writing $k(x, h) = (k_1(x, h), \ldots, k_d(x, h))$, the map A: $(x, h) \mapsto (x, k(x, h))$ is an equivariant analytic isomorphism in some invariant neighbourhood of $(x = 0) \subset \mathcal{O} \times \mathbf{R}^d$, since $(\alpha_{ij}(0))$ is invertible. By $(*)$, A defines an isomorphism of the germs of $(\sum_{i=1}^d h_i F_i(x) = 0)$ and $(\sum_{i=1}^d h_i G_i(x) = 0)$ at $(x = 0)$.

Now let $F_1, \ldots, F_k$ be any set of polynomial generators for $\mathcal{C}_G^\infty(\mathcal{O}, W)$. By Proposition 3.4 we can assume that $F_1, \ldots, F_d$ generate $\mathfrak{M}$ over $\mathfrak{A}$. There are G-invariant convergent power series $f_{ji}(x)$, $j = 1, \ldots, d$, $i = d + 1, \ldots, k$ (which are, again, algebraic over the ring of polynomials), such that $F_i(x) = \sum_{j=1}^d f_{ji}(x) F_j(x)$, $i = d + 1, \ldots, k$. Then

$$\sum_{i=1}^k h_i F_i(x) = \sum_{i=1}^d \left(h_i + \sum_{j=d+1}^k f_{ij}(x) h_j \right) F_i(x).$$

The map A: $(x, h) \mapsto (x, k(x, h))$, defined by

$$k_i(x, h) = h_i - \sum_{j=d+1}^k f_{ij}(x) h_j, \qquad i = 1, \ldots, d,$$

$$k_i(x, h) = h_i, \qquad i = d + 1, \ldots, k,$$

is an equivariant analytic isomorphism in an invariant neighbourhood of $(x = 0) \subset \mathcal{O} \times \mathbf{R}^k$, inducing an isomorphism from $(\sum_{i=1}^d h_i F_i(x) = 0) \times \mathbf{R}^{k-d}$ to $(\sum_{i=1}^k h_i F_i(x) = 0)$. This completes the proof.

5. **Equivariant general position.** We begin by recalling some basic facts about stratified sets. A *stratification* of a subset of E of $\mathbf{R}^q$ is a locally finite partition of E into connected manifolds, called the *strata*, such that frontier $\mathrm{Cl}(X) - X$ of each stratum X is the union of a set of lower dimensional strata ($\mathrm{Cl}(X)$ denotes the closure of X in E). A *Whitney stratification* of E has the additional property that the following Condition B of Whitney is satisfied by every pair (X, Y) of strata, with X in the frontier of Y:

Condition B. Let x be any point of X. Let $\{x_i\}$, $\{y_i\}$ be any sequences of points in X, Y (respectively), such that each sequence converges to x, the line

joining x_i and y_i converges (in projective space $\mathbf{P}^{q-1}$) to a line l, and TY_{y_i} converges (in the Grassmannian of (dim Y)-planes) to a plane τ. Then $l \subset \tau$.

To a Whitney stratification $\mathcal{S}$ of E is associated a *filtration by dimension*, defined by letting E_j be the union of the strata of dimension at most j [16]. Suppose $\mathcal{S}$ and $\mathcal{S}'$ are two Whitney stratifications of E, and $\{E_j\}$, $\{E_j'\}$ the associated filtrations. Say that $\mathcal{S} < \mathcal{S}'$ if there exists an integer j such that $E_j \subset E_j'$, and $E_k = E_k'$ for $k > j$.

A semialgebraic subset of E of $\mathbf{R}^q$ has a canonical "minimum" Whitney stratification $\mathcal{S}$, as constructed inductively by Mather [16] or Lojasiewicz [11]. Each stratum of $\mathcal{S}$ is a semialgebraic submanifold. "Minimum" means that if $\mathcal{S}'$ is any other Whitney stratification by smooth manifolds, then $\mathcal{S} \leq \mathcal{S}'$.

If X is a smooth manifold then by transversality of a smooth map f: $X \to \mathbf{R}^q$ to an algebraic subvariety E of $\mathbf{R}^q$, we mean transversality of f to each stratum of the minimum Whitney stratification of E ([22], [20, §II, D]). By Whitney's Condition A, if f is transverse to a stratum Y of E, then f is transverse to each stratum in the star of Y, in some neighbourhood of Y (since Condition A is weaker than B [16], we skip its definition here).

Now let V, W be linear G-spaces, and $\mathcal{O}$ an invariant open neighbourhood of $0 \in V$. We again consider a smooth equivariant map F: $\mathcal{O} \to W$, and write $F = U \circ \text{graph } h$, where $h = (h_1, \ldots, h_k)$, $h_i \in \mathcal{C}_G^\infty(\mathcal{O})$, and $U(x, h) = \Sigma_{i=1}^k h_i F_i(x)$, with $F_1, \ldots, F_k$ a set of polynomial generators for $\mathcal{C}_G^\infty(\mathcal{O}, W)$ over $\mathcal{C}_G^\infty(\mathcal{O})$. The G-invariant affine algebraic subvariety $(U(x, h) = 0)$ of $\mathcal{O} \times \mathbf{R}^k$ has a canonical minimum Whitney stratification, which is clearly G-invariant (cf. the proof of Proposition 6.1 below).

