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THE CENTRALIZER OF A CARTAN SUBALGEBRA OF A JORDAN ALGEBRA(1)

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ABSTRACT. If L is a diagonable subspace of an associative algebra A over a field Φ (L is spanned by commuting elements and the linear transformations $\operatorname{ad} x$: $a \mapsto x - xa$, $x \in L$, are simultaneously diagonalizable), then a map λ: $L \to \Phi$ is said to be a weight of L on an A-module V if the space $V_\lambda = \{v \in V: vx = \lambda(x)v \text{ for all } x \in L\}$ is nonzero. It is shown that if A is finite dimensional semisimple and the characteristic of Φ is zero then the centralizer of L in A is the centralizer of an element $x \in A$ if and only if x distinguishes the weights of L on every irreducible A-module. This theorem can be used to show that for each representative V of an isomorphism class of irreducible A-modules and for each weight λ of L on V, the centralizer of L contains the matrix ring D_{n_λ}, $D = \operatorname{End}_A V$, $n_\lambda = \dim_D V_\lambda$ and in fact is the direct sum of all such algebras. If J is a finite dimensional simple reduced Jordan algebra, one can determine precisely those x in J whose centralizer in the universal enveloping algebra of J coincides with the centralizer of a Cartan subalgebra. The simple components of such a centralizer can also be found and in fact are listed for the degree $J > 3$ case.

0. **Introduction.** Suppose an associative algebra A with 1 over a field Φ possesses a *diagonable subspace* L; that is, L is spanned by commuting elements and the linear transformations $\operatorname{ad} x$: $a \mapsto ax - xa$ for $x \in L$ are simultaneously diagonalizable. Then a map λ: $L \to \Phi$ (necessarily linear) is a *weight* of L on an A-module V and V is λ-*weighted*, if the *weight space* $V_\lambda = \{v \in V: vx = \lambda(x)v \text{ for all } x \in L\}$ is nonzero. If H is a Cartan subalgebra of a finite dimensional simple Lie or Jordan algebra, then considered as a subspace of the universal enveloping algebra, H is diagonable and in the Lie case, our definition of weighted $U(L)$-module is equivalent to the classical definition of weighted L-module. The significance of the weighted representations of A lies in the fact that for each fixed linear functional λ: $L \to \Phi$, there is a one-one correspondence between the isomorphism classes of λ-weighted irreducible representations of A and those of the subalgebra

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$C(L) = \{a \in A\colon ax = xa \text{ for all } x \in L\}$, the centralizer of L in A. It is known that if A is semisimple so is $C(L)$, and in the presence of a finiteness condition such as finite dimensionality, $C(L)$ is actually the centralizer $C(x)$ of some x in L (provided Φ is infinite). In this paper we show that if A is finite dimensional semisimple over Φ of characteristic 0 and all irreducible A-modules are weighted, then the x's for which $C(L) = C(x)$ are precisely those x's which distinguish the weights of L on every irreducible A-module. The structure of $C(L)$, moreover, can be completely determined from the knowledge of the weights of L on the irreducible modules of A, and the corresponding weight spaces. If J is a finite dimensional simple reduced Jordan algebra, then the weight theory of J is known, and so we are able to list the simple components of the centralizer of a Cartan subalgebra of J in its universal enveloping algebra. We refer the reader to [1], [2] for further information about the ideas we have discussed in this paragraph.

1. **Centralizers in semisimple algebras.** If A is an algebra over Φ, we call an element $x \in A$ *diagonable* if Φx is a diagonable subspace of A, equivalently, ad x is a diagonalizable linear transformation of A.

