NEARNESSES, PROXIMITIES, AND $T_1$-COMPACTIFICATIONS

BY

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ABSTRACT. Gagrat, Naimpally, and Thron together have shown that separated Lodato proximities yield $T_1$-compactifications, and conversely. This correspondence is not 1-1, since nonequivalent compactifications can induce the same proximity. Herrlich has shown that if the concept of proximity is replaced by that of nearness then all principal (or strict) $T_1$-extensions can be accounted for. (In general there are many nearnesses compatible with a given proximity.) In this paper we obtain a 1-1 correspondence between principal $T_1$-extensions and cluster-generated nearnesses. This specializes to a 1-1 match between principal $T_1$-compactifications and contigual nearnesses.

These results are utilized to obtain a 1-1 correspondence between Lodato proximities and a subclass of $T_1$-compactifications. Each proximity has a largest compatible nearness, which is contigual. The extension induced by this nearness is the construction of Gagrat and Naimpally and is characterized by the property that the dual of each clan converges. Hence we obtain a 1-1 match between Lodato proximities and clan-complete principal $T_1$-compactifications. When restricted to EF-proximities, this correspondence yields the usual map between $T_2$-compactifications and EF-proximities.

0. Notation. Let $\mathcal{E}$ and $\mathfrak{B}$ be families of subsets of a topological space $X$. Let $\lambda$ be a collection of families of subsets of $X$, and let $A \subseteq X$. 
(1) $\mathcal{E} \subset \mathfrak{B}$ iff each set in $\mathfrak{B}$ contains a set in $\mathcal{E}$;
(2) $\mathcal{E} \vee \mathfrak{B} = \{A \cup B : A \in \mathcal{E} \text{ and } B \in \mathfrak{B}\}$;
(3) $c_\lambda(A) = \{x \in X : \{A, \{x\}\} \in \lambda\}$;
(4) $c_\lambda(\mathcal{E}) = \{c_\lambda(A) : A \in \mathcal{E}\}$;
(5) $A^\sim$ is the topological closure of $A$;
(6) $\mathcal{E}^\sim = \{A^\sim : A \in \mathcal{E}\}$.

1. Nearnesses and extensions.

A. Obtaining nearnesses from extensions. In this part we will develop a map $\mathcal{T}_1$ from the $T_1$-extensions of a topological space to its compatible nearnesses.

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The main result states that principal $T_1$-extensions give rise (under $T_r$) to cluster-generated compatible Lodato nearnesses.

1.1 DEFINITION. A nearness on a set $X$ is a collection $\nu$ of families of subsets of $X$ such that

(N1) $\bigcap \mathcal{A} \neq \emptyset \Rightarrow \mathcal{A} \in \nu$;

(N2) If $\mathcal{A} \in \nu$ and $\mathcal{A} \subset \mathcal{B}$ then $\mathcal{B} \in \nu$;

(N3) If $\mathcal{A} \cup \mathcal{B} \in \nu$ then $\mathcal{A} \in \nu$ or $\mathcal{B} \in \nu$;

(N4) If $\mathcal{A} \in \nu$ then $\emptyset \notin \mathcal{A}$.

1.2 DEFINITION. A grill on a set $X$ is any family $\mathcal{G}$ of subsets of $X$ satisfying

(G1) $A \in \mathcal{G}$ and $A \subseteq B \Rightarrow B \in \mathcal{G}$;

(G2) $A \cup G \in \mathcal{G} \Rightarrow A \in \mathcal{G}$ or $B \in \mathcal{G}$;

(G3) $\emptyset \notin \mathcal{G}$.

1.3 REMARK. A grill is simply any union of ultrafilters. For more background, see Thron [9].

1.4 DEFINITION. Let $\nu$ be a nearness on a set $X$.

a. A $\nu$-clan is a grill which is a member of $\nu$.

b. A $\nu$-bunch is a $\nu$-clan $\sigma$ satisfying

$$A \in \sigma \Rightarrow \mathcal{A} \in \sigma.$$  

c. A $\nu$-cluster is a maximal element of $\nu$.

1.5 DEFINITIONS–KINDS OF NEARNESSES. Let $\nu$ be a nearness on a set $X$ and let $\mathcal{T}$ be a topology on $X$.

a. $\nu$ is compatible with $\mathcal{T}$ iff $c_\nu$ is the closure operator determined by $\mathcal{T}$; i.e.

$$c_\nu(A) = A^-$$ for $A \subseteq X$.

b. $\nu$ is Lodato if $\mathcal{A} \in \nu$ whenever $c_\nu(\mathcal{A}) \in \nu$.

c. Let $\chi \subseteq \nu$. We say $\nu$ is $\chi$-generated iff every member of $\nu$ is contained in a member of $\chi$. In particular then $\nu$ is cluster-generated iff each member of $\nu$ is contained in a $\nu$-cluster.

1.6 PROPOSITION. Let $\nu$ be a Lodato nearness on a set $X$. For $\mathcal{A} \subseteq \mathcal{P}(X)$, define

$$\nu \mathcal{A} = \{ A \subseteq X : c_\nu(A) \in \mathcal{A} \}.$$  

Then

(1) if $\sigma$ is a $\nu$-clan then $\nu \sigma$ is a $\nu$-bunch containing $\sigma$;

(2) every maximal $\nu$-clan is a $\nu$-bunch.