PROPOSITION 5.1. *The Definition* 1.1 *of equivariant general position is independent of the choice of F_i and h_i.*

PROOF. STEP 1. In the notation preceding Proposition 3.4 and in the proof of Proposition 4.1, let $F_1, \ldots, F_d$ and $G_1, \ldots, G_d$ be 2 minimal sets of polynomial generators for $\mathfrak{M}$ over $\mathcal{Q}$. By the first part of the proof of Proposition 4.1, the map $x \mapsto (x, h(x))$ is transverse to the affine algebraic variety $(\Sigma_{i=1}^d h_i F_i(x) = 0)$ at $x = 0$, if and only if the map $x \mapsto (x, k(x, h(x)))$ is transverse to $(\Sigma_{i=1}^d h_i G_i(x) = 0)$ at $x = 0$.

STEP 2. Let $F_1, \ldots, F_k$ be a set of polynomial generators for $\mathcal{C}_G^\infty(\mathcal{O}, W)$ over $\mathcal{C}_G^\infty(\mathcal{O})$, and assume that $F_1, \ldots, F_d$ generate $\mathfrak{M}$ over $\mathcal{Q}$. By the second part of the proof of Proposition 4.1, $x \mapsto (x, h_1(x), \ldots, h_k(x))$ is transverse to $(\Sigma_{i=1}^k h_i F_i(x) = 0)$ at $x = 0$, if and only if

$$x \mapsto \left(x, h_1(x) + \sum_{j=d+1}^{k} f_{1j}(x) h_j(x), \ldots, h_d(x) + \sum_{j=d+1}^{k} f_{dj}(x) h_j(x)\right)$$

is transverse to $(\Sigma_{i=1}^d h_i F_i(x) = 0)$ at $x = 0$.

STEP 3. We now show that Definition 1.1 is independent of the choice of h_i for a special choice of generators $G_1, \ldots, G_d$ for $\mathfrak{M}$ over $\mathcal{Q}$. We write $V = V^G \oplus V'$, $W = W^G \oplus W'$, where V', W' are the unique linear G-subspaces complementary to the fixed point sets V^G, W^G in V, W. Denote points $x \in V$ by pairs $x = (x_1, x_2) \in V^G \oplus V'$. Then $G_1, \ldots, G_d$ are defined as follows: $G_i(x_1, x_2) = e_i$, $i = 1, \ldots, b$, where $e_1, \ldots, e_b$ is a basis for W^G, and $G_i(x_1, x_2) = G_i(x_2)$, $i = b + 1, \ldots, d$, where $G_{b+1}(x_2), \ldots, G_d(x_2)$ form a minimal set of polynomial generators for germs of equivariant maps $V' \to W'$ at $0 \in V'$. By Propositions 3.3 and 3.4, any $F \in \mathfrak{M}$ can be written

$$F(x_1, x_2) = \sum_{i=1}^{b} h_i(x_1, x_2)e_i + \sum_{i=b+1}^{d} h_i(x_1, x_2)G_i(x_2),$$

where $h_i(x_1, x_2)$, $i = 1, \ldots, b$, and $h_i(x_1, 0)$, $i = b + 1, \ldots, d$, are uniquely determined by the G_i and F. On the other hand, since $(V')^G = \{0\}$, we can write $h_i(x_1, x_2) = H_i(x_1, p_1(x_2), \ldots, p_\mu(x_2))$, $i = 1, \ldots, d$, where $p_1, \ldots, p_\mu$ is a set of homogeneous generators, of degree at least 2, for the algebra of invariant polynomials on V'. It follows that the derivative of $h = (h_1, \ldots, h_d)$ with respect to $x = (x_1, x_2)$ at $x = 0$ is uniquely determined.

STEP 4. Finally, let $F_1, \ldots, F_k$ and $F_1', \ldots, F_l'$ be 2 sets of polynomial generators for $\mathcal{C}_G^\infty(\mathcal{O}, W)$ over $\mathcal{C}_G^\infty(\mathcal{O})$, and assume that $F_1, \ldots, F_d$ and $F_1', \ldots, F_d'$ are sets of generators for $\mathfrak{M}$ over $\mathcal{Q}$. Suppose $F(x) = \Sigma_{i=1}^k h_i(x)F_i(x) = \Sigma_{i=1}^l h_i'(x)F_i'(x)$. Define

$$H_i(x) = h_i(x) + \sum_{j=d+1}^{k} f_{ij}(x)h_j(x), \qquad i = 1, \ldots, d,$$

and

$$H_i'(x) = h_i'(x) + \sum_{j=d+1}^{l} f_{ij}'(x)h_j'(x), \qquad i = 1, \ldots, d,$$