1.1. PROPOSITION. *Let A be a simple finite dimensional algebra over its centre F. Then any F-linear combination of orthogonal idempotents of A is diagonable. The converse holds in characteristic* 0, *or more generally, for diagonable elements whose minimal polynomial over F has degree less than the characteristic of F.*

PROOF. If $x = \sum \alpha_i e_i$, the e_i orthogonal idempotents which we can assume sum to 1, then any $a \in A$ can be written $a = \sum e_i a e_j$ and $(e_i a e_j)$ ad $x = (\alpha_j - \alpha_i) e_i a e_j$, so x is diagonable. Conversely, the minimal polynomial of any element x can be written $\prod_1^s q_i^{n_i}$, $q_1, \ldots, q_s$ being distinct irreducible monic polynomials, and if x is diagonable, each $n_i = 1$ because $C(x)$ is semiprime [1, Theorem 5.4]. Using a well-established argument, $1 = \sum_1^s e_i$, where $e_1, \ldots, e_s$ are orthogonal idempotents which commute with x and satisfy $q_i(x)e_i = 0$. Thus q_i is the minimal polynomial of $x_i = xe_i$ which is a diagonable element of $e_i A e_i$ because this subalgebra of A is invariant under ad x [1, 2.2]. Now if (u)ad $x_i = \alpha u$ for some nonzero $u \in e_i A e_i$ and $\alpha \in F$, then $0 = u q_i(x_i) = q_i(x_i + \alpha)u$. Since also $q_i(x_i)u = 0$ and $u \neq 0$, the polynomials $q_i(t)$ and $q_i(t + \alpha)$ cannot be relatively prime and because they are monic, they are equal. Thus $\xi + n\alpha$ is a root of q_i (in some extension field of F) whenever ξ is a root, an impossible situation for F of characteristic 0 or greater than deg q_i unless $\alpha = 0$; i.e. x_i is in the centre of $e_i A e_i$. We can think of A as the ring of endomorphisms of some vector space V finite dimensional over a division ring D, and in this setting it is clear that $e_i A e_i$ is just $\mathrm{End}_D V e_i$ and hence has centre F, more correctly $F e_i$. Thus $x_i = \alpha_i e_i$ for some $\alpha_i \in F$

and $x = x1 = \sum xe_i = \sum \alpha_i e_i$ as required.

The following corollary is easy.

1.2. COROLLARY. *An element x of any algebra A with 1 can be written $x = \sum_i \alpha_i e_i$, the e_i's orthogonal idempotents summing to 1 and the α_i's distinct scalars if and only if the minimal polynomial of x is $\prod_i (t - \alpha_i)$. In this case x is diagonable and $C(x) = \sum_i e_i A e_i$ and so if $A = D_n$ is the ring of endomorphisms of a vector space V finite dimensional over division ring D, then*

$$C(x) = \bigoplus \sum_i D_{n_i}, \qquad n_i = \dim_D Ve_i.$$

Before continuing, we remark that the restrictions on the characteristic of F given in the proposition are absolutely necessary, for in the algebra F_2 of 2×2 matrices over a field F of characteristic 2, the element $x = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is diagonable. It is easy to check that $\operatorname{ad} x$ has eigenvalues 0 and 1, each eigenspace having dimension 2. The one corresponding to 0 has basis $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and that corresponding to 1, $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. However, x is not a linear combination of orthogonal idempotents because its minimal polynomial $t^2 + t + 1$ is irreducible over F.

For finite dimensional algebras over infinite fields, the centralizer of a diagonable subspace L is of the form $C(x)$ if and only if $C(x)$ is minimal (with respect to inclusion) in the set of centralizers of elements of L [1, Theorem 6.2]. While the results that follow hold for infinite fields of suitably big characteristic, for the sake of simplicity we will henceforth assume that all fields have characteristic 0.

1.3. PROPOSITION. *Let A be the algebra D_n, D a division algebra of finite dimension over its centre F (of characteristic 0). For any $x \in A$, let p_x denote the minimal polynomial of x over F. Then if x and y are diagonable elements of A, $C(x) \subseteq C(y)$ implies $\deg p_y \leqslant \deg p_x$. Hence if L is a diagonable subspace of A, $C(L) = C(x)$ if and only if $\deg p_x$ is maximal in $\{\deg p_y : y \in L\}$.*

