CHARACTERISTIC NUMBERS OF G-MANIFOLDS
AND MULTIPLICATIVE INDUCTION

BY

MICHAEL BIX(') AND TAMMO TOM DIECK

ABSTRACT. We determine those finite groups $G$ for which characteristic
numbers determine $G$-equivariant bordism in the unoriented and unitary
cases.

It was shown in [6], [7] that suitable global characteristic numbers deter-
mine the bordism classes of unoriented $G$-manifolds if $G \cong (\mathbb{Z}_2)^k$ and of
unitary $G$-manifolds if $G$ is cyclic. We show in this paper that there are no
other cases in which characteristic numbers determine bordism classes. The
proof is based on an explicit computation of the equivariant characteristic
numbers of certain manifolds and on the use of a new construction in
bordism theory, which we call multiplicative induction, and which should
have many more applications.

In §1 we review some well-known facts about characteristic numbers and
explain the notation used in §2 to state the main results of this paper. §3
contains calculations. The definition of multiplicative induction and some
applications are given in §4.

1. Characteristic numbers. Let $G$ be a compact Lie group. We denote by
$\mathcal{G}(X, A)$ (respectively, $\mathcal{G}(X, A)$) the geometric bordism group of $n$-dimen-
sional (unitary) singular $G$-manifolds in $(X, A)$, where $X$ is any $G$-space and
$A$ is a $G$-subspace of $X$. If $X$ is a point and $A$ is the empty set, we write
$\mathcal{G}(\text{point}, \emptyset) = \mathcal{G}$ and $\mathcal{G}(\text{point}, \emptyset) = \mathcal{G}$.

We suppress the $G$ if $G$ is the trivial group. Similarly, we use $\mathcal{G}(X, A),
\mathcal{G}(X, A), \mathcal{G}(X, A)$, and $\mathcal{G}(X, A)$ to denote the homotopical bordism groups, defined
by means of equivariant Thom spectra [4]. We also use the corresponding
cohomology groups $\mathcal{G}(X, A)$ and $\mathcal{G}(X, A)$. The Pontrjagin-Thom
construction induces natural transformations of equivariant homology
theories

$$i: \mathcal{G}(X, A) \rightarrow \mathcal{G}(X, A) \quad \text{and} \quad i: \mathcal{G}(X, A) \rightarrow \mathcal{G}(X, A).$$

') The first author was partially supported by NSF Grant MPS 71-03109 A06.
The bundling transformations
\[ \alpha: N^n_G (X, A) \to N^n (EG \times_G X, EG \times_G A) \]
and
\[ \alpha: U^n_G (X, A) \to U^n (EG \times_G X, EG \times_G A) \]
are as defined in [4], where \( EG \) is a universal free \( G \)-space with orbit space \( EG/G = BG \).

The main ingredient in the definition of equivariant characteristic numbers is the Boardman map. In the case of unoriented bordism, this is a natural transformation of multiplicative equivariant cohomology theories
\[ B: N^* (EG \times_G X) \to H^* (EG \times_G X) [[a_1, a_2, \ldots ]] , \]
where cohomology is taken with \( \mathbb{Z}_2 \) coefficients and \( a_i \) has degree \( -i \). The definition of \( B \) and its basic properties can be found in X. Kapitel of [2]. For the unitary case we consider characteristic numbers lying in equivariant \( K \)-theory. The Boardman map
\[ B: U^n_G (X) \to K^n_G (X) [[a_1, a_2, \ldots ]] , \]
where we use \( \mathbb{Z}_2 \)-graded \( K_G \)-theory, is a natural transformation of multiplicative cohomology theories and is described in [7].

The characteristic number map for \( n \)-dimensional \( G \)-equivariant unoriented bordism, denoted \( \chi_n^G \), is defined to be the composition
\[ B a: \mathfrak{N}_n^G \to N_n^G \cong N_G^{-n} \to N^{-n} (BG) \to H^* (BG) [[a_1, a_2, \ldots ]] . \]
The characteristic number map \( \chi_n^G \) for \( n \)-dimensional \( G \)-equivariant unitary bordism is the composition
\[ B i : \mathfrak{U}_n^G \to U_n^G \cong U_G^{-n} \to R (G) [[a_1, a_2, \ldots ]] , \]
where \( R (G) \cong K_G^0 (\text{point}) \) is the complex representation ring of \( G \).