PROOF. (1) Let $\sigma$ be a $\nu$-clan. Since $\nu$ is Lodato, we have that $c_\nu$ is a Kuratowski closure operator. Clearly then $\nu \sigma$ is a grill. Note $c_\nu \sigma \subseteq \sigma \subseteq c_\nu \sigma$. Since $\sigma \in \nu$, we have $c_\nu \sigma \in \nu$. But $\nu$ is Lodato so $\nu \sigma \in \nu$. Thus $\nu \sigma$ is a $\nu$-clan containing $\sigma$. Since $c_\nu$ is idempotent, $\nu \sigma$ is a bunch.

(2) Now let $\mu$ be a maximal $\nu$-clan. Then by (1), $\nu \mu$ is a $\nu$-bunch containing $\mu$. Since $\mu$ is maximal, $\nu \mu = \mu$, and so $\mu$ is a $\nu$-bunch. □
1.7 Proposition. For any nearness \( v \), a \( v \)-cluster is a maximal \( v \)-clan. If \( v \) is cluster-generated, then every maximal \( v \)-clan is a \( v \)-cluster.

Proof. Let \( v \) be a nearness. Then every \( v \)-cluster is a grill. This follows readily from the nearness axioms. Therefore each \( v \)-cluster is a maximal \( v \)-clan. (See Gagrat and Thron [4].)

Assume now that \( v \) is cluster-generated. Let \( \sigma \) be a maximal \( v \)-clan. Choose a \( v \)-cluster \( \mathcal{M} \) such that \( \sigma \subseteq \mathcal{M} \). Then by the above remark, \( \mathcal{M} \) is a \( v \)-clan, and so \( \mathcal{M} = \sigma \). □

The next theorem sets up a map \( \text{Tr} \) from \( T_1 \)-extensions of a space \( X \) to bunch-generated Lodato nearnesses compatible with \( X \). It is similar to a result of Bentley [2] which reaches the same conclusion under a slightly weaker hypothesis [2, Theorems 2.8 and 3.6]. The proof is given here for convenience.

1.8 Theorem. Let \( \kappa = (e, Y) \) be a \( T_1 \)-extension of a topological space \( X \). Let \( \text{Tr}(\kappa) = v_\kappa = \{ \mathcal{C} : e(\mathcal{C})^- \neq \emptyset \} \).

Then \( v_\kappa \) is a bunch-generated Lodato nearness compatible with the topology on \( X \).

Proof. Clearly (N1), (N2), and (N4) hold. Now suppose \( \mathcal{C} \cup \mathcal{B} \in v_\kappa \) with \( \mathcal{C} \notin v_\kappa \). Let \( \mathcal{C} = \mathcal{C} \cup \mathcal{B} \). We claim \( \bigcap e(\mathcal{C})^- \subseteq \bigcap e(\mathcal{B})^- \).

Let \( y \in \bigcap e(\mathcal{C})^- \). Since \( \mathcal{C} \notin v_\kappa \), choose \( A \in \mathcal{C} \) such that \( y \notin e(A)^- \). Now let \( B \in \mathcal{B} \). Then \( y \in e(A \cup B)^- = e(A)^- \cup e(B)^- \). Thus, \( y \in e(B)^- \), as desired.

To see that \( v_\kappa \) is compatible with the topology on \( X \), note that for \( A \subseteq X \) we have

\[
e((A)^-) = e(A)^- \cap e(X).
\]

Therefore \( x \in A^- \iff e(x) \in e(A)^- \Rightarrow \{e(x)\}^- \cap e(A)^- \neq \emptyset \iff \{x\} \subseteq A \in v_\kappa \). We obtain the missing implication from the fact that \( Y \) is \( T_1 \).

By virtue of the continuity of \( e \) we have \( e(A)^- = e(A^-)^- \), for \( A \subseteq X \). This is sufficient to establish that \( v_\kappa \) is Lodato.

Finally, we claim that \( v_\kappa \) is bunch-generated. For \( y \in Y \) let \( \tau(y) = \{ A \subseteq X : y \in e(A)^- \} \). Note if \( y \in \bigcap e(\mathcal{C})^- \), then \( \mathcal{C} \subseteq \tau(y) \). Thus every member of \( v_\kappa \) is a subset of some \( \tau(y) \). From definitions it is easy to see that \( \tau(y) \) is a \( v_\kappa \)-clan. Since \( e(A)^- = e(A^-)^- \) we have that \( \tau(y) \) is a \( v_\kappa \)-bunch. Hence \( v_\kappa \) is bunch-generated. □

For completeness we will show that equivalent extensions induce the same nearness, so that \( \text{Tr} \) is defined on equivalence classes of \( T_1 \)-extensions.

1.9 Lemma. If \( \kappa_1 \) and \( \kappa_2 \) are equivalent extensions of a space \( X \) then \( v_{\kappa_1} = v_{\kappa_2} \).
Proof. Say $\kappa_1 = (e_1, Y_1)$ and $f: Y_1 \to Y_2$ is an equivalence between $\kappa_1$ and $\kappa_2$. Let $g = f^{-1}$. Then for $A \subseteq X$ we have $e_2(A^{-}) = f(e_1(A^{-}))$ and similarly $e_1(A^{-}) = g(e_2(A^{-}))$. From this it is easy to see that $\nu_{\kappa_1} = \nu_{\kappa_2}$.

