TOPOLOGICAL ENTROPY AT AN $\Omega$-EXPLOSION

BY

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Abstract. In this paper an example is given of a $C^2$ map $g$ from the circle onto itself, which permits an $\Omega$-explosion. It is shown that topological entropy (considered as a map from $C^2(S^1, S^1)$ to the nonnegative real numbers) is continuous at $g$.

1. Introduction. Let $g$ denote any $C^2$ mapping of the circle onto itself which satisfies the following properties (see Figure 1):

1. $g$ has an expanding fixed point $e$ and a contracting fixed point $c$, and these are the only fixed points of $g$.
2. $g$ preserves orientation at $e$ and $c$.
3. $g$ has nondegenerate singularities $t$ and $s$, and these are the only singularities of $g$.
4. The points $e$, $t$, $s$, $g(s)$, and $c$ are distinct and in order on the circle in the counterclockwise direction.
5. $g$ is one-to-one on each of the intervals $(e, t)$, $(t, s)$, and $(s, c)$. Here we

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use the notation \((a, b)\) to denote the open arc from a counterclockwise to \(b\).

(6) \(g(t) = e\).

These properties imply that \(\Omega(g) = \{e, c\}\), where \(\Omega(g)\) denotes the nonwandering set (see [2], [5], or [7] for definition).

It is easy to see that \(g\) permits an \(\Omega\)-explosion. By this we mean that for any neighborhood \(N\) of \(g\) in \(C^2(S^1, S^1)\), there is a map \(f \in N\) with \(\Omega(f)\) infinite. See Proposition 9 in §4 for a proof.

Let \(\text{ent}\) denote topological entropy (see §2 for the definition). Our main result is the following:

**Theorem A.** The map \(\text{ent}: C^2(S^1, S^1) \rightarrow R\) is continuous at \(g\).

Theorem A implies that for any bifurcation through \(g\), at the map \(g\) there is no sudden jump in the amount of action.

To prove Theorem A we first, in §3, obtain an upper bound for entropy (Theorem 6). Then, in §4, we construct a sequence \((f_n)\) of maps with \(\text{ent}(f_n) \rightarrow 0\) as \(n \rightarrow \infty\). Finally, in §5, we prove Theorem A by using Theorem 6 to show that for arbitrarily large \(n\) if \(f\) is close enough to \(g\) in \(C^2(S^1, S^1)\), then \(\text{ent}(f) < \text{ent}(f_n)\).

2. Preliminary definitions and results. We first review the definition of topological entropy given in [1]. Let \(X\) be a compact space and \(f: X \rightarrow X\) a continuous map. For any two open covers \(\mathcal{C}\) and \(\mathcal{B}\) of \(X\), let \(\mathcal{C} \vee \mathcal{B}\) denote \(\{A \cap B: A \in \mathcal{C} \text{ and } B \in \mathcal{B}\}\), and let \(f^{-1}(\mathcal{C})\) denote \(\{f^{-1}(A): A \in \mathcal{C}\}\). Let \(M_n(f, \mathcal{C})\) denote the minimum cardinality of a subcover of \(X\) of

\[
\mathcal{C} \vee f^{-1}(\mathcal{C}) \vee \cdots \vee f^{-(n+1)}(\mathcal{C}).
\]

We set

\[
\text{ent}(f, \mathcal{C}) = \lim_{n \rightarrow \infty} \frac{1}{n} \left( \ln \left( M_n(f, \mathcal{C}) \right) \right),
\]

where \(\ln\) denotes the natural logarithm. It is easy to see that this limit exists and is finite (see [1]). Finally, we define the topological entropy of \(f\) by

\[
\text{ent}(f) = \sup(\text{ent}(f, \mathcal{C}))
\]

where the supremum is taken over all open covers \(\mathcal{C}\) of \(X\). If \(X\) is a metric space it suffices to consider any sequence of open covers whose diameter approaches zero (see [1]). By the diameter of an open cover \(\mathcal{C}\) we mean the supremum of the diameters of the open sets in \(\mathcal{C}\).