PROOF. Any diagonable element x can be written $x = \sum_1^t \alpha_i e_i$ where $e_1, \ldots, e_t$ are orthogonal idempotents with sum 1 and $\alpha_1, \ldots, \alpha_t$ are distinct scalars. If y is diagonable and $C(x) \subseteq C(y)$, then y is in the centre of $C(x) = \sum e_i A e_i$ and so is of the form $y = \sum \beta_i e_i$, $\beta_i \in F$, since the centre of $e_i A e_i$ is Fe_i (cf. the proof of 1.1). It follows that the roots of p_y are precisely $\{\beta_1, \ldots, \beta_t\}$, a set of cardinality at most $t = \deg p_x$. Thus $\deg p_y \leqslant \deg p_x$, and furthermore, $\deg p_y = \deg p_x$ if and only if $\beta_1, \ldots, \beta_t$ are all different in which case $C(y) = \sum e_i A e_i = C(x)$. This remark shows that if the degree of p_x is maximal in $\{\deg p_y : y \in L\}$, L some diagonable subspace of A, and $C(y) \subseteq C(x)$, then $\deg p_x \leqslant \deg p_y$ forces equality and so $C(y) = C(x)$; i.e. $C(x)$ is minimal. On the other hand if $C(x)$ is minimal, $C(x) = C(L)$ and so

$C(x) \subseteq C(y)$ and $\deg p_y \leqslant \deg p_x$ for any $y \in L$.

Now let A, F and L be as in the proposition and suppose V is an irreducible weighted A-module. Then V is the direct sum of its weight spaces $V_\lambda \equiv \{v \in V\colon vx = \lambda(x)v \text{ for all } x \in L\}$. If $0 \neq v \in V_\lambda$ and $x \in L$, then $v(x - \lambda(x)1) = 0$ implies that $\lambda(x)$ must be a root of the minimal polynomial p_x of x. Conversely, since x is diagonable, p_x splits into distinct linear factors. Thus let $\alpha \in F$ be a root of p_x. There is certainly a nonzero element of A annihilated by $x - \alpha 1$ and since A is the direct sum of irreducibles, any two irreducible A-modules are isomorphic, and two isomorphic A-modules must have identical weight sets, we can in fact assume that $v(x - \alpha 1) = 0$ for some nonzero $v \in V$. Decomposing $v = \sum_{\lambda \in \Lambda} v_\lambda$ relative to the set Λ of weights on V, we have $\alpha \sum_\lambda v_\lambda = \alpha v = vx = \sum_\lambda \lambda(x) v_\lambda$. Now for any $y \in L$, the v_λ's are eigenvectors for each of the linear transformations $R_y\colon u \mapsto uy$ of V and because F is infinite and each $\lambda \in \Lambda$ is a linear functional on L, for some $y \in L$ the corresponding eigenvalues $\{\lambda(y)\colon \lambda \in \Lambda\}$ are distinct. It follows immediately from $\sum_\lambda (\alpha - \lambda(x)) v_\lambda = 0$ that $\alpha = \lambda(x)$ (for at least one λ). Consequently, $p_x(t) = \prod(t - \lambda(x))$, the product being taken over the set $\{\lambda(x)\colon \lambda \in \Lambda\}$. In particular, $\deg p_x \leqslant |\Lambda|$ with equality achieved exactly for those x which distinguish the weights of Λ. For such an x, we have $x = \sum_{\lambda \in \Lambda} \lambda(x) e_\lambda$, $\{e_\lambda\colon \lambda \in \Lambda\}$ a set of orthogonal idempotents with sum 1, and $V_{\lambda_0} \subset V e_{\lambda_0}$ for any $\lambda_0 \in \Lambda$. To see this, let $v \in V_{\lambda_0}$ and observe that $\sum_\lambda \lambda_0(x) v e_\lambda = \lambda_0(x) v = vx = \sum_\lambda \lambda(x) v e_\lambda$ (because $v = v1 = \sum_\lambda v e_\lambda$ of course). Since $V = \sum_\lambda V e_\lambda$ is a direct sum and $\lambda(x) \neq \lambda_0(x)$ unless $\lambda = \lambda_0$, $v e_\lambda = 0$ for $\lambda \neq \lambda_0$ and $v = v e_{\lambda_0} \in V e_{\lambda_0}$. It now follows from the directness of $\sum_\lambda V e_\lambda$ again that in fact we have $V_\lambda = V e_\lambda$ for all $\lambda \in \Lambda$.