Next we will show that in the case of a principal $T_1$-extension the associated nearness is cluster-generated. In fact in this case the trace of each point of the extension is a cluster; and the set of all these traces generates the nearness.

To proceed, we will develop several characterizations of a principal extension.

1.10 Definitions and notation. Let $\kappa = (e, Y)$ be an extension of a space $X$.

(a) For $y \in Y$ the trace of $y$ in $X$ is $\tau(y) = \{ A \subseteq X : y \in e(A^{-}) \}$.

(b) For $G$ an open set in $Y$, $G^+ = \{ y \in Y : e^{-1}(G) \subseteq e^{-1}(\tau(y)) \}$.

1.11 Theorem. Let $\kappa = (e, Y)$ be an extension of a space $X$. The following conditions are equivalent:

(1) $\{ e(A)^- : A \subseteq X \}$ is a base for the closed sets of $Y$;

(2) $\{ G^+ : G \text{ open in } Y \}$ is a base for the open sets of $Y$.

Proof. The proof follows from the following facts.

(i) If $G$ is open in $Y$ then $G^+$ is open in $Y$.

(ii) If $G$ is open in $Y$ then $Y \setminus G^+ = ee^{-1}(X \setminus G)^-$.  

(iii) For $A \subseteq X$, and $G = Y \setminus e(A)^-$ we have $G = G^+$.  

1.12 Definition. An extension is principal, or strict, iff it satisfies one of the conditions of the preceding theorem.

Note that these are the same as the strict extensions of Banaschewski [1] and the principal extensions of Thron [8].

1.13 Lemma. Let $\kappa = (e, Y)$ be a principal extension of a space $X$.

(1) If $\tau(y) \subseteq \tau(z)$ then $\mathcal{R}_x \subseteq \mathcal{R}_y$ for all $y, z$ in $Y$;

(2) if $Y$ is $T_1$ then each $\tau(y)$ is a $\nu_x$-cluster.

Proof. (1) If $\tau(y) \subseteq \tau(z)$ then $\mathcal{R}_x \subseteq \mathcal{R}_y$.

Let $G$ be open in $Y$ with $z \in G$. Then $z \not\in Y \setminus G$. Since $\kappa$ is a principal extension, there is a set $A \subseteq X$ such that $Y \setminus G \subseteq e(A)^-$ and $z \not\in e(A)^-$. Thus $A \not\in \tau(z)$ and so $A \not\in \tau(y)$. From this we obtain $y \in Y \setminus e(A)^- \subseteq G$.

(2) If $Y$ is a $T_1$-space then each $\tau(y)$ is a $\nu_x$-cluster.

Note $\tau(y) \subseteq \nu_y$ since $y \in \cap e\tau(y)^-$. Now let $A \subseteq X$ and suppose $\mathcal{B} = \tau(y) \cup \{ A \}$ is in $\nu_x$. We wish to show $\mathcal{A} \in \tau(y)$. Let $z \in \cap e(\mathcal{B})^-$. Then $\tau(y) \subseteq \tau(z)$ and so by (1) we obtain $\mathcal{R}_x \subseteq \mathcal{R}_y$. Since $Y$ is $T_1$, $z = y$ and so $\tau(z) = \tau(y)$. Note $A \in \tau(z)$ since $\mathcal{B} \subseteq \tau(z)$. Thus $A \in \tau(y)$ as desired.

1.14 Theorem. If $\kappa = (e, Y)$ is a principal $T_1$-extension of a space $X$ then $\nu_x$ is a cluster-generated compatible Lodato nearness on $X$.  

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Proof. From Theorem 1.8 we know that $\nu_\xi$ is a compatible Lodato nearness on $X$. From the preceding lemma we have that each $\tau(y)$ is a $\nu_\xi$-cluster. Since each member of $\nu_\xi$ is contained in some $\tau(y)$ we conclude that $\nu_\xi$ is cluster-generated. □

B. Obtaining extensions from nearnesses. In what follows we will construct a map Ext from the Lodato nearnesses on a space $X$ to extensions of $X$. The main result states that if $\nu$ is a Lodato nearness compatible with a $T_1$-space $X$ then Ext($\nu$) is a principal $T_1$-extension of $X$.

1.15 Construction of Ext. Let $\nu$ be a Lodato nearness compatible with a $T_1$-space $X$. Let $Y_\nu$ be the set of all $\nu$-clusters. For $A \subset X$, let $A^\nu = \{ \sigma \in Y_\nu: A \in \sigma \}$. Then $\{ A^\nu: A \text{ is closed in } X \}$ is a base for the closed sets of a topology $T_\nu$ on $Y_\nu$. Let $e_\nu(x) = \sigma_x = \{ A \subset X: x \in A^- \}$. We define $\text{Ext}(\nu) = \kappa_\nu = (e_\nu, (Y_\nu, T_\nu))$.

In the next theorem we will show that $\kappa_\nu$ is a principal $T_1$-extension of $X$. This theorem is actually a consequence of work of Herrlich [5], but a proof is given here, for completeness.

1.16 Lemma. Let $\nu$ be a Lodato nearness compatible with a $T_1$-space $X$. Then for $A \subset X$ we have

$$A^\nu = (A^-)^\nu.$$ 

In particular then the closed sets of $\kappa_\nu$ are generated by $\{ A^\nu: A \subset X \}$.