We now state some basic facts about topological entropy which will be used later. In each of the four propositions, \(f\) is a continuous map of a compact space \(X\) into itself. Proposition 2 follows immediately from the definition, and Proposition 4 follows from Proposition 3.

**Proposition 1** (see [5]). If \(X\) is a metric space, \(\text{ent}(f) = \text{ent}(f|\Omega(f))\).
Proposition 2. If $X$ is finite, $\text{ent}(f) = 0$.

Proposition 3 (see [1]). If $X_1$ and $X_2$ are closed subsets of $X$, with $X_1 \cup X_2 = X$ and $f(X_1) \subset X_1$ and $f(X_2) \subset X_2$, then

$$\text{ent}(f) = \max\{\text{ent}(f|X_1), \text{ent}(f|X_2)\}.$$ 

Proposition 4. If $K$ is a closed subset of $X$, with $f(K) \subset K$, then $\text{ent}(f|K) \leq \text{ent}(f)$.

We will assume the reader is familiar with the following terminology (see [2] or [7]); nonwandering set, expanding fixed point, contracting fixed point, and stable manifold of a contracting fixed point (denoted $W^s(c)$). For any point $x \in W^s(c)$, we will use the notation $\text{slsm}(x)$ to denote the component of $W^s(c)$ which contains $x$.

3. An upper bound for entropy. The proof of Theorem 6 (which modifies a theorem of [2]) uses the following lemma from [2] (see [2, §3, Lemma 5]).

Lemma 5. Let $f \in C^0(S^1, S^1)$. Let $K_1, \ldots, K_n$ be proper closed intervals of $S^1$, such that for each $i = 1, \ldots, n - 1$, $f|K_i$ is a homeomorphism and $f(K_i) \subset K_{i+1}$. Let $\mathcal{E}$ be a covering of $K_1 \cup \cdots \cup K_n$ by finitely many open intervals, such that $\forall A \in \mathcal{E}$ and $i = 1, \ldots, n$, $A \cap K_i$ is an interval (or empty). Let $\text{card}(\mathcal{E}) = k$ (card denotes cardinality). Then there is a subset $\mathcal{B}$ of $(\ast)$ which covers $K_1$, with $\text{card}(\mathcal{B}) \leq n \cdot k$.

In Theorems 6 and A we will use the following definition. Let $\mathcal{C}$ be a finite collection of closed intervals on $S^1$, $\mathcal{C} = \{I(1), \ldots, I(p)\}$, and let $f \in C^0(S^1, S^1)$. We denote by $K_n(f, \mathcal{C})$ the number of distinct nonempty sets of the form

$$I(j_1) \cap f^{-1}(I(j_2)) \cap \cdots \cap f^{-n+1}(I(j_n)),$$

where $j_i \in \{1, \ldots, p\}$ for $i = 1, \ldots, n$.

Theorem 6. Let $f \in C^1(S^1, S^1)$, and let $\mathcal{C} = \{I(1), \ldots, I(p)\}$ be a finite collection of proper closed intervals on $S^1$. Let $W = S^1$ or $W = S^1 - (O_1 \cup O_2 \cup \cdots \cup O_m)$ where $O_i$ is a component of the stable manifold of a contracting periodic point $c_i$ for $i = 1, \ldots, m$. Suppose the following conditions hold.

1. $I(1) \cup \cdots \cup I(p) = W$.
2. For $j = 1, \ldots, p$, $f$ maps $I(j)$ homeomorphically onto its image.
3. For any $i = 1, \ldots, p$ and $j = 1, \ldots, p$, $f(I(i)) \cap I(j)$ is an interval.

Then

$$\text{ent}(f) \leq \lim_{n \to \infty} \frac{1}{n} (\ln(K_n(f, \mathcal{C}))).$$

Proof. Let $\delta$ denote the minimum length of the intervals $S^1 - I(j)$ where $j = 1, \ldots, p$. Let $\mathcal{E}$ be any finite cover of $I(1) \cup \cdots \cup I(p)$ by open
intervals with the diameter of $\mathfrak{O}$ less than $\delta$. Let $k = \text{card}(\mathfrak{O})$.