where the f_{ij} (respectivly f_{ij}') are chosen for the F_i (respectively F_i') as in Step 2. By Step 2, the map $x \mapsto (x, h(x))$ (respectively $x \mapsto (x, h'(x))$) is transverse to $(\Sigma_{j=1}^k h_i F_i(x) = 0)$ (respectively to $(\Sigma_{i=1}^l h_i F_i'(x) = 0)$) at $x = 0$, if and only if the map $x \mapsto (x, H(x))$ (respectively $x \mapsto (x, H'(x))$) is transverse to $(\Sigma_{i=1}^d h_i F_i(x) = 0)$ (respectively to $(\Sigma_{i=1}^d h_i F_i'(x) = 0)$) at $x = 0$. Choose $G_1, \ldots, G_d$ as in Step 3, and $k(x, h)$ (respectively $k'(x, h)$) as in Step 1 for $F_1, \ldots, F_d$ (respectively $F_1', \ldots, F_d'$). By Step 1, $x \mapsto (x, H(x))$ (respectively $x \mapsto (x, H'(x))$) is transverse to $(\Sigma_{i=1}^d h_i F_i(x) = 0)$ (respectively to $(\Sigma_{i=1}^d h_i F_i'(x) = 0)$) at $x = 0$, if and only if $x \mapsto (x, k(x, H(x)))$ (respectively $x \mapsto (x, k'(x, H'(x))))$ is transverse to $(\Sigma_{i=1}^d h_i G_i(x) = 0)$ at $x = 0$. But the latter two conditions are equivalent by Step 3. This completes the proof.

The following two propositions show that the definition of general position

is invariant under equivariant diffeomorphisms of the source and target.

PROPOSITION 5.2. *If $F \in \mathcal{C}_G^\infty(\mathcal{O}, W)$ is in general position with respect to $0 \in W$ at $0 \in \mathcal{O}$, and α is a G-diffeomorphism of $\mathcal{O}$ into V with $\alpha(0) = 0$, then $F \circ \alpha$ is in general position with respect to $0 \in W$ at $0 \in \mathcal{O}$.*

PROOF. Let $F_1, \ldots, F_d$ be a minimal set of polynomial generators for $\mathfrak{M}$ over $\mathcal{Q}$, and $F(x) = \sum_{i=1}^d h_i(x)F_i(x)$, where $h_i \in \mathcal{Q}$, $i = 1, \ldots, d$. We can write $F_i \circ \alpha(x) = \sum_{j=1}^d \gamma_{ji}(x)F_j(x)$, $i = 1, \ldots, d$, where $\gamma_{ij} \in \mathcal{Q}$, $i, j = 1, \ldots, d$. It is easily checked that the matrix $(\gamma_{ij}(0))$ is invertible. Now

$$F \circ \alpha(x) = \sum_{i=1}^d h_i(\alpha(x))F_i(\alpha(x)) = \sum_{i=1}^d k_i(x)F_i(x),$$

where $k_i(x) = \sum_{j=1}^d \gamma_{ij}(x)h_j(\alpha(x))$, $i = 1, \ldots, d$. We must show that graph k is transverse to $(\sum_{i=1}^d h_iF_i(x) = 0)$ at $x = 0$. Consider the equivariant map $A(x, h) = (\alpha^{-1}(x), K(x, h))$, where $K = (K_1, \ldots, K_d)$ and $K_i(x, h) = \sum_{j=1}^d \gamma_{ij}(\alpha^{-1}(x))h_j$, $i = 1, \ldots, d$. A is a G-diffeomorphism of $\mathcal{U} \times \mathbf{R}^d$ into $V \times \mathbf{R}^d$, for some invariant open neighbourhood $\mathcal{U}$ of 0 in V, and A restricts to an isomorphism of $(\sum_{i=1}^d h_iF_i(x) = 0)$, since

$$\sum_{i=1}^d h_iF_i(x) = \sum_{i=1}^d \sum_{j=1}^d \gamma_{ij}(\alpha^{-1}(x))h_jF_i(\alpha^{-1}(x)).$$

The result follows since graph $k = A \circ (\text{graph } h) \circ \alpha$.

PROPOSITION 5.3. *If $F \in \mathcal{C}_G^\infty(\mathcal{O}, W)$ is in general position with respect to $0 \in W$ at $0 \in \mathcal{O}$, and β is a G-diffeomorphism of W with $\beta(0) = 0$, then $\beta \circ F$ is in general position with respect to $0 \in W$ at $0 \in \mathcal{O}$.*

PROOF. Let $F_1, \ldots, F_d$ be a minimal set of polynomial generators for $\mathfrak{M}$ over $\mathcal{Q}$, and $F(x) = \sum_{i=1}^d h_i(x)F_i(x)$, where $h_i \in \mathcal{Q}$, $i = 1, \ldots, d$. We can write $\beta(\sum_{i=1}^d h_iF_i(x)) = \sum_{i=1}^d \gamma_i(x, h)F_i(x)$, with γ_i smooth invariant functions in a neighbourhood of $(x = 0) \subset V \times \mathbf{R}^d$. Then $(x, h) \mapsto (x, \gamma(x, h))$ is a G-diffeomorphism near $(x = 0) \subset V \times \mathbf{R}^d$, which restricts to an isomorphism of $(\sum_{i=1}^d h_iF_i(x) = 0)$. If $x \mapsto (x, h(x))$ is transverse to $(\sum_{i=1}^d h_iF_i(x) = 0)$ at $x = 0$, then so is $x \mapsto (x, \gamma(x, h(x)))$, so that $\beta \circ F(x) = \sum_{i=1}^d \gamma_i(x, h(x))F_i(x)$ is in general position with respect to $0 \in W$ at $0 \in \mathcal{O}$.