Proof. Note that each $\nu$-cluster is a $\nu$-bunch, by Propositions 1.6 and 1.7. Thus $A^\nu = (A^-)^\nu$. Now let $\sigma \in A^\nu$. We wish to show $\sigma \in e_\nu(A)^\nu$. Let $B \subset X$ such that $e_\nu(A) \subset B^\nu$. Then $A \subset B^-$ and so $\sigma \in (B^-)^\nu = B^\nu$. This establishes $\sigma \in e_\nu(A)^\nu$.

Now suppose $\sigma \in e_\nu(A)^\nu$. Then $e_\nu(A) \subset A^\nu$. Since $A^\nu$ is closed, we have $\sigma \in A^\nu$. □

1.17 Theorem. Let $\nu$ be a Lodato nearness compatible with a $T_1$-space $X$. Then $\kappa_\nu$ is a principal $T_1$-extension of $X$.

Proof. (1) $e_\nu$ is an injection from $X$ to $Y_\nu$.

Clearly each $\sigma_x$ is a grill. Since $\nu$ is Lodato, $\sigma_x \in \nu$. Since $\nu$ is compatible with the topology on $X$, each $\sigma_x$ is a maximal element of $\nu$. Thus $e_\nu: X \to Y_\nu$.

Since $X$ is $T_1$, $e_\nu$ is 1-1.

(2) $e_\nu$ is a dense embedding of $X$ into $Y_\nu$.

Note that if $K$ is closed in $X$ then $e_\nu^{-1}(K^\nu) = K$. Hence $e_\nu$ is continuous.

Recall that for $A \subset X$, $A^\nu = e_\nu(A)^\nu$ (Lemma 1.16). Hence for $K$ closed we have

$$e_\nu(K) = e_\nu(K^-) \cap e_\nu(X).$$

This establishes that $e_\nu: X \to e_\nu(X)$ is a closed map.
Finally, $e_s(X)$ is dense in $Y$; for if $K$ is closed in $X$ and $e_s(X) \subseteq K$ then $X = K$.

(3) $Y_s$ is $T_1$. Let $\sigma \in Y_s$. We claim $\{\sigma\}^- = \{\sigma\}$. Let $\sigma_1 \in Y_s$ such that $\sigma_1 \neq \sigma$. Since $\sigma$ is a $\nu$-cluster we have $\sigma \not\subseteq \sigma_1$ and we choose $S \in \sigma \setminus \sigma_1$. Then $Y_s \setminus S$ is a neighborhood of $\sigma_1$ which misses $\{\sigma\}$.

(4) $K_s$ is a principal extension of $X$. This follows easily from the relation $A^* = e_s(A)^-$. □

C. Correspondence between nearnesses and extensions. In this section we will show that Ext and Tr are inverses if we restrict ourselves to cluster-generated nearnesses and principal extensions. Moreover, these maps yield a 1-1 correspondence between contigual nearnesses and $T_1$-compactifications. More precisely, Ext restricted to contigual nearnesses is a bijection to the set of principal $T_1$-compactifications.

1.18 Theorem. If $\nu$ is a cluster-generated Lodato nearness compatible with a $T_1$-space $X$, then $\nu^* = \nu$.

Proof. Let $\nu' = \nu^*$. In general we have $\nu' \subseteq \nu$. For let $\mathcal{B} \in \nu'$. Then we can choose $\sigma \in \bigcap e_s(\mathcal{B})^-$. From Lemma 1.16 we have $\mathcal{B} \subseteq \sigma$. Since $\sigma \in \nu$ we have $\mathcal{B} \in \nu$.

To get that $\nu \subseteq \nu'$ we will use the fact that $\nu$ is cluster-generated. Let $\mathcal{B} \in \nu$ and let $\sigma$ be a $\nu$-cluster containing $\mathcal{B}$. Then $\sigma \in \bigcap e_s(\mathcal{B})^-$ (Lemma 1.16). Hence $\mathcal{B} \in \nu'$. □

The next theorem is due to Herrlich [5], but a direct proof is given here for convenience.

1.19 Theorem. If $\kappa$ is a principal $T_1$-extension of a space $X$ then $\kappa$ is equivalent to $\kappa^*$.

Proof. Suppose $\kappa = (e, Y)$. Recall $\tau(y) = \{A \subseteq X: y \in e(A)^-\}$. We claim $\tau$ gives the desired equivalence.

Let $\kappa'$ be denoted by $\kappa' = (e', (Y', \tau'))$. We note first that $\kappa'$ is a principal $T_1$-extension of $X$ (Theorems 1.8 and 1.17).

(1) $\tau$ is a bijection from $Y$ to $Y'$.

Note that each $\tau(y)$ is a $\nu$-cluster, by Lemma 1.13. Thus $\tau: Y \to Y'$. Now let $\sigma$ be a $\nu$-cluster. Choose $y \in \bigcap e(\sigma)^-$. Clearly then $\sigma \subseteq \tau(y)$. Since $\sigma$ is a maximal element of $\nu$, and $\tau(y) \in \nu$, we have $\sigma = \tau(y)$. Thus $\tau$ is a surjection.

Now suppose $\tau(y) = \tau(z)$. From Lemma 1.13 it follows that $\mathcal{R}_y = \mathcal{R}_z$. Since $Y$ is $T_1$, we have $y = z$. Thus $\tau$ is an injection.

(2) $\tau$ is a homeomorphism.