1.4. THEOREM. *Let L be a diagonable subspace of a finite dimensional semisimple algebra A over a field Φ of characteristic* 0. *Assume all irreducible A-modules are weighted. Then $C(L) = C(x)$ if and only if x distinguishes the weights of L on each irreducible A-module. Moreover, if $\mathfrak{M}$ is a complete set of nonisomorphic A-modules and for $V \in \mathfrak{M}$, Λ_V is the set of weights of L on V and $D_V = \operatorname{End}_A V$, then*

$$C(L) = \bigoplus \sum_{V \in \mathfrak{M}} \sum_{\lambda \in \Lambda_V} (D_V)_{n_\lambda}, \quad n_\lambda = \dim_{D_V} V_\lambda \quad \textit{for each } \lambda \in \Lambda_V.$$

PROOF. The above discussion together with 1.2 and 1.3 establish the theorem in case A is simple, the only point that perhaps should be made being that if L is a diagonable subspace of an algebra A over Φ and F is a subfield of the centre of A containing Φ then L is still diagonable in the F-algebra A. In general A is the direct sum of simple algebras A_V, $V \in \mathfrak{M}$, V a representative of the (unique) isomorphism class of irreducible A_V-modules. Let $\pi_V\colon A \to A_V$ denote the canonical projections, $L_V = \pi_V(L)$ and $x_V =$

$\pi_V(x)$ for any $x \in A$. Then L_V is a diagonable subspace of A_V and $C(L) = C(x)$ implies $C(L_V) = C(x_V)$ for all $V \in \mathfrak{M}$. (By $C(L_V)$ we mean the centralizer of L_V in A_V and similarly for $C(x_V)$.) Thus by the simple case, $C(L) = C(x)$ implies that each x_V distinguishes the weights of L_V on V. But for any $x \in L$,

$$(1) \qquad \{\mu(x_V): \mu \text{ a weight of } L_V \text{ on } V\} = \{\lambda(x): \lambda \in \Lambda_V\}$$

and so x distinguishes the weights of L on every irreducible A-module as claimed. To see why (1) is true, note that if μ is a weight of L_V on V, then $v\pi_V(y) = \mu(\pi_V(y))v$ for some nonzero $v \in V$ and all $y \in L$. But for $y \in L$, $v\pi_V(y) = vy$ and thus the map λ: $L \to \Phi$ defined by $\lambda(y) = \mu(\pi_V(y))$ is a weight of L on V whose value at x is the value of μ at x_V. Conversely, given a weight $\lambda \in \Lambda_V$, $vy = \lambda(y)v$ for some nonzero $v \in V$ and all $y \in L$. Again $vy = v\pi_V(y)$ and μ: $L_V \to \Phi$ defined by $\mu(\pi_V(y)) = \lambda(y)$ is well defined because $v \neq 0$ and a weight of L_V on V whose value at x_V is the value of λ at x. Now (1) also says that if x distinguishes the weights of L on each $V \in \mathfrak{M}$, then x_V distinguishes the weights of L_V on V and so $C(L_V) = C(x_V)$. Hence $C(L) = \bigoplus \sum_V C(L_V) = \bigoplus \sum C(x_V) = C(x)$. For the last statement of the theorem, we know that $C(L_V) = \bigoplus \sum_{\mu \in \Omega} D_{n_\mu}$ where Ω is the set of weights of L_V on V and $n_\mu = \dim_D V_\mu$, $D = \mathrm{End}_{A_V} V$. But clearly $D = D_V$ and the proof of (1) showed that Ω is in one-one correspondence with Λ_V in such a way that corresponding weight spaces are identical and most surely therefore have the same dimension. The proof is now complete.