Let M, N be smooth G-manifolds, and P a smooth G-submanifold of N. Let $x \in M$ be a fixed point. Recall Definition 1.1′.

DEFINITION 5.4. A smooth equivariant map $F: M \to N$ is *in general position with respect to $P \subset N$ at $x \in M$* if either $F(x) \notin P$, or $F(x) \in P$ and the following condition is satisfied: if ϕ, ψ are G-diffeomorphisms of linear G-spaces V, W (respectively) into M, N (respectively), such that (i) $\phi(0) = x$, and (ii) W is a G-direct sum $W = W_1 \oplus W_2$, with $F(x) \in \psi(W_1) = P \cap$

$\psi(W)$, then $\psi^{-1} \circ F \circ \phi$ (defined in an invariant neighbourhood of $0 \in V$) is in general position with respect to $W_1 \subset W$ at $0 \in V$.

It is easily verified, using Propositions 5.2 and 5.3, that Definition 5.4 is independent of the choice of ϕ, ψ.

We finally show that Definition 1.1″ of general position with respect to an invariant submanifold, is independent of the choice of slice. Let H be a closed subgroup of G, and W a linear H-space. The G-vector bundle $G \times_H W \to G/H$ is the bundle with fibre W associated to the principal bundle $G \to G/H$. Here $G \times_H W$ denotes the orbit space of $G \times W$, under the action of H defined by $h(g, w) = (gh^{-1}, hw)$, where $h \in H$ and $(g, w) \in G \times W$; $[g, w]$ denotes the orbit of (g, w).

LEMMA 5.5. *There is an H-invariant neighbourhood U of the identity coset $1H \in G/H$, and an H-vector bundle equivalence $G \times_H W|U \to U \times W$ over U, taking $G \times_H Q$ onto $U \times Q$, for any H-submanifold Q of W.*

PROOF. The canonical projection $G \to G/H$ is equivariant with respect to the action of H on G by conjugation, and on G/H by left translation. There is an H-invariant open neighbourhood U of $1H \in G/H$, and an H-equivariant local section σ: $U \to G$, with $\sigma(1H) = 1$ (this follows, for example, from [4, p. 622]). The map $[g, w] \mapsto (gH, \sigma(gH)^{-1}gw)$ of $G \times_H W|U$ onto $U \times W$ defines the required H-vector bundle equivalence (its inverse is $(gH, w) \mapsto [\sigma(gH), w]$).

PROPOSITION 5.6. *The Definition 1.1″ of general position is independent of the choice of slice.*

PROOF. Let M, N be G-manifolds, P a G-submanifold of N, and F: $M \to N$ a smooth equivariant map. Note that if S is a slice for the orbit Gx at $x \in M$, then $F|S$: $S \to N$ is in general position with respect to P at x, if and only if (graph $F)|S$: $S \to M \times N$ is in general position with respect to $M \times P$ at x.

Fix G-invariant Riemannian metrics on M, N. Let $H = G_x$, the isotropy subgroup of x. Denote by V the orthogonal H-space $TM_x/T(Gx)_x$, and by $D(V)$ the closed unit disk in V, Int $D(V)$ the open unit disk. Consider 2 invariant tubular neighbourhoods Φ, Ψ: $G \times_H V \to M$ for the orbit Gx (mapping the identity coset $1H$ of the zero section G/H to x). The images of $\Phi|V$, $\Psi|V$, in other words, are 2 slices for the orbit Gx at x. Let $\phi = \Phi|D(V)$, $\psi = \Psi|D(V)$. By the uniqueness of invariant tubular neighbourhoods [7, Chapter VI, Theorem 2.6] and the equivariant ambient isotopy theorem [7, Chapter VI, Theorem 3.1], there is an orthogonal G-vector bundle equivalence $\Theta = G \times_H \theta$ of $G \times_H V$ (θ: $V \to V$ is an orthogonal H-isomorphism), and a smooth equivariant isotopy ω_t: $M \to M$, $t \in [0, 1]$, such that $\omega_0 = \mathrm{id}$ and $\psi \circ \theta = \omega_1 \circ \phi$. Let $\phi_t = \omega_t \circ \phi$, so that $\phi_0 = \phi$, $\phi_1 = \psi \circ \theta$.

Assuming that $E_0 = (\text{graph } F) \circ \phi_0 | \text{Int } D(V)$: $\text{Int } D(V) \to M \times N$ is in general position with respect to $M \times P$ at $0 \in V$, we must show that $E_1 = (\text{graph } F) \circ \phi_1 | \text{Int } D(V)$ is in general position with respect to $M \times P$ at $0 \in V$.