For $A \subseteq X$ let $A^*$ denote $A^*$. Recall that $\{A^*: A \subseteq X\}$ is a base for the closed sets of $Y$; and $\{e(A)^-: A \subseteq X\}$ is a base for the closed sets of $Y$. 

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The continuity of $\tau$ follows from the relation

$$\tau^{-1}(A^*) = e(A)^{\sim}.$$ 

To see that $\tau$ is a closed map, note $\tau(e(A)^{\sim}) = A^*$.  

(3) $\tau e = e'$.  

We need to show that $\tau e(x) = \{ A: x \in A^\sim \}$. Note $A \in \tau e(x) \iff e(x) \in e(A)^{\sim} \iff x \in A^\sim$.  

1.20 Corollary. Let $X$ be a fixed $T_1$-space. The map $Ext$ is a bijection from the set of cluster-generated compatible Lodato nearnesses to the set of (equivalence classes of) principal $T_1$-extensions of $X$. The maps $Ext$ and $Tr$ are inverses on these two sets.  

Proof. Theorems 1.18 and 1.19.  

Next we will show that this correspondence extends to contigual nearnesses and $T_1$-compactifications. To proceed, we will define a contigual nearness and show its relation to other nearnesses.

1.21 Definition. A nearness $\nu$ is contigual iff for $\mathcal{G} \subset \mathcal{P}(X)$ we have $\mathcal{G} \in \nu$ whenever each finite subset of $\mathcal{G}$ is in $\nu$.  

1.22 Proposition. Let $\nu$ be a nearness on a set $X$. Let

$$av = \{ \mathcal{G} \subset \mathcal{P}(X): \text{every finite subset of} \ \mathcal{G} \ \text{is in} \ \nu \}.$$ 

Then $av$ is a nearness which determines the same closure operator as $\nu$. Moreover, $av$ is the smallest contigual nearness larger than $\nu$.  

Proof. (1) $av$ is a nearness which contains $\nu$. Clearly $\nu \subset av$ and (N1), (N2), and (N4) hold.  

Suppose $\mathcal{G} \cup \mathcal{F} \in av$ and $\mathcal{G} \in av$. Let $\mathcal{G}_0$ be a finite subset of $\mathcal{G}$ such that $\mathcal{G}_0 \not\in \nu$. We claim $\mathcal{F} \in av$. Let $\mathcal{T}$ be any finite subset of $\mathcal{F}$. Then $\mathcal{G}_0 \cup \mathcal{T}$ is a finite subset of $\mathcal{G} \cup \mathcal{F}$ and hence $\mathcal{G}_0 \cup \mathcal{T} \in \nu$. Since $\mathcal{G}_0 \not\in \nu$, we have $\mathcal{T} \in \nu$. Therefore $\mathcal{F} \in av$.  

(2) $\nu$ and $av$ determine the same closure operator.  

Note for $x \in X$ and $A \subset X$ we have

$$\{ \{ x \}, A \} \in \nu \iff \{ \{ x \}, A \} \in av.$$ 

(3) $av$ is contigual, since every finite family $\mathcal{F}$ which is in $av$ is also in $\nu$.  

(4) If $\nu_1$ is a contigual nearness containing $\nu$ then $\nu_1$ contains $av$.  

If $\mathcal{G} \in av$ then every finite subset of $\mathcal{G}$ is in $\nu$ and hence in $\nu_1$; consequently $\mathcal{G} \in \nu_1$.  

1.23 Lemma. Every contigual nearness is cluster-generated.  

Proof. If $\nu$ is contigual then each nonempty chain in $\nu$ has its union in $\nu$. Using Zorn’s lemma then it is easy to prove that $\nu$ is cluster-generated.
The next two results are due to Herrlich [5].

1.24 Theorem. If ν is a contiguous Lodato nearness on a $T_1$-space $X$ then $κ_ν$ is compact.

Proof. Let $C$ be any family of closed subsets of $X$. Suppose $C = \{C^* : C \in C\}$ has the f.i.p. To show $κ_ν$ is compact it is sufficient to show $C$ has nonempty intersection.

We claim that every finite subset of $C$ is in $ν$. For if $C \subseteq C_1 \cap \cdots \cap C_n$ then $(C_1, \ldots, C_n) \in ν$. Since $ν$ is contiguous, $C \in ν$. But $ν$ is cluster-generated (Lemma 1.23) and so we can choose a $ν$-cluster $μ$ such that $C \subseteq μ$. Clearly then $μ \in \cap C^*$. □

1.25 Theorem. If $ν = (e, Y)$ is a $T_1$-compactification of a space $X$ then $ν_κ$ is contiguous.

Proof. Let $E \in av_ν$. Then $e(E)^- \in ν$. Since $Y$ is compact, $e(E)^-$ has nonempty intersection. By construction, $C \in ν_κ$. □

1.26 Corollary. Let $X$ be a fixed $T_1$-space. The map $Ext$ is a bijection from the set of contiguous compatible Lodato nearnesses to the set of (equivalence classes of) principal $T_1$-compactifications of $X$. The maps $Ext$ and $Tr$ are inverse maps on these two sets.