REMARKS. 1. The characteristic numbers of an \( n \)-manifold \( M \), lying in \( H^* (BG) \) or \( R (G) \), are the coefficients in \( \chi_n^G [M] \) of monomials in the \( a_i \)'s.

2. We call the characteristic numbers defined above "global", because the definition does not involve the orbit structures of manifolds. One could, alternatively, define characteristic numbers using the normal bundles to various fixed point sets, as in the work of M. Rothenberg. Such numbers would combine the orbit structures with the global characteristic numbers.

3. Of course, one could define characteristic numbers with values in other equivariant cohomology rings by an analogous procedure.

2. Characteristic numbers and bordism. The notation in this section is as above. We only consider finite groups \( G \).

THEOREM 1. (a) The composition
G-manifolds

\[ B\alpha: N^*_G \to N^*(BG) \to H^*(BG)[[a_1, a_2, \ldots ]] \]

is injective if and only if \( G \cong (\mathbb{Z}_2)^k \).

(b) The characteristic number map

\[ \chi^G_*: \Omega^G_* \to H^*(BG)[[a_1, a_2, \ldots ]] \]

is injective if and only if \( G \cong (\mathbb{Z}_2)^k \).

**Theorem 2.** (a) The Boardman map

\[ B: U^*_G \to R(G)[[a_1, a_2, \ldots ]] \]

is injective if and only if \( G \) is a cyclic group.

(b) The characteristic number map

\[ \chi^G_*: \Omega^G_* \to R(G)[[a_1, a_2, \ldots ]] \]

is injective if and only if \( G \) is a cyclic group.

**Remarks.** 1. We are mainly interested in the second parts of the above theorems. But the algebraic reasons why the results hold are clearer for the homotopical theories than for the geometric ones.

2. We conjecture that the above theorems are true for compact Lie groups. The proofs below for Theorems 1(a) and 2(a) are valid for such a generalization.

The injectivity statements are already known. If \( G \cong (\mathbb{Z}_2)^k \), it was shown in [6] that \( B\alpha \) and \( \chi^G_* \) in Theorem 1 are injective. If \( G \) is a finite cyclic group (more generally, the product of a torus and a finite cyclic group), it was proved in [7] that \( B \) and \( \chi^G_* \) in Theorem 2 are injective.

**Proofs of Theorems 1(a) and 2(a).** We begin with Theorem 1(a). If \( G \) is not isomorphic to \((\mathbb{Z}_2)^k\), \( G \) has an irreducible real representation of dimension greater than one. Let \( PV \) be the real projective space associated with such a \( G \)-module \( V \), furnished with the \( G \)-action inherited from the linear \( G \)-action on \( V \). Then the \( G \)-action on \( PV \) has no fixed points. Consider the commutative diagram

\[
\begin{array}{ccc}
N^*_G & \xrightarrow{B\alpha} & H^*(BG)[[a_1, a_2, \ldots ]] \\
\downarrow{i_1} & & \downarrow{i_2} \\
N^*_G(PV) & \xrightarrow{B\alpha(PV)} & H^*(EG \times_G PV)[[a_1, a_2, \ldots ]] 
\end{array}
\]

where \( i_1 \) and \( i_2 \) map \( x \) to \( x \cdot 1 \). The map \( i_2 \) is injective, because \( H^*(EG \times_G PV) \) is a free \( H^*(BG) \)-module. Assume that \( B\alpha \) is injective. Then \( i_1 \) is injective, also. Now localize away from the multiplicatively closed subset \( S_G \) which contains 1 and the Euler classes of \( G \)-modules with no trivial
summands. Since localization is an exact functor,

\[ S_G^{-1}i_1 : S_G^{-1}N_G^* \to S_G^{-1}N_G^* (PV) \]

is injective. But \( S_G^{-1}N_G^* \neq 0 \), by an analogue of Theorem 3.1 of [4], while

\[ S_G^{-1}N_G^* (PV) \cong S_G^{-1}N_G^* (PV^G) = S_G^{-1}N_G^* (\emptyset) = 0, \]

by Satz 1 of [5]. This is a contradiction. Hence \( B_\alpha \) cannot be injective if \( G \) is not isomorphic to \((\mathbb{Z}_2)^k\).

Theorem 2(a) can be proved similarly. If \( B \) is injective, then \( S_G^{-1}B \) is, too. But, by Lemma 1 of [7], \( S_G^{-1}R(G)[[a_1, a_2, \ldots]] \) is nonzero if and only if \( G \) is cyclic.