We may assume that E_0 is an H-equivariant embedding of $\text{Int } D(V)$ in a slice for the orbit $G(x, F(x))$ at $(x, F(x)) \in M \times N$, and may identify this slice with an orthogonal H-space W. Identify the invariant tubular neighbourhood corresponding to the slice W with $G \times_H W$, and let P: $G \times_H W | U \to W$ be the composition of the H-equivalence $G \times_H W | U \to U \times W$ given by Lemma 5.5, and the projection $U \times W \to W$. We may assume also that E_1 embeds $\text{Int } D(V)$ in $G \times_H W | U$. Note that an H-equivariant map α: $\text{Int } D(V) \to G \times_H W | U \subset M \times N$, with $\alpha(0) = [1H, 0]$, is in general position with respect to $M \times P$ at $0 \in V$, if and only if $P \circ \alpha$: $\text{Int } D(V) \to W$ is in general position with respect to $W \cap (M \times P)$ at $0 \in V$.

Again by the equivariant ambient isotopy theorem, let Ω_t be a smooth equivariant isotopy of $M \times N$, such that $\Omega_t \circ \text{graph } F = (\text{graph } F) \circ \omega_t$ (in a suitable compact neighbourhood of $\Phi(G \times_H D(V)) \cup \Psi(G \times_H D(V))$). Then $E_1 = (\text{graph } F) \circ \omega_1 \circ \phi_0 = \Omega_1 \circ E_0$. We must show that $P \circ E_1$: $\text{Int } D(V) \to W$ is in general position with respect to $W \cap (M \times P)$ at $0 \in V$. Write this map as the composition of E_0 on $E_0^{-1} \circ P \circ \Omega_1 \circ E_0$. Note that, by the property of P given in Lemma 5.5, the composition $E_0^{-1} \circ P \circ \Omega_1 \circ E_0$ is well defined, and the present proposition follows from Proposition 5.2 if we show that $P \circ \Omega_1 | W$ is an H-diffeomorphism in a neighbourhood of $0 \in W$. Identify $G \times_H W | U$ with $U \times W$ by Lemma 5.5. The derivative of the H-diffeomorphism Ω_1 at $(1H, 0) \in U \times W$ can be written in the following block form with respect to the product $U \times W$:

$$\left(\begin{array}{c|c} \text{id} & * \\ \hline 0 & D(P \circ \Omega_1 | W)(0) \end{array}\right).$$

Hence the derivative $D(P \circ \Omega_1 | W)(0)$ is nonsingular, and the result follows from the inverse function theorem.

6. **Elementary properties.** Let M, N be smooth G-manifolds, and P a G-submanifold of N. Recall (Definition 1.2) that a smooth equivariant map F: $M \to N$ is in general position with respect to P if it is in general position at each point of M. It is immediate from Definition 1.1 that if $G = 1$, then general position = transversality.

PROPOSITION 6.1. *Let V, W be linear G-spaces, and $\mathcal{O}$ an invariant open neighbourhood of $0 \in V$. Let $F_1, \ldots, F_k$ be a set of polynomial generators for $\mathcal{C}_G^\infty(\mathcal{O}, W)$ over $\mathcal{C}_G^\infty(\mathcal{O})$, and $F = U \circ \text{graph } h \in \mathcal{C}_G^\infty(\mathcal{O}, W)$, where $h(x) =$*

$(h_1(x), \ldots, h_k(x)) \in \mathcal{C}_G^\infty(\mathcal{O})^k$, *and* $U(x, h) = \sum_{i=1}^k h_i F_i(x)$. *Then* F *is in general position with respect to* $0 \in W$ *if and only if* graph h: $\mathcal{O} \to \mathcal{O} \times \mathbf{R}^k$ *is transverse to the affine algebraic subvariety* $(U(x, h) = 0)$ *of* $\mathcal{O} \times \mathbf{R}^k$.

PROOF. Let $x_0 \in \mathcal{O}$, and $H = G_{x_0}$. Let T be a linear H-space complementary to the tangent space $T(Gx_0)_{x_0}$ in V, so that the image of the map $y \mapsto x_0 + y$, of some invariant neighbourhood S of $0 \in T$ into $\mathcal{O}$, is a slice for the orbit Gx_0 at x_0. We identify the corresponding invariant tubular neighbourhood of Gx_0 with $G \times_H S$. Then $(G \times_H S) \times \mathbf{R}^k = G \times_H (S \times \mathbf{R}^k)$ is an invariant tubular neighbourhood of the orbit of $(x_0, h(x_0))$ in $\mathcal{O} \times \mathbf{R}^k$. Note that the maps $y \mapsto F_i(x_0 + y)$ of S into W form a set of generators for the module of germs at $0 \in S$ of smooth equivariant maps $S \to W$, over the ring of germs of smooth H-invariant functions at $0 \in S$. Let X be the affine algebraic subvariety $(\sum_{i=1}^k h_i F_i(x) = 0)$ of $\mathcal{O} \times \mathbf{R}^k$, and Y the germ at $(y = 0)$ of the affine algebraic subvariety $(\sum_{i=1}^k h_i F_i(x_0 + y) = 0)$ of $S \times \mathbf{R}^k$. Note that (the germ at $Gx_0 \times \mathbf{R}^k$ of) $X \cap G \times_H (S \times \mathbf{R}^k)$ is $G \times_H Y$. Since $x \mapsto h(x)$ is invariant, it maps the orbit Gx_0 into a single point. We see immediately that $x \mapsto (x, h(x))$ is transverse to X at x_0, if and only if the map $y \mapsto (y, h(x_0 + y))$ of S into $S \times \mathbf{R}^k$ is transverse to Y at $y = 0$.