Proof. From Theorems 1.17 and 1.24 it follows that $Ext$ maps contiguous Lodato nearnesses compatible with $X$ to principal $T_1$-compactifications of $X$. Similarly by Theorems 1.14 and 1.25 we have that $Tr$ maps principal $T_1$-compactifications to contiguous Lodato nearnesses on $X$. But $Ext$ and $Tr$ are inverses on these two sets, by Theorems 1.18 and 1.19 and Lemma 1.23. □

2. Proximities and extensions. Each proximity $π$ carries with it a whole band of nearnesses. However there is always a largest one, $ν_G(π)$, which is always contiguous and is Lodato if $π$ is Lodato. If $π$ is an $EF$-proximity then $Ext(ν_G(π))$ is the $T_2$-compactification corresponding to $π$. Moreover if $π$ is Lodato then $Ext(ν_G(π))$ is the compactification obtained by Gagrat and Naimpally [3]. These compactifications are characterized by the property that the dual of each clan converges. Using results of the previous section then we obtain a 1-1 correspondence between the Lodato proximities on a $T_1$-space and its clan-complete principal $T_1$-compactifications.

2.1 Definition. A proximity on a set $X$ is a family $π$ of subsets of $P(X)$ such that

(P0) If $C \in π$ then $C$ has at most two members;
(P1) $A \cap B \neq \emptyset$ $⇒$ $\{\{A, B\} \in π\}$;
(P2) For fixed $A \subseteq X$ the set $π(A) = \{B : \{A, B\} \in π\}$ is a grill.
A proximity \( \pi \) is Lodato iff for \( C \subseteq \mathcal{P}(X) \) we have \( C \in \pi \) whenever \( c_* C \subseteq \pi \).

2.2 Remark. Lodato proximities defined as above correspond to the usual definition as follows:

\( A \delta B \text{ iff } \{A, B\} \in \pi \).

For a proof, see Thron [9].

2.3 Theorem. Let \( \nu \) be a nearness on a set \( X \) and let \( \pi_\nu \) be the set of all members of \( \nu \) which have at most two members. Then \( \pi_\nu \) is a proximity on \( X \) with the same closure operator as \( \nu \). If \( \nu \) is Lodato then \( \pi_\nu \) is Lodato.

Proof. See Gagrat and Thron [4]. □

2.4 Definition. A proximity \( \pi \) on a set \( X \) is compatible with a nearness \( \nu \) iff \( \pi_\nu = \pi \). It is compatible with a topology \( \mathcal{F} \) on \( X \) iff \( c_* \) is the closure operator determined by \( \mathcal{F} \).

2.5. Proposition. Let \( \nu \) be a cluster-generated Lodato nearness on a \( T_1 \)-space \( X \). Then

\[ \{A, B\} \in \pi_\nu \text{ iff } e_\nu(A)^- \cap e_\nu(B)^- \neq \emptyset. \]

Proof. Let \( \{A, B\} \in \pi_\nu \). Then \( \{A, B\} \in \nu \) and there is a \( \nu \)-cluster \( \sigma \) such that \( A, B \in \sigma \). By Lemma 1.16, \( \sigma \in e_\nu(A)^- \cap e_\nu(B)^- \).

Conversely, if \( \sigma \in e_\nu(A)^- \cap e_\nu(B)^- \) then by Lemma 1.16 we have \( A, B \in \sigma \). Since \( \sigma \in \nu \) we have \( \{A, B\} \in \pi_\nu \). □

2.6 Definition. Let \( \pi \) be a proximity on a set \( X \).

(a) A \( \pi \)-clan is a grill \( \sigma \) with the property that if \( A, B \in \sigma \) then \( \{A, B\} \in \pi_\nu \).

(b) A \( \pi \)-bunch is a \( \pi \)-clan \( \sigma \) such that \( c_* A \in \sigma \Rightarrow A \in \sigma \).

2.7 Theorem. Let \( \nu \) be a proximity on a set \( X \). Let

\[ \nu_\pi(\pi) = \{C \subseteq \mathcal{P}(X): C \text{ is contained in a } \pi \text{-clan}\}. \]

Then \( \nu_\pi(\pi) \) is the largest nearness compatible with \( \pi \). Moreover, \( \nu_\pi(\pi) \) is contigual. Finally, \( \nu_\pi(\pi) \) is Lodato if \( \pi \) is Lodato.

Proof. Let \( \nu_\pi(\pi) \) be denoted by \( \nu_\pi \).

(1) \( \nu_\pi \) is a nearness.

Let \( C \subseteq \mathcal{P}(X) \) and suppose \( \cap C \neq \emptyset \).

Let \( x \in \cap C \). Let \( \hat{x} = \{S \subseteq X: x \in S\} \). Then \( \hat{x} \) is a \( \pi \)-clan containing \( C \).

Suppose \( C \subseteq \nu_\pi \) and \( C < \beta \). Let \( \sigma \) be a \( \pi \)-clan such that \( C \subseteq \sigma \). Then \( \beta \subseteq \sigma \) and so \( \beta \subseteq \nu_\pi \).

If \( \beta \cup \beta \subseteq \sigma \), where \( \sigma \) is a \( \pi \)-clan, and if \( \beta \not\subseteq \sigma \) then \( \sigma \subseteq \beta \). Hence \( N3 \) holds. It is clear that \( N4 \) holds.

(2) \( \nu_\pi \) is compatible with \( \pi \).

Clearly if \( \{A, B\} \in \nu_\pi \) then \( \{A, B\} \in \pi_\nu \). The converse was actually proved
by Thron [10]. However, the proof is short and will be given here.