Proof of Theorem 2(b). It remains to show that \( \chi_*^G \) is not injective if \( G \) is not cyclic. As in [11], a set \( F \) of subgroups of \( G \) is called a family if it includes all of the subgroups and conjugates of each of its elements. Let \( EF \) denote the terminal object in the \( G \)-homotopy category of numerable \( G \)-spaces whose isotropy groups belong to \( F \). Given a pair of families \( F \supset F' \) and a \( G \)-homology theory \( h_*^G \), let

\[ h_*^G[F, F'](X, A) = h_*^G(EF \times X, (EF \times A) \cup (EF' \times X)). \]

For a subgroup \( H \) of \( G \), let \( h_*^G(X, A)^{G\text{-inv}} \) denote the “invariant elements”, that is, the cokernel of the map

\[ h_*^G((G/H) \times (G/H) \times X, (G/H) \times (G/H) \times A) \to h_*^G((G/H) \times X, (G/H) \times A) \]

defined by \((\text{pr}_i \times 1_X)_* - (\text{pr}_2 \times 1_X)_*\), where \( \text{pr}_i : (G/H) \times (G/H) \to G/H \) for \( i = 1, 2 \) are the projections. Now if \( \chi_*^G \) is injective, the rationalization \( \chi_*^G \otimes \mathbb{Q} \) is injective, because the range of \( \chi_*^G \) is torsion-free. There is a splitting of \( \mathbb{Q} \chi_*^G \otimes \mathbb{Q} \), described in more general terms in Theorem 1 of [8], which is given by

\[ \mathbb{Q} \chi_*^G \otimes \mathbb{Q} \cong \bigoplus_{(H)} \mathbb{Q} \chi_*^H[\text{All, Prop}](\text{point}, \emptyset)^{G\text{-inv}} \otimes \mathbb{Q}, \]

where \( \text{All} \) is the family of all subgroups of \( G \), \( \text{Prop} \) the family of proper subgroups of \( G \), and \( H \) ranges over a complete set of conjugacy class representatives of the subgroups of \( G \). Because the splitting is compatible with the natural transformations \( i \otimes \mathbb{Q} \) and \( B \otimes \mathbb{Q} \), there is a commutative diagram

\[
\begin{array}{ccc}
\mathbb{Q} \chi_*^G \otimes \mathbb{Q} & \xrightarrow{i \otimes \mathbb{Q}} & U_*^G \otimes \mathbb{Q} \\
\downarrow{s_1} & & \downarrow{s_2} \\
\bigoplus_{(H)} \mathbb{Q} \chi_*^H[\text{All, Prop}]^{G\text{-inv}} \otimes \mathbb{Q} & \xrightarrow{B \otimes \mathbb{Q}} & R(G)[[a_1, a_2, \ldots]] \otimes \mathbb{Q} \\
\end{array}
\]

\[
\begin{array}{ccc}
\downarrow{s_3} & & \\
\bigoplus_{(H)} S_*^1U_*^H(X)^{G\text{-inv}} \otimes \mathbb{Q} & \xrightarrow{s_3} & \bigoplus_{(H)} S_*^1R(H)[[a_1, a_2, \ldots]] \otimes \mathbb{Q} \\
\end{array}
\]
where the vertical maps are the splitting maps. The group $U_H^H\text{[All, Prop]}$ may be thought of, as in [9], as $S_H^{-1}U_H^H$. The maps $s_1$ and $s_2$ are isomorphisms. The elements of $\mathcal{U}_G \otimes \mathbb{Q}$ corresponding to summands $\mathcal{U}_G^{\text{Prop}} \otimes \mathbb{Q}$ for which $H$ is not cyclic are contained in the kernel of $\chi_*^G \otimes \mathbb{Q}$.

The remainder of this paper is chiefly concerned with a proof of the "only if" part of Theorem 1(b), that is, that $\chi_*^G$ is not injective when $G$ does not have the form $(\mathbb{Z}_2)^k$. In fact, we exhibit explicit bordism classes which lie in the kernel of $\chi_*^G$.

3. Computations of equivariant characteristic numbers. A convenient and conceptually simple method of calculating characteristic numbers uses exponential characteristic classes and the map given by evaluation on the fundamental class. In this section we describe that method.