Let

$$I(j_1, j_2, \ldots, j_n) = I(j_i) \cap f^{-1}(I(j_2)) \cap \cdots \cap f^{(-n+1)}(I(j_n))$$

where $j_i \in \{1, 2, \ldots, p\}$ $\forall i = 1, \ldots, n$. Then each nonempty $I(j_1, j_2, \ldots, j_n)$ is a closed interval and $f$ maps $I(j_1, j_2, \ldots, j_n)$ homeomorphically into $I(j_2, \ldots, j_n)$. For any fixed $I(j_1, j_2, \ldots, j_n)$, by Lemma 5 (with $K_i = I(j_1, \ldots, j_n)$), there is a subset of $(\ast)$ which covers $I(j_1, j_2, \ldots, j_n)$ of cardinality at most $n \cdot k$.

Let $X = S^1 - (\bigcup_{i=1}^m W^s(c_i))$. Then $X$ is a compact set with $f(X) \subset X$ and $f^{-1}(X) \subset X$.

Let $\mathfrak{O}(X)$ be the open cover of $X$ defined by $\mathfrak{O}(X) = \{A \cap X: A \in \mathfrak{O}\}$. Then for any $I(j_1, j_2, \ldots, j_n)$, the minimal number of open sets of $\mathfrak{O}(X)$ needed to cover $X \cap I(j_1, j_2, \ldots, j_n)$ is equal to the minimal number of open sets of $(\ast)$ needed to cover $X \cap I(j_1, j_2, \ldots, j_n)$. Also $X$ is contained in the union of all the $I(j_1, j_2, \ldots, j_n)$. Hence

$$M_n (f|X, \mathfrak{O}(X)) \leq (K_n(f, \mathfrak{C})) \cdot n \cdot k.$$

This implies that

$$\text{ent}(f|X, \mathfrak{O}(X)) < \lim_{n \to \infty} \frac{1}{n} \left( \ln(K_n(f, \mathfrak{C})) \right).$$

Since the diameter of $\mathfrak{O}(X)$ may be taken to be arbitrarily small, we have

$$\text{ent}(f|X) < \lim_{n \to \infty} \frac{1}{n} \left( \ln(K_n(f, \mathfrak{C})) \right).$$

But $X$ contains all nonwandering points of $f$ except for the finite set \{c_1, c_2, \ldots, c_m\}. Thus, using Propositions 1-4,

$$\text{ent}(f) = \text{ent}(f|\mathfrak{O}(f)) = \text{ent}(f|X) < \lim_{n \to \infty} \frac{1}{n} \left( \ln(K_n(f, \mathfrak{C})) \right).$$

Q.E.D.

4. Construction of the sequence $f_n$. Let $f_n$ be any map in $C^2(S^1, S^1)$ which satisfies properties (1)-(4) of $g$ in §1 and the following:

(5') There are points $l \in (e, t)$ and $k \in (t, s)$ with $f(l) = f(k) = e$.

(6') $g$ is one-to-one on each of the intervals $(e, l)$, $(l, t)$, $(t, s)$, and $(s, e)$.

(7') $f_n(t) \in \text{slnsm}(c^{-n})$ where $c^{-n}$ is defined as follows. Let $c^0 = c$. Then for $i = 1, \ldots, n$ let $c^{-i}$ denote the unique inverse image (under $f_n$) of $c^{-i+1}$ in $(e, l)$. Recall that slsm($c^{-n}$) denotes the component of $W^s(c)$ which contains $c^{-n}$.

The map $f_2$ is pictured in Figure 2. Let $H_1$, $H_2$, $H_3$, $H_4$, $H_5$ be the disjoint
closed intervals which form the complement of

$$\text{slsm}(c) \cup \text{slsm}(c^{-1}) \cup \text{slsm}(c^{-2}) \cup \text{slsm}(c^{-3}) \cup \text{slsm}(t).$$

Note that $\text{slsm}(c) = (k, e)$. We can define a $5 \times 5$ matrix $A_3$ by $A_3(i,j) = 1$ if $f_3(H_i) \cap H_j \neq \emptyset$ and $A_3(i,j) = 0$ otherwise. Note that $f_3(H_i) \cap H_j \neq \emptyset$ implies $f_3(H_i) \supset H_j$. It is easy to see from Figure 2 that

$$A_3 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We can define a matrix $A_n$ analogously, and $A_n$ is the $(n + 2) \times (n + 2)$ matrix

$$A_n = \begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}$$
where the missing rows have ones on the super diagonal and zeros elsewhere.