Let \( \{A, B\} \in \pi \). Then \( B \in \pi(A) \), which is a grill. Let \( \mathcal{U} \) be an ultrafilter such that \( B \in \mathcal{U} \subset \pi(A) \). Now \( \pi(\mathcal{U}) \) is also a grill.

\[
\{D \subset X: \{D, U\} \in \pi \text{ for } U \in \mathcal{U}\}
\]

Since \( A \in \pi(\mathcal{U}) \) we can choose an ultrafilter \( \mathcal{V} \) such that \( A \in \mathcal{V} \subset \pi(\mathcal{U}) \). Then \( \mathcal{U} \cup \mathcal{V} \) is a \( \pi \)-clan containing \( \{A, B\} \).

(3) \( \nu_G \) is the largest nearness compatible with \( \pi \).

Let \( \nu \) be any nearness compatible with \( \pi \). It is easy to see that \( \nu \) is also compatible with \( \pi \). (See Proposition 1.22 for a definition of \( \nu \).) Now, \( \nu \) is contigual, and hence cluster-generated. (See Lemma 1.23.) But every grill in \( \nu \) is a \( \pi \)-clan, and so we have \( \nu \subset \nu \subset \nu_G \).

(4) \( \nu_G \) is contigual.

Note \( \nu_G \) is compatible with \( \pi \). Thus by (3) we have \( \nu_G \subset \nu_G \).

(5) If \( \pi \) is Lodato then \( \nu_G \) is Lodato.

Let \( A^- = c_\pi(A) = c_\rho(A) \). Suppose \( \mathcal{G} \subset \mathcal{P}(X) \) and \( \mathcal{G}^- \subset \sigma \) where \( \sigma \) is a \( \pi \)-clan. We wish to show \( \mathcal{G} \) is contained in some \( \pi \)-clan. Let

\[
\mathcal{G} = \{A \subset X: A^- \in \sigma\}.
\]

Then \( \mathcal{G} \) is a grill containing \( \mathcal{G} \). Since \( \pi \) is Lodato, \( \mathcal{G} \) is a \( \pi \)-clan. \( \square \)

2.8 REMARK AND DEFINITION. If \( \pi \) is a Lodato proximity compatible with a \( T_1 \)-space \( X \) then \( \text{Ext}(\nu_G(\pi)) \) is a \( T_1 \)-compactification of \( X \) (Corollary 1.26).

The next theorem verifies that this is the construction of Gagrard and Naimpally [3, Theorem 3.13]. For this reason we define a \( \text{GN-compactification} \) to be any compactification which is equivalent to some \( \text{Ext}(\nu_G(\pi)) \), where \( \pi \) is a compatible Lodato proximity. We will denote \( \text{Ext}(\nu_G(\pi)) \) by \( \kappa_G(\pi) \), or \( \kappa_G \).

The following theorem establishes that \( \kappa_G \) is indeed Gagrard-Naimpally construction.

2.9 THEOREM. Let \( \pi \) be a Lodato proximity compatible with a \( T_1 \)-space \( X \).

(1) The set of \( \nu_G(\pi) \)-clusters is the set of maximal \( \pi \)-bunches.

(2) The set of maximal \( \pi \)-bunches is the set of maximal \( \pi \)-clans.

(3) For \( \omega \in Y_{\nu_G} \) we have \( \omega^- = \cap \{\sigma: \cap \omega \subset \sigma\} \).

PROOF. (1) The set of \( \nu_G \)-clusters is the set of maximal \( \pi \)-clans.

Since \( \nu_G \) is cluster-generated, the \( \nu_G \)-clusters are the maximal \( \nu_G \)-clans (Proposition 1.7). But the \( \nu_G \)-clans are just the \( \pi \)-clans. Thus \( \nu_G \)-clusters are maximal \( \pi \)-clans, conversely.

(2) Every maximal \( \pi \)-clan is a maximal \( \pi \)-bunch. This follows from the fact \( \pi \) is Lodato. The proof is nearly identical with that of Proposition 1.6.

(3) Every maximal \( \pi \)-bunch is a maximal \( \pi \)-clan. Let \( \sigma \) be a maximal \( \pi \)-bunch. By a Zorn’s lemma argument, \( \sigma \) is contained in a maximal \( \pi \)-clan \( \mu \). By (2), \( \mu \) is a \( \pi \)-clan, and hence \( \mu = \sigma \). Thus \( \sigma \) is a maximal \( \pi \)-clan.
(4) For $\omega \subset Y_{n_0}$ we have $\omega^{-} = \{\sigma: \cap \omega \subset \sigma\}$. Let $\nu_\sigma$ be denoted by $\nu$. By Lemma 1.16, the closed sets of $Y_{n_0}$ are generated by $\{A^\sigma: A \subset X\}$. Let $\sigma \in \omega^{-}$ and $A \in \cap \omega$. Then $\omega \subset A^\sigma$ and hence $\omega^{-} \subset A^\sigma$. Thus $A \in \sigma$, and $\cap \omega \subset \sigma$.

Conversely, suppose $\cap \omega \subset \sigma$. To show $\sigma \in \omega^{-}$ it is sufficient to show $\sigma \in A^\sigma$ whenever $\omega \subset A^\sigma$. If $\omega \subset A^\sigma$ then $A \in \cap \omega \subset \sigma$. Thus $\sigma \in A^\sigma$ as desired. □

2.10 Definition. Let $\pi$ be a proximity on a set $X$ and let $\mathcal{C} \subset \mathcal{P}(X)$. Let $A, B \subset X$.