If $\eta: E \to X$ is a real $G$-vector bundle, let $\eta_G$ denote the bundle

$$1_{EG} \times_G \eta: EG \times_G E \to EG \times_G X.$$ 

If $A[[a_1, a_2, \ldots]]$ is a power series ring, define conjugate generators $a_1, a_2, \ldots$ by

$$1 + \tilde{a}_1 t + \tilde{a}_2 t^2 + \ldots = (1 + a_1 t + a_2 t^2 + \ldots)^{-1},$$

where $t$ is an indeterminate. An exponential characteristic class for real $G$-vector bundles associates to each such bundle $\xi$ an element $v(\xi) \in H^*(EG \times_G X)[[a_1, a_2, \ldots]]$ of degree zero (recall that $a_i$ has degree $-i$) with constant term 1. The characteristic class must be natural with respect to $G$-bundle maps and must satisfy the exponential law $v(\xi \oplus \eta) = v(\xi)v(\eta)$.

**Proposition 1.** There is exactly one exponential characteristic class $v$ (respectively, $\tilde{v}$) for real $G$-vector bundles which satisfies the condition that

$$v(\eta) = 1 + (w_1(\eta_G))a_1 + (w_1(\eta_G))^2 a_2 + \ldots$$

$$+ \ldots (\tilde{v}(\eta) = 1 + (w_1(\eta_G))\tilde{a}_1 + (w_1(\eta_G))^2 \tilde{a}_2 + \ldots)$$

for every real $G$-line bundle $\eta$, where $w_1$ denotes the first Stiefel-Whitney class. In general, $\tilde{v}(\xi)$ can be obtained from $v(\xi)$ by applying the conjugation automorphism for a power series ring.

**Proof.** It suffices to construct $v$ for universal bundles. This can be done by using the splitting principle, as in VIII.X. Kapitel of [2].

**Remark.** An analogous proposition is true for complex $G$-vector bundles, where now the characteristic classes lie in $K_G^*(X)[a_1, a_2, \ldots]$, as in formula (4) on p. 35 of [7].

Now let $M$ be a closed $n$-dimensional $G$-manifold. Let $x \mapsto x[M]$ be the $H^*(BG)$-linear map
known variously as the integration along the fibre map, evaluation on the fundamental class, or the Gysin homomorphism (recall we are using $\mathbb{Z}_2$ coefficients for cohomology). Extend this map linearly to power series rings, still denoting it $x \mapsto x[M]$. Let $\bar{v}$ be the exponential characteristic class defined in Proposition 1. We use $\tau_M$ to denote the tangent bundle of $M$, an $n$-dimensional real $G$-vector bundle.

**Proposition 2.** $\chi^G_*(M) = (\bar{v}(\tau_M))[M]$.

**Proof.** Observing that $v(\tau_M)v(\nu_M) = v(V)$, where $\nu_M$ is the normal bundle of a $G$-equivariant embedding of $M$ in a real $G$-module $V$, the proof is formally the same as that of Satz 5.9 of [2].

Now let $V$ be a real $G$-module and $\varepsilon$ the trivial real $G$-line bundle over $PV$ with trivial $G$-action on the fibre. Then

$$\tau_{PV} \oplus \varepsilon \cong V \otimes \eta,$$

where $\eta$ is the Hopf line bundle over $PV$ and $V$ denotes the trivial vector bundle over $PV$ with fibre $V$. To compute the characteristic numbers of $PV$, we use the following special case of the standard projective bundle theorem.

**Proposition 3.** $H^*(EG \times_G PV)$ is a free $H^*(BG)$-module on generators $1$, $b$, $b^2$, ..., $b^{n-1}$, where $b = w_1(\eta_G) \in H^1(EG \times_G PV)$ and $n$ is the dimension of $V$. Furthermore,

$$b^n = \sum_{i=1}^{n} (w_i(V_G))b^{n-i},$$

where $V_G$ is the real $n$-plane bundle $EG \times_G V \to BG$, the bundle map being projection on the first factor. The map $x \mapsto x[PV]$ sends $1$, $b$, $b^2$, ..., $b^{n-2}$ to zero and $b^{n-1}$ to 1.