LEMMA 6.2. *If a smooth equivariant map* F: $M \to N$ *is in general position with respect to an invariant submanifold* P *of* N *at* $x \in M$, *then* F *is in general position with respect to* P *at* gx, *for any* $g \in G$.

The proof is trivial.

PROPOSITION 6.3. *If* F: $M \to N$ *is in general position with respect to* P *at* $x \in M$, *then it is in general position in a neighbourhood of* x.

PROOF. In view of Lemma 6.2, it suffices to prove that if V, W are linear G-spaces, and a smooth equivariant map F: $V \to W$ is in general position with respect to $0 \in W$ at $0 \in V$, then F is in general position with respect to $0 \in W$ in a neighbourhood of $0 \in V$. This follows from the corresponding result for transversality to a Whitney stratified set, using Proposition 6.1.

PROPOSITION 6.4. *If a smooth equivariant map* F: $M \to N$ *is in general position with respect to an invariant submanifold* P *of* N, *then it is stratumwise transverse to* P. *In other words, for every isotropy subgroup* H *of* M, $F|M_H$: $M_H \to N^H$ *is transverse to* P^H, *or, equivalently,* (graph $F)|M_{(H)}$: $M_{(H)} \to (M \times N)_{(H)}$ *is transverse to* $(M \times P)_{(H)}$.

PROOF. The proof reduces to the case of a smooth equivariant map F: $V \to W$ between linear G-spaces V, W. We must show that if F is in general position with respect to $0 \in W$ at $0 \in V$, then $F|V^G$: $V^G \to W^G$ is transverse to $0 \in W^G$. The result follows immediately from Definition 1.1, using the

special set of generators given in Proposition 5.1, Step 3, for the module of equivariant maps.

Thom [21] defines a *strongly stratified* set E as a Hausdorff space such that for any $x \in E$, there is a local presentation of E as the transversal intersection of a semialgebraic subset A of $\mathbf{R}^n$, by a diffeomorphism $g\colon \mathbf{R}^m \to \mathbf{R}^n$. A strongly stratified set E clearly has a minimum Whitney stratification.

PROPOSITION 6.5. *Let $F\colon M \to N$ be a smooth equivariant map, in general position with respect to an invariant submanifold P of N. Then $F^{-1}(P)$ is a strongly stratified subset of M, and the strata are invariant submanifolds. In fact $F^{-1}(P)$ is locally presented as the transversal intersection of an algebraic set by a diffeomorphism.*

PROOF. A compact Lie group G has the structure of a real algebraic linear group. If V is a linear G-space, then the orbits of G in V are algebraic. If H is a closed subgroup of G, and X an H-invariant real affine algebraic subvariety of a linear H-space V, then the twisted products $G \times_H X \subset G \times_H V$ are algebraic. The proposition is an immediate consequence of Proposition 6.1.

7. The openness theorem.

PROOF OF THEOREM 1.3. We consider the Whitney topology case; with trivial modifications, the same proof handles the $\mathcal{C}^\infty$ case. We will show that the complement C in $\mathcal{C}^\infty_G(M, N)$, of the set of smooth equivariant maps which are in general position with respect to P, is closed; in other words, if the sequence $F_1, F_2, \ldots$ in C converges in the Whitney topology to $F \in \mathcal{C}^\infty_G(M, N)$, then $F \in C$. For each $i = 1, 2, \ldots,$ there is a point $x_i \in M$, such that $y_i = F_i(x_i) \in P$, but F_i is not in general position with respect to P at x_i. Since the functions in a Whitney convergent sequence eventually agree off a compact set, we can assume that the F_i all agree outside a compact invariant subset K of M, and that $\{x_i\} \subset K$. The sequence $x_1, x_2, \ldots$ has a subsequence $x_{i_1}, x_{i_2}, \ldots$ which converges to a point $x \in K$. Then $y_{i_1}, y_{i_2}, \ldots$ converges to $y = F(x)$, and $y \in P$ since P is closed.

Suppose that F is in general position with respect to P at x, and hence, by Lemma 6.2 and Proposition 6.3, in some invariant neighbourhood of the orbit Gx. Since Whitney convergence implies $\mathcal{C}^\infty$ convergence, it follows from Propositions 6.1 and 3.2 (or the remarks preceding it), together with the openness theorem for transversality to a Whitney stratified set, that there is an invariant neighbourhood of Gx, in which F_{i_j} is in general position with respect to P, for j sufficiently large. This is a contradiction, so that $F \in C$, as required.

8. Density theorems.

PROOF OF THEOREM 1.4. We again argue only the Whitney topology case. Let $Z_1, Z_2, \ldots$ be a countable cover of $M \times P$ by compact invariant sets.