The following proposition follows from Theorem D of [2].

**Proposition 7.** $\text{ent}(f_n) = \ln(\lambda_n)$ where $\lambda_n$ denotes the largest eigenvalue of $A_n$.

**Theorem 8.** $\text{ent}(f_n) \to 0$ as $n \to \infty$.

**Proof.** A straightforward calculation shows that for $n \geq 3$ the characteristic polynomial of $A_n$ is $p_n(x) = (-1)^n(x)(x^{n+1} - x^n - 2)$. Now $\lambda_n$ is the largest root of $p_n$, and it is easy to see that $\lambda_n \to 1$ as $n \to \infty$. The theorem now follows from Proposition 7. Q.E.D.

**Proposition 9.** For any neighborhood $N$ of $g$ in $C^2(S^1, S^1)$, there is a map in $N$ with positive entropy (and hence by Propositions 1 and 2 infinite nonwandering set).

**Proof.** Let $N$ be any neighborhood of $g$ in $C^2(S^1, S^1)$. There is (for large enough $n$) a map $h \in N$ satisfying properties (1)–(4) and (5')–(7') of the map $f_n$ in the sequence defined above. Hence $\text{ent}(h) = \ln(\lambda_n)$ where $\lambda_n$ is the largest root of $p_n(x) = (-1)^n(x)(x^{n+1} - x^n - 2)$. Clearly $\lambda_n > 1$, so $\ln(\lambda_n) > 0$. Q.E.D.

5. Proof of Theorem A.

**Theorem A.** The map $\text{ent}: C^2(S^1, S^1) \to \mathbb{R}$ is continuous at $g$.

**Proof.** Let $\epsilon > 0$. Choose $N$ large enough that $\text{ent}(f_N) < \epsilon$ where $f_N$ is the $N$th term of the sequence defined in §4. Choose $\delta > 0$ such that if $d(g, f) < \delta$, where $d$ denotes a metric on $C^2(S^1, S^1)$, then the following hold.

1. $f$ has an expanding fixed point $e(f)$ and a contracting fixed point $c(f)$ and these are the only fixed points of $f$.
2. $f$ preserves orientation at $e(f)$ and $c(f)$.
3. $f$ has nondegenerate singularities $t(f)$ and $s(f)$ and these are the only singularities of $f$.
4. The points $e(f), t(f), s(f), f(s(f))$, and $c(f)$ are distinct and in order on the circle in the counterclockwise direction.
5. Either (5A) holds or (5B) and (5C) hold.
   5A) $f(t(f)) \in [c(f), e(f)]$ and $f$ is one-to-one on each of the intervals $(e(f), t(f)), (t(f), s(f))$ and $(s(f), e(f))$.
   5B) There are points $l(f) \in (e(f), t(f))$ and $k(f) \in (t(f), s(f))$ with $f(l(f)) = f(k(f)) = e(f)$.
   Also $f$ is one-to-one on each of the intervals $(e(f), l(f)), (l(f), t(f)), (t(f), s(f))$, and $(s(f), e(f))$.
   5C) $f(t(f)) \in (e(f), c^{-N}(f))$ where $c^{-N}(f)$ is defined as follows. Let
\(c^0(f) = c(f)\). Then for \(i = 1, \ldots, N\) let \(c^{-i}(f)\) denote the unique inverse image (under \(f\)) of \(c^{-i+1}(f)\) in \((e(f), l(f))\).

Let \(f \in C^2(S^1, S^1)\) with \(d(g, f) < \delta\). We will show that \(\text{ent}(f) < \varepsilon\). If property (5A) above holds, it follows that \(\Omega(f) = \{e(f), c(f)\}\), and \(\text{ent}(f) = 0\). Hence we may assume that (5B) and (5C) hold.