Now let $\xi: E \to M$ be a two-dimensional $G$-vector bundle with $w_1(\xi_G) = 0$ and $w_2(\xi_G) = x$. As usual, we can write Stiefel-Whitney classes formally as elementary symmetric polynomials $w_1 = y_1 + y_2$ and $w_2 = y_1y_2$. Then

$$v(\xi) = (1 + y_1a_1 + y_2a_2 + \ldots)(1 + y_1a_1 + y_2a_2 + \ldots)$$

$$= 1 + y_1y_2a_1^2 + y_1^2y_2^2a_2^2 + \ldots$$

$$= 1 + xa_1^2 + x^2a_2^2 + \ldots,$$

computing modulo 2 and using $y_1 + y_2 = 0$. We next apply this computation and Proposition 3 to the case in which

$$V = V(1) \oplus \cdots \oplus V(r) \oplus W(1) \oplus \cdots \oplus W(s),$$

where $V(i)$ and $W(j)$ are irreducible real $G$-modules, each $V(i)$ has dimension one, and each $W(j)$ has dimension two. Assume $w_1(V(i)_G) = x(i)$,
$w_1(W(j)_G) = 0$, and $w_2(W(j)_G) = y(j)$. Then using the notation of Proposition 3 we have

**Proposition 4.**

$$v(\tau_{PV}) = \prod_{i=1}^{r} \left(1 + (x(i) + b)a_i + (x(i) + b)^2a_2 + \ldots \right)$$

$$\times \prod_{j=1}^{s} \left(1 + (y(j) + b^2)a_1^2 + (y(j) + b^2)^2a_2^2 + \ldots \right).$$

**Proof.** By Proposition 1 and (1),

$$v(\tau_{PV}) = v(\tau_{PV})v(\varepsilon) = v(\tau_{PV} \oplus \varepsilon) = v(V \otimes \eta)$$

$$= v((V(1) \oplus \cdots \oplus V(r) \oplus W(1) \oplus \cdots \oplus W(s)) \otimes \eta)$$

$$= v((V(1) \otimes \eta) \oplus \cdots \oplus (V(r) \otimes \eta) \oplus (W(1) \otimes \eta) \oplus \cdots \oplus (W(s) \otimes \eta))$$

$$= \prod_{i=1}^{r} v(V(i) \otimes \eta) \prod_{j=1}^{s} v(W(j) \otimes \eta).$$

Since $V(i) \otimes \eta$ is a real $G$-line bundle, by Proposition 1 we have

$$v(V(i) \otimes \eta) = 1 + \left(w_1\left((V(i) \otimes \eta)_G\right)\right)a_1 + \left(w_1\left((V(i) \otimes \eta)_G\right)\right)^2a_2 + \ldots$$

$$= 1 + \left(w_1(V(i)_G) + w_1(\eta_G)\right)a_1 + \left(w_1(V(i)_G) + w_1(\eta_G)\right)^2a_2 + \ldots$$

$$= 1 + (x(i) + b)a_1 + (x(i) + b)^2a_2 + \ldots.$$  

Similarly, $v(W(j) \otimes \eta)$ can be computed from (2) and the formula for the Stiefel-Whitney classes of a tensor product.

We apply Proposition 4 to the case in which $G = \mathbb{Z}_4$. Let 1 denote the trivial one-dimensional real representation of $\mathbb{Z}_4$, $-1$ its nontrivial one-dimensional real representation, and $i$ the two-dimensional real representation which comes from the standard representation of $\mathbb{Z}_4 \subset S^1$ on $\mathbb{C}$. Let $u$ denote the unique nonzero element of $H^1(B\mathbb{Z}_4)$, so that $u^2 = 0$, and $d$ the unique nonzero element of $H^2(B\mathbb{Z}_4)$. Let $V = 1^{2n+1} \oplus i$ and $W = 1^{2n} \oplus (-1) \oplus i$.

By Proposition 3, $H^*(EZ_4 \times_{\mathbb{Z}_4} PV)$ is a free $H^*(B\mathbb{Z}_4)$-module on generators $1, b, b^2, \ldots, b^{2n+2}$, where $b^{2n+3} = db^{2n+1}$. The relation implies:

$$b^{2m} = d^{m-n-1}b^{2n+2}, \quad \text{if } m \geq n + 1,$$

$$b^{2m+1} = d^{m-n}b^{2n+1}, \quad \text{if } m > n.$$

Similarly, $H^*(EZ_4 \times_{\mathbb{Z}_4} PW)$ has free $H^*(B\mathbb{Z}_4)$-generators $1, b, b^2, \ldots, b^{2n+2}$, where $b^{2n+3} = udb^{2n} + db^{2n+1} + ub^{2n+2}$. Thus for $m \geq n + 1$ we have:
\[ b^{2m} = d^{m-n-l}b^{2n+2} \quad \text{and} \quad b^{2m+1} = ud^{m-n-l}b^{2n} + d^{m-n-l}b^{2n+1} + ud^{m-n-l}b^{2n+2}. \]