For each $r = 1, 2, \ldots,$ let $T(Z_r)$ be the set of maps $F \in \mathcal{C}_G^\infty(M, N)$ such that graph $F = (\mathrm{id}, F)$: $M \to M \times N$ is in general position with respect to $M \times P$ on Z_r. It suffices to show each $T(Z_r)$ is open and dense.

By (a trivial modification of) Theorem 1.3, the set T_r of maps $F \in \mathcal{C}_G^\infty(M, M \times N)$, such that F is in general position with respect to $M \times P$ on Z_r, is open. Since the map graph: $\mathcal{C}_G^\infty(M, N) \to \mathcal{C}_G^\infty(M, M \times N)$ is continuous, then $T(Z_r) = \mathrm{graph}^{-1}(T_r)$ is open.

Let π_M: $M \times N \to M$, π_N: $M \times N \to N$ be the projections. Given $F \in \mathcal{C}_G^\infty(M, N)$, we choose for each $x \in \pi_M(Z_r)$:

(1) a linear $G_{F(x)}$-space W, and a $G_{F(x)}$-embedding ψ: $W \to N$, such that $\psi(0) = F(x)$ and (a) if $F(x) \notin \pi_N(Z_r)$, then $\psi(W) \cup \pi_N(Z_r) = \varnothing$; (b) if $F(x) \in \pi_N(Z_r)$, then W is a $G_{F(x)}$-direct sum $W = W^1 \oplus W^2$, with $\psi(W^1) = \psi(W) \cap P$;

(2) an invariant tubular neighbourhood Φ: $G \times_{G_x} V \to M$ of Gx, such that $\phi(V)$ is a slice at x and $F(\phi(V)) \subset W$, where $\phi = \Phi|V$;

(3) closed G_x-invariant neighbourhoods $D_1(V), D_2(V)$ of $0 \in V$, with $D_1(V) \subset \mathrm{Int}\, D_2(V)$.

There is a finite subset $x_1, x_2, \ldots, x_K$ of $\pi_M(Z_r)$ such that the corresponding $\Phi(G \times_{G_x} \mathrm{Int}\, D_1(V))$ cover $\pi_M(Z_r)$. Denote by ϕ_q, V_q, ψ_q, W_q the ϕ, V, ψ, W corresponding to x_q, $q = 1, \ldots, K$, as above. For $q = 1, \ldots, K$, choose smooth G_{x_q}-invariant functions ξ_q: $\phi_q(V_q) \to [0, 1]$, η_q: $\psi_q(W_q) \to [0, 1]$, such that $\xi_q = 1$ in a neighbourhood of $\phi_q(D_1(V_q))$, $\xi_q = 0$ outside $\phi_q(D_2(V_q))$, and $\eta_q = 1$ in a neighbourhood of $F(\phi_q(D_1(V_q)))$, $\eta_q = 0$ outside a larger compact neighbourhood. We now define $f \in \mathcal{C}_G^\infty(M, N)$, arbitrarily close to F in the Whitney topology, such that graph f is in general position with respect to $M \times P$ on Z_r, by the following induction:

For $q = 0, \ldots, K$, f_q is defined by:

(1) $f_0 = F$;

(2) (i) If $W_q \cap \pi_N(Z_r) = \varnothing$, then $f_q = f_{q-1}$; (ii) if $W_q \cap \pi_N(Z_r) \neq \varnothing$, then f_q is the unique equivariant map such that $f_q(x) = f_{q-1}(x)$ for $x \notin G \cdot \phi_q(V_q)$, and

$$f_q(x) = \psi_q\left(\psi_q^{-1}(f_{q-1}(x)) + \xi_q(x)\eta_q(f_{q-1}(x)) \sum_{i=1}^{k} b_i F_i(\phi_q^{-1}(x))\right),$$

for $x \in \phi_q(V_q)$, where $F_1, \ldots, F_k$ generate $\mathcal{C}_{G_{x_q}}^\infty(V_q, W_q^2)$ over $\mathcal{C}_{G_{x_q}}^\infty(V_q)$, and $b = (b_1, \ldots, b_k) \in \mathbf{R}^k$ is chosen arbitrarily small so that

(a) graph f_q is in general position with respect to $M \times P$ on Z_r, at each point of $\bigcup_{j=1}^{q-1} G \cdot \phi_j(D_1(V_j))$;

(b) $x \mapsto (x, h(x) + b)$ is transverse to the affine algebraic subvariety $(\Sigma_{i=1}^k h_i F_i(v) = 0)$ of $V_q \times \mathbf{R}^k$, where $\psi_q^{-1}(F(\phi_q(v))) = \Sigma_{i=1}^k h_i(v) F_i(v)$, with $h_i \in \mathcal{C}_{G_{x_q}}^\infty(V_q)$, $i = 1, \ldots, k$, and $h(v) = (h_1(v), \ldots, h_k(v))$;

(c) $f_q(\phi_j(D_1(V_j))) \subset \operatorname{Int} \eta_j^{-1}(1), j = q + 1, \ldots, K$.