2. **Applications to Jordan algebras.** In this section we assume that J is a finite dimensional simple reduced Jordan algebra of degree n over a field Φ of characteristic different from 2 (see [3, pp. 197–203] for the definition, properties and classification of these algebras). Then J possesses a Cartan subalgebra $H = \sum_1^n \Phi e_i$ where the e_i are orthogonal idempotents with sum 1. The universal enveloping algebra $U(J)$ of J is a semisimple associative algebra of finite dimension which contains J in such a way that H is a diagonable subspace [1]. The significance of the universal enveloping algebra lies in the fact that any irreducible unital module of $U(J)$ is an irreducible J-bimodule (and conversely), the identification being such that $vh = \lambda(h)v$, $h \in H$, v in a J-bimodule V, if and only if the same equation holds in the unital $U(J)$-module V. In other words, V is an irreducible J-bimodule with weight λ if and only if V is an irreducible $U(J)$-module with weight λ, and the weight spaces of V are well defined regardless of whether one is thinking of V as a J-bimodule or $U(J)$-module. Recently, a detailed study of the weights on J-bimodules has been completed [2]. It turns out that every irreducible J-bimodule (and hence $U(J)$-module) is weighted and has weight

set one of $\{\zeta_1, \ldots, \zeta_n\}$, $\{\lambda_{ij}, i, j = 1, \ldots, n\}$ and $\{\lambda_i, \lambda_{ij}: i, j = 1, \ldots, n\}$ where $\zeta_i(e_k) = \frac{1}{2}\delta_{ik}$, $\lambda_i = 2\zeta_i$, $\lambda_{ij} = \zeta_i + \zeta_j$. Any weight is a linear functional on H so these equations define ζ_i, λ_i and λ_{ij} completely.

2.1. THEOREM. *Let J be a finite dimensional simple reduced Jordan algebra of degree n over a field Φ of characteristic* 0 *with Cartan subalgebra $H = \sum_1^n \Phi e_i$. Then the centralizer of H in $U(J)$ is the centralizer of an element $x = \sum \alpha_i e_i \in H$ if and only if the scalars $\alpha_i - \alpha_j$, $i < j$, are nonzero and in pairs, neither equal nor additive inverses.*

PROOF. For $x = \sum_1^n \alpha_k e_k \in H$, $\zeta_i(x) = \frac{1}{2}\alpha_i$, $\lambda_i(x) = \alpha_i$ and $\lambda_{ij}(x) = \frac{1}{2}(\alpha_i + \alpha_j)$. By Theorem 1.4, the centralizer of H in $U(J)$ is the centralizer of x if and only if x distinguishes the weights in each of the sets $\{\zeta_1, \ldots, \zeta_n\}$, $\{\lambda_{ij}: i, j = 1, \ldots, n\}$ and $\{\lambda_i, \lambda_{ij}: i, j = 1, \ldots, n\}$ and thus if and only if it distinguishes the weights of the third set; i.e., if and only if the scalars α_i, $\frac{1}{2}(\alpha_i + \alpha_j)$, $i < j = 1, \ldots, n$, are distinct. This condition is equivalent to the one stated above.

One of the motivating factors behind the study of any universal envelope is its role as universal object for the representations of the underlying algebra. For any given subclass of representations, it is reasonable to expect a more suitable universal object, of smaller dimension and simpler structure. The major result of [1] states that the centralizer of a diagonable subspace of an algebra is a universal object for the λ-weighted irreducible representations. In the present context, the centralizer of H in $U(J)$ is universal for the λ-weighted irreducible representations of J. Jacobson determines the representations of various Jordan algebras in [3] and uses these to give the structure of $U(J)$. In an analogous way, our Theorem 1.4 allows the ready determination of the structure of $C(H)$, the direct calculation of which seems rather complicated, because the weight spaces of each irreducible J-bimodule which is unital appear in [2], while the weight spaces of the remaining ones can easily be found. It is interesting to see an example of just how simple a universal object for a restricted class of representations can be.