Using the above notation and following [1], we have

**Proposition 5.** \( \chi_{2n+2}^{Z_4}[PV] = \chi_{2n+2}^{Z_4}[PW] \).

**Proof.** Notice that \( w_1((-1)z_4) = u, \ w_1(i_{Z_4}) = 0, \) and \( w_2(i_{Z_4}) = d \).

Propositions 1, 2, and 4 imply that \( \chi_{2n+2}^{Z_4}[PV] \) is the conjugate of the coefficient of \( b^{2n+2} \) in \( (\sum_{j=0}^{\infty} a_j(b^2 + d)^j)(\sum_{j=0}^{\infty} a_j(b + u)^j) \). And now using (3) we find that \( \chi_{2n+2}^{Z_4}[PV] \) has nonzero homogeneous components only in even dimensions. Similarly, \( \chi_{2n+2}^{Z_4}[PW] \) is the conjugate of the coefficient of \( b^{2n+2} \) in

\[
\left( \sum_{j=0}^{\infty} a_j b^j \right)^{2n} \left( \sum_{j=0}^{\infty} a_j^2(b^2 + d)^j \right) \left( \sum_{j=0}^{\infty} a_j(b + u)^j \right).
\]

Because \( u^2 = 0 \), in even dimensions this product is

\[
\left( \sum_{j=0}^{\infty} a_j b^j \right)^{2n+1} \left( \sum_{j=0}^{\infty} a_j^2(b^2 + d)^j \right).
\]

Since \( b^{2m} = d^{m-n-l}b^{2n+2}, \) if \( m > n + 1 \), in both \( H^*(EZ_4 \times Z_4 PV) \) and \( H^*(EZ_4 \times Z_4 PW) \), we see that \( \chi_{2n+2}^{Z_4}[PV] \) and \( \chi_{2n+2}^{Z_4}[PW] \) are the same in even dimensions. Thus it only remains to show that \( b^{2n+2} \) has no nonzero odd-dimensional coefficients in

\[
\left( \sum_{j=0}^{\infty} a_j b^j \right)^{2n} \left( \sum_{j=0}^{\infty} a_j^2(b^2 + d)^j \right) \left( \sum_{j=0}^{\infty} a_j(b + u)^j \right).
\]

But such a term would have to be of the form \( a^k d^j(b + u)^{j+1}b^{2k} \), where \( k + j > n + 1 \). In all such cases \( b^{2n+2} \) has a zero coefficient, since

\[ b^{2m+1} + ub^{2m} = ud^{m-n-l}b^{2n} + d^{m-n-l}b^{2n+1}, \quad \text{if} \ m > n, \]

by (4).

As in [1], Proposition 5 implies

**Proposition 6.** \( \chi_{Z_4}^{Z_4} \) is not injective.

**Proof.** The fixed point set of \( PV \) is \( RP^{2n} \). The fixed point set of \( PW \) is the disjoint union of a point and \( RP^{2n-1} \). Since, if \( n > 1 \), these two fixed point sets represent different bordism classes in the unoriented bordism ring \( \Omega_* \). \( PV \) and \( PW \) cannot represent the same equivariant bordism class in \( \Omega_* \). But their \( Z_4 \)-characteristic numbers are the same, by Proposition 5.

**Proposition 7.** If \( G \) is not a 2-group, \( \chi_G^{Z_4} \) is not injective.
Proof. Let $H$ be a 2-Sylow subgroup of $G$. Since $|N_G(H) : H| \equiv 1 \pmod{2}$, $\mathfrak{N}_H^G$ is a $\mathbb{Z}_2$-vector space having at least two generators, a point with trivial $G$-action and $G/H$ with the $G$-action defined by left multiplication (Proposition 13.1 of [11]). But if $M$ is any zero-dimensional $G$-manifold,

$$\chi^G_0[M] \in (H^*(BG)[[a_1, a_2, \ldots]])^0 \cong \mathbb{Z}_2$$

is nonzero if and only if $M$ consists of an odd number of points. Since $|G : H| \equiv 1 \pmod{2}$, the two generators of $\mathfrak{N}_H^G$ described above are both mapped into 1 by $\chi^G_0$.

It follows from Proposition 7 that in order to prove Theorem 1(b) we need only consider 2-groups. We start with the examples used to prove Proposition 6 and construct new manifolds by a method which we call “multiplicative induction”.