We can make this choice of $b \in \mathbf{R}^k$ because:

(1) f_{q-1} has the property that graph f_{q-1} is in general position with respect to $M \times P$ on Z_r, at each point of $\bigcup_{j=1}^{q-1} G \cdot \phi_j(D_1(V_j))$, and the set of such maps is open;

(2) the set of points $b \in \mathbf{R}^k$, such that $x \mapsto (x, h(x) + b)$ is transverse to $(\sum_{i=1}^k h_i F_i(x) = 0)$, is dense in $\mathbf{R}^k$.

Let $f = f_K$. Note that $f = F$ outside a compact set, so that f can be made arbitrarily close to F in the Whitney topology, by choosing b sufficiently small at each stage of the induction. This completes the proof of the density of $T(Z_r)$.

As in the case $G = 1$ (cf. [10, Chapter II, §4]), if a parametrized family of equivariant maps is in general position with respect to an invariant submanifold, then for a dense set of parameters, the individual maps are also in general position:

PROPOSITION 8.1. *Let S, M, N be smooth G-manifolds, with G acting trivially on S, and let P be a G-submanifold of N. Let F: $S \times M \to N$ be a smooth equivariant map, and $F_s(x) = F(s, x)$. If F is in general position with respect to P, then $\{s \in S \mid F_s$ is in general position with respect to $P\}$ is dense in S.*

PROOF. $F^{-1}(P)$ is a Whitney stratified subset of $S \times M$, with countably many strata. Let π: $S \times M \to S$ be the projection map. We will show that if $s \in S$ is a regular value for the restriction of π to each stratum of $F^{-1}(P)$, then F_s is in general position with respect to P at each $x \in M$. The result then follows from Sard's Theorem [1, Theorem 16.1].

If $(s, x) \notin F^{-1}(P)$, then $F_s(x) \notin P$, so that F_s is in general position with respect to P at x. If $(s, x) \in F^{-1}(P)$, we can reduce to the following local situation: M, N are linear H-spaces V, W (respectively), where $H = G_x$, S is a linear space, and F: $S \times V \to W$ is an H-equivariant map with $F(s, 0) = 0$. Given that F is in general position with respect to $0 \in W$ at $(s, 0) \in S \times V$, we must show that F_s is in general position with respect to $0 \in W$ at $0 \in V$. By Proposition 3.3, we can write $F(s, x) = \sum_{i=1}^k h_i(s, x) F_i(x)$, where $F_1, \ldots, F_k$ generate $\mathcal{C}_H^\infty(V, W)$ over $\mathcal{C}_H^\infty(V)$, and $h_i \in \mathcal{C}_H^\infty(S \times V)$, $i = 1, \ldots, k$. Let Q be the stratum containing $(0, h(s, 0))$ in the (minimum Whitney stratification of) the algebraic subvariety $(\sum_{i=1}^k h_i F_i(x) = 0)$ of $V \times \mathbf{R}^k$, so that $S \times Q$ is the stratum containing $(s, 0, h(s, 0))$ in the subvariety $(\sum_{i=1}^k h_i F_i(x) = 0)$ of $S \times V \times \mathbf{R}^k$. Given that $(t, x) \mapsto (t, x, h(t, x))$ is transverse to $S \times Q$ at $(s, 0)$, we must show that $x \mapsto (x, h(s, x))$ is transverse to Q at 0.

Since s is a regular value for $\pi|(\text{graph } h)^{-1}(S \times Q)$, then

$$T(S \times V)_{(s,0)} = T\big((\text{graph } h)^{-1}(S \times Q)\big)_{(s,0)} + T(\{s\} \times V)_{(s,0)}.$$

Since graph h is transverse to $S \times Q$ at $(s, 0)$, then

$$\begin{aligned} T(S \times V \times \mathbf{R}^k)_{(s,0,h(s,0))} &= T(S \times Q)_{(s,0,h(s,0))} + d(\text{graph } h)_{(s,0)} T(S \times V)_{(s,0)} \\ &= T(S \times Q)_{(s,0,h(s,0))} + \{0\} \times d(\text{graph } h_s)_0 TV_0 \end{aligned}$$

(from the previous statement). Hence $T(V \times \mathbf{R}^k)_{(0,h(s,0))} = TQ_{(0,h(s,0))} + d(\text{graph } h_s)_0 TV_0$, as required.

9. The isotopy theorem.

PROOF OF THEOREM 1.5. Since by Proposition 3.3, the set of parameters $s \in S$, such that F_s is in general position with respect to P, is open, we may assume that F_s is in general position with respect to P for all $s \in S$. Since P is closed, $S \times M$ is Whitney stratified by invariant submanifolds, so that $F^{-1}(P)$ is a Whitney stratified subset, and the projection $S \times M \to S$ maps each stratum submersively onto S. This follows (using Propositions 6.1 and 3.3) from the corresponding statement for parametrized families of maps transverse to a Whitney stratification (just an application of the implicit function theorem). The theorem now follows from (an equivariant version of) Thom's First Isotopy Lemma [16, Theorem 8.1].

We leave to the reader the verification that Thom's First Isotopy Lemma can be made equivariant. One merely works through the machinery of stratified sets, as presented by Mather [16], [13], using systems of invariant tubular neighbourhoods.

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