We define a collection of proper closed intervals \(C(f) = \{I_1, \ldots, I_{N+2}\}\) as follows. Let \(I_1, \ldots, I_N\) be the components of the complement in \([e(f), c^{-1}(f)]\) of

\[\text{slsm}(c^{-1}(f)) \cup \cdots \cup \text{slsm}(c^{-N}(f)).\]

Let \(I_{N+1} = [I(f), t(f)]\) and \(I_{N+2} = [t(f), k(f)]\). Then if \(W\) is the complement in \(S^1\) of

\[\text{slsm}(c(f)) \cup \text{slsm}(c^{-1}(f)) \cup \cdots \cup \text{slsm}(c^{-N}(f))\]

we have \(I_1 \cup \cdots \cup I_{N+2} = W\). Hence by Theorem 6,

\[\text{ent}(f) < \lim_{n \to \infty} \frac{1}{n} \left(\ln(K_n(f, C(f)))\right).\]

Let \(H_1, \ldots, H_{N+2}\) be the components of the complement in \(S^1\) of the following set (defined with respect to \(f_N\)):

\[\text{slsm}(c) \cup \text{slsm}(c^{-1}) \cup \cdots \cup \text{slsm}(c^{-N}) \cup \text{slsm}(t).\]

Let \(h = f_N\) and \(\mathcal{D}(h) = \{H_1, \ldots, H_{N+2}\}\).

It will be helpful for the reader to see Figure 2, in which \(N = 3\) and \(H_1, H_2, H_3, H_4,\) and \(H_5\) are as indicated. In the case \(N = 3\) one may also use Figure 2 for a picture of the intervals \(I_1, I_2, I_3, I_4,\) and \(I_5\). To do this, of course, we must replace \(e, t, c,\) etc., by \(e(f), t(f), c(f),\) etc. Then in the modified figure, \(I_1, I_2,\) and \(I_3\) are intervals corresponding to \(H_1, H_2,\) and \(H_3,\) while \(I_4 = [t(f), k(f)]\) and \(I_5 = [t(f), k(f)].\)

We may assume that the \(H_i\) are numbered as in Figure 2, and the \(I_i\) are numbered analogously. We claim that for each positive integer \(n, K_n(f, C(f)) \leq K_n(h, \mathcal{D}(h)).\) To prove this claim, suppose that

\[I(j_1) \cap f^{-1}(I(j_2)) \cap \cdots \cup f^{(-n+1)}(I(j_n)) \neq \emptyset.\]

Then \(f(I(j_i)) \cap I(j_{i+1}) \neq \emptyset\) for \(i = 1, \ldots, n - 1\). By construction, whenever \(f(I(i)) \cap I(k) \neq \emptyset, A_N(i, k) = 1\) where \(A_N\) is the matrix defined in §4. Hence \(h(H(j_i)) \supset H(j_{i+1})\) for \(i = 1, \ldots, n - 1.\) This implies that

\[H(j_1) \cap h^{-1}(H(j_2)) \cap \cdots \cap h^{(-n+1)}(H(j_n)) \neq \emptyset.\]

This proves our claim that for each positive integer \(n, K_n(f, C(f)) < K_n(h, \mathcal{D}(h)).\)

Let \(X\) be the complement in \(S^1\) of the stable manifold of \(c\) (with respect to \(h = f_N\)). Let \(\mathcal{D}(X) = \{H_1 \cap X, \ldots, H_{N+2} \cap X\}\). Then \(\mathcal{D}(X)\) is an open
cover of $X$, and for each positive integer $n$ (since the $H_i$ are pairwise disjoint),
$$K_n(h, \mathcal{D}(h)) = M_n(h|X, \mathcal{D}(X)).$$
We have for each positive integer $n$,
$$K_n(f, \mathcal{C}(f)) < K_n(h, \mathcal{D}(h)) = M_n(h|X, \mathcal{D}(X)).$$
Also,
$$\text{ent}(f) < \lim_{n \to \infty} \frac{1}{n} \left( \ln(K_n(f, \mathcal{C}(f))) \right).$$
Hence
$$\text{ent}(f) < \text{ent}(h|X, \mathcal{D}(X)) < \text{ent}(h) < \varepsilon. \quad \text{Q.E.D.}$$

**References**


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