4. Multiplicative induction. Let $G$ be a finite group and denote by $G$ the category whose objects are $G$-homeomorphism classes of left $G$-spaces and whose morphisms are continuous $G$-maps. If $H$ is a subgroup of $G$, the restriction functor $r_H^G : G \to H$ forgets the actions of the elements of $G$ which are not in $H$. The functor $r_H^G$ has a left adjoint which maps an $H$-space $X$ to $G \times_H X$ with $G$-action given by $g_1(g_2, x) = (g_1g_2, x)$. This left adjoint is additive, but is not, in general, multiplicative. A right adjoint $m_H^G$ to $r_H^G$ can be defined on objects of $H$ by $m_H^G(X) = \text{Hom}_H(G, X)$, where we consider $G$ an $H$-space via left multiplication. The $G$-action on $m_H^G(X)$ is given by $(g_1 \cdot f)(g_2) = f(g_2g_1)$. An $H$-map $f : X \to Y$ induces a $G$-map

$$m_H^G(f) : m_H^G(X) \to m_H^G(Y)$$

by composition. The usual relationship between adjoint functors is, in our particular case, that

$$\text{Hom}_G(X, m_H^G(Y)) \cong \text{Hom}_H(r_H^G(X), Y)$$

for any $G$-space $X$ and $H$-space $Y$. We call the functor $m_H^G$ “multiplicative induction” because

$$m_H^G(X_1 \times X_2) \cong m_H^G(X_1) \times m_H^G(X_2)$$

for any $H$-spaces $X_1$ and $X_2$. Observe, also, that $m_H^G(X)$ is homeomorphic to the set of all continuous maps $f : G/H \to G \times_H X$ such that $\pi_1 \circ f = 1_{G/H}$, where $\pi_1 : G \times_H X \to G/H$ is the projection on the first factor and the function space is given the compact-open topology. And now that space is, in turn, homeomorphic to the product of $|G : H|$ copies of $X$.

If $M$ is an $n$-dimensional smooth $H$-manifold, then $m_H^G(M)$ is, in a natural way, a smooth $G$-manifold of dimension $n|G : H|$. It is surprising that the functor $m_H^G$ is compatible with the bordism relation. That is, let $f : M \to X$ be
a singular $H$-manifold in $X$. Then $m^G_H(f): m^G_H(M) \to m^G_H(X)$ is a singular $G$-manifold in $m^G_H(X)$.

**Proposition 8.** The function $f \mapsto m^G_H(f)$ induces a well-defined map

$$m^G_H: \mathfrak{N}_n^H(X) \to \mathfrak{N}_{n|G:H|}^G(m^G_H(X))$$

which takes products to products but is, in general, not additive.

**Proof.** A proof can be found in [10]. The idea is to apply a suitable Pontrjagin-Thom construction, as in [12], to convert the bordism relation into a homotopy relation, and then to use the fact that the image under $m^G_H$ of an $H$-homotopy is a $G$-homotopy.

**Remarks.** 1. It is, in general, extremely difficult to compute the effect of $m^G_H$ on characteristic numbers. One approach would be to elaborate on the work of Evans.

2. The Steenrod power operation is a special case of multiplicative induction. For its effect on characteristic numbers, see 16.5 of [3].

3. Since $m^G_H$ is not, in general, additive, a more general construction would start with nonhomogeneous bordism elements, that is, sums of manifolds of different dimensions.

To use $m^G_H$ for our purposes, we study its effect on the Pontrjagin-Thom construction and the bundling map. Using a construction similar to $m^G_H$ for $H$-spaces with base points, we construct induction maps for $N^*_H(X)$ and $N^*(EH \times_H X)$.

Let $X_0$ be a pointed $H$-space. Define an $H$-map $p: X = G \times_H X_0 \to G/H$ by $p(g, x) = gH$. Let

$$n^G_H(X) = \bigwedge_{a \in G/H} p^{-1}(a),$$

where $\bigwedge$ denotes the smash product, using the base point which $p^{-1}(a)$ inherits from $X_0$. Define a $G$-action on $n^G_H(X)$ by $g \cdot p^{-1}(a) = p^{-1}(ga)$. Recall that $m^G_H(X)$ is naturally $G$-homeomorphic to a similarly defined object, $\prod_{a \in G/H} p^{-1}(a)$. Clearly $n^G_H$ extends to a functor from pointed $H$-spaces to pointed $G$-spaces and is compatible with equivariant pointed homotopies and smash products. That is, $n^G_H(X \wedge Y) = n^G_H(X) \wedge n^G_H(Y)$. Letting $M(\xi)$ denote the Thom space of the vector bundle $\xi$, there is a natural isomorphism between $n^G_H(M(\xi))$ and $M(m^G_H(\xi))$.