The case where J has degree 2 is not particularly illuminating: it is already known that $C(H)$ is a direct sum of matrix rings over division rings and our theorem gives the sizes and number of these rings as rather complex functions of the dimension of J. For $n \geqslant 3$ however, the matrix rings are small, of size independent of n, and in one case they are even all 1×1 over Φ. Thus as our final result we state:

2.2. THEOREM. *Let $J = \mathfrak{H}(\mathfrak{D}_n, J_\gamma)$ be a Jordan matrix algebra with entries in a composition algebra $\mathfrak{D}$ over Φ and $H = \sum_1^n \Phi e_{ii}$, e_{ij} denoting the usual matrix units of $\mathfrak{D}_n$. Then the centralizer of H in $U(J)$ is the direct sum of*

(i) $n(n+1)$ *copies of* Φ,
(ii) $n(2n+3)$ *copies of* Φ *and* $\frac{1}{2}n(n-1)$ *copies of* Φ_2,
(iii) n *copies of* Φ, $n(n+1)$ *copies of* $\mathfrak{D}$ *and* $\frac{1}{2}n(n-1)$ *copies of* Φ_2,
(iv) n *copies of each of* Φ, $\mathfrak{D}$ *and* Φ_3, *and* $n(n-1)$ *copies of* Φ_4,
(iv′) 3 *copies each of* Φ *and* Φ_3, *and* 6 *copies each of* $\mathfrak{D}$ *and* Φ_4,
(v) n *copies of each of* Φ, Φ_2 *and* Φ_3, *and* $n(n-1)$ *copies of* Φ_4,
(v′) 3 *copies each of* Φ *and* Φ_3, *and* 6 *copies each of* Φ_2 *and* Φ_4,
(vi) 3 *copies each of* Φ *and* Φ_8,
according as $\mathfrak{D}$ *is* (i) Φ, (ii) *split quadratic*, (iii) *a quadratic field*, (iv) *a quaternion division algebra and* $n \geqslant 4$ *or* (iv′) $n = 3$, (v) *split quaternion and* $n \geqslant 4$ *or* (v′) $n = 3$, *and* (vi) *a Cayley algebra.*

PROOF. (i) If $\mathfrak{D} = \Phi$, there are three nonisomorphic irreducible bimodules, two of which are unital with weight sets $\{\lambda_i, \lambda_{ij}: i, j = 1, \ldots, n\}$ and $\{\lambda_{ij}: i, j = 1, \ldots, n\}$. All the weight spaces are one-dimensional over Φ, the division ring of endomorphisms in each case, contributing $n + \frac{1}{2}n(n-1) + \frac{1}{2}n(n-1) = n^2$ copies of Φ to $C(H)$. The third irreducible bimodule V corresponds to the simple component Φ_n of $U(J)$ and can be taken to be any row of Φ_n. The action of J on V is such that ζ_i is a weight with weight space the row whose only nonzero entries come in column i. Thus V_{ζ_i} is again one-dimensional and we obtain a final n copies of Φ as direct summands of $C(H)$.

(v) If $\mathfrak{D}$ is split quaternion there are again three irreducible bimodules two of which are unital. The latter each have weight set $\{\lambda_i, \lambda_{ij}: i, j = 1, \ldots, n\}$, the λ_{ij} weight space having dimension 4 over Φ, the λ_i space having dimension 1 in one of the bimodules and 3 in the other. These bimodules therefore contribute $2 \times \frac{1}{2}n(n-1)$ copies of Φ_4, n copies of Φ and n of Φ_3 to $C(H)$. The third irreducible bimodule V corresponds to the simple component $\mathfrak{D}_n$ of $U(J)$ as in case (i). This time though the entries in the row representing V must come from an irreducible $\mathfrak{D}$-module (of dimension 2 over Φ) and so $\dim V_{\zeta_i} = 2$, hence n copies of Φ_2. If $n \geqslant 4$ these are all the irreducible J-bimodules (up to isomorphism) but if $n = 3$, there is an additional module described explicitly in [2] possessing weight set $\{\lambda_{ij}: i, j = 1, 2, 3\}$ with $\dim V_{\lambda_{ij}} = 2$. The additional three copies of Φ_2 listed in (v′) arise from this module. The proofs of (i) and (v) are representative of the proofs required for all the cases given and we therefore omit the remaining details.

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