Recall that the group $\tilde{N}^*_H(X)$ is defined (at least for compact $X$) as a direct limit of pointed $H$-homotopy sets $[V^c \wedge X, M(\xi^k)]^0_H$, where $V^c$ is the one-point compactification of the $G$-module $V$ of dimension $|V|$ and $\xi^k_H$ is the universal $k$-dimensional $H$-vector bundle. Applying the functor $n^G_H$, we get maps
the last map being induced by the classifying map for $m_H^G(\xi^k_H)$. Passing to the limit, we obtain the multiplicative induction map for homotopical bordism,

$$n_H^G: \tilde{N}_H^n(X) \to \tilde{N}_G^n[G:H|^1(n_H^G(X)).$$

If $X$ is a free $G$-space, there is a natural isomorphism $N_G^*(X) \cong N^*(X/G)$. Using this, the bundling map $\alpha: N_G^0(X) \to N^0(EG \times_G X)$ is induced by the projection $EG \times X \to X$. The multiplicative induction $n_H^G$ for $N^*(EG \times_G X)$ is defined in these terms to be the composition

$$N_k^H(EG \times X) \to N_G^k[G:H|^1(m_H^G(EG \times X))$$

$$\cong N_G^k[G:H|^1(m_H^G(EG) \times m_H^G(X))$$

$$\to N_G^k[G:H|^1(EG \times m_H^G(X)),$$

where the last map is induced by the map $EG \to m_H^G(r_H^G(EG))$ which corresponds to the identity map of $r_H^G(EG)$ under the adjunction map mentioned above (we may take $EH$ to be $r_H^G(EG)$). Now one only needs to apply the definitions to prove

**Proposition 9.** The following diagrams are commutative:

**Corollary.** If $M_1$ and $M_2$ are $H$-manifolds with the same $H$-equivariant characteristic numbers, then $m_H^G(M_1)$ and $m_H^G(M_2)$ are $G$-manifolds with the same $G$-equivariant characteristic numbers.
We can now prove Theorem 1(b), using the examples and notation of Proposition 5. Let $G$ be a 2-group which is not of the form $(\mathbb{Z}_2)^k$. Then $G$ has at least one subgroup $H$ such that $H \cong \mathbb{Z}_4$. We fix one such $H$ and consider the multiplicative induction $m_H^G$.

**Proposition 10.** For each positive integer $n$, the $G$-manifolds $m_H^G(PV)$ and $m_H^G(PW)$ represent different bordism classes in $\mathcal{M}_{2n+2}(G:H)$.

**Proof.** It suffices to show that the $H$-fixed point sets of $m_H^G(PV)$ and $m_H^G(PW)$ represent different bordism classes in $\mathcal{M}_n$. Let $G/H = \bigsqcup_i M(i)$ be the decomposition of the $H$-space $G/H$ into its orbits. Then $M(i) \cong H/H(i)$, where $H(i) \cong \{1\}$, $\mathbb{Z}_2$, or $\mathbb{Z}_4$. If $M$ is any $H$-manifold,

$$(m_H^G(M))^H \cong \text{Hom}_G(G/H, m_H^G(M)) \cong \text{Hom}_H(G/H, M) \cong \prod_i \text{Hom}_H(M(i), M) \cong \prod_i M^H(i).$$

Now suppose that $H(i) \cong \{1\}$ for $b$ values of $i$, $\cong \mathbb{Z}_2$ for $c$ values, and $\cong \mathbb{Z}_4$ for $d$ values. Then

$$(m_H^G(PV))^H \cong (\mathbb{R}P^{2n+2})^b (\mathbb{R}P^{2n})^c (\text{point})^d$$

and

$$(m_H^G(PW))^H \cong (\mathbb{R}P^{2n+2})^b (\mathbb{R}P^{2n})^c (\text{point})^d.$$ 

And these manifolds are not bordant if $d > 0$ and $n > 0$.

Proposition 10 and the corollary to Proposition 9 complete the proof of Theorem 1(b).

**References**


**DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN 53706**

**MATHEMATISCHES INSTITUT DER UNIVERSITÄT, D-3400 GÖTTINGEN, BUNSENSTRASSE 3-5, FEDERAL REPUBLIC OF GERMANY** (Current Address of Tammo tom Dieck)

**Current address** (Michael Bix): 1977 York Lane, Highland Park, Illinois 60035