(a) $A \prec_{\pi} B$ iff $\{A, X \setminus B\} \not\in \pi$;
(b) $r_{\pi} \mathcal{C} = \{S: \exists A \in \mathcal{C} \text{ such that } A \prec_{\pi} S\}$;
(c) $c_{\mathcal{C}} = \{S: X \setminus S \in \mathcal{C}\}$.

We call $c_{\mathcal{C}}$ the dual of $\mathcal{C}$.

2.11 Remark. If $\sigma$ is a grill then $c_{\sigma}$ is a filter and if $\mathcal{F}$ is a filter, $c_{\mathcal{F}}$ is a grill. Moreover, for any $\mathcal{C} \subset \mathcal{P}(X)$, $c_{\mathcal{C}} = \mathcal{C}$. Finally, if $\mathcal{C} \subset \mathcal{B}$ then $c_{\mathcal{B}} \subset c_{\mathcal{C}}$. For proofs see Thron [9].

In general if $\mathcal{C}$ is closed under supersets then $r_{\sigma} \mathcal{C} \subset \mathcal{C}$. If $\mathcal{F}$ is a filter then $r_{\sigma} \mathcal{F}$ is also a filter. However if $\sigma$ is a grill then $r_{\sigma} \sigma$ need not be a filter or a grill.

2.12 Lemma. Let $\pi$ be a Lodato nearness compatible with a $T_{1}$-space $X$, and let $\sigma$ be a $\pi$-cluster. Then

$$c_{\sigma} = e_{\pi}^{-1}(\mathcal{C}_{\sigma}).$$

Proof. Let $A \in c_{\sigma}$. Then $Y \setminus (X \setminus A)^\pi$ is an open neighborhood $G_{\sigma}$ of $\sigma$. But $e_{\pi}^{-1}(G_{\sigma}) \subset A$.

Conversely, suppose $A \supset e_{\pi}^{-1}(U_{\sigma})$ for some open neighborhood $U_{\sigma}$ of $\sigma$. Since $\sigma \not\in Y \setminus U_{\sigma}$ there is a closed set $K$ in $X$ such that $\sigma \not\in K^\pi$ and $Y \setminus U_{\sigma} \subset K^\pi$. Then $A \supset X \setminus K \in c_{\sigma}$. □

2.13 Lemma. Let $\pi$ be a Lodato proximity on a set $X$, and let $\sigma$ be a grill on $X$. Then $\sigma$ is a maximal $\pi$-clan iff $c_{\sigma}$ is the filter generated by $r_{\sigma} \sigma$.

Proof. (1) Assume $\sigma$ is a maximal $\pi$-clan. Note first that since $\sigma$ is a $\pi$-clan we have $r_{\sigma} \sigma \subset c_{\sigma}$. Since $c_{\sigma}$ is a filter, this says the filter generated by $r_{\sigma} \sigma$ is contained in $c_{\sigma}$. Now suppose $A$ is not in the filter generated by $r_{\sigma} \sigma$. We wish to show $A \not\in c_{\sigma}$. Note that $r_{\sigma} \sigma \cup (X \setminus A)$ has the f.i.p. Let $\mathcal{U}$ be an ultrafilter containing $r_{\sigma} \sigma \cup (X \setminus A)$. We claim that $\sigma \cup \mathcal{U}$ is a $\pi$-clan.

Note $\sigma \cup \mathcal{U}$ is a grill, since it is a union of ultrafilters. Clearly $\sigma$ and $\mathcal{U}$ are separately $\pi$-compatible. Let $S \in \sigma$ and $U \in \mathcal{U}$ and suppose $(S, U) \not\in \pi$. Then $S \prec_{\pi} X \setminus U$. Hence $X \setminus U \in r_{\sigma} \sigma \subset \mathcal{U}$, which violates $U \in \mathcal{U}$. Therefore $(S, U) \in \pi$ as desired.
Since \( \sigma \) is a maximal \( \pi \)-clan, we have \( \mathcal{U} \subset \sigma \). Hence \( X \setminus A \in \sigma \), and \( A \not\in \sigma \).

(2) Conversely, assume \( \sigma \) is generated by \( r_\sigma \). Since \( r_\sigma \subset \sigma \) we have that \( \sigma \) is a \( \pi \)-clan. To show that \( \sigma \) is a maximal \( \pi \)-clan it is sufficient to show that for every ultrafilter \( \mathcal{U} \), if \( \sigma \cup \mathcal{U} \) is a \( \pi \)-clan then \( \mathcal{U} \subset \sigma \).

Assume \( \sigma \cup \mathcal{U} \) is a \( \pi \)-clan. Suppose \( A \notin \sigma \). Then \( X \setminus A \subset \sigma \), which is generated by \( r_\sigma \). Let \( S_1, \ldots, S_n \in \sigma \) such that \( S_i \subset T_i \) and \( \bigcap_i T_i \subset X \setminus A \). Since \( \sigma \cup \mathcal{U} \) is a \( \pi \)-clan, \( X \setminus T_i \in \mathcal{U} \) for all \( i \). Hence each \( T_i \) is in \( \mathcal{U} \), and \( X \setminus A \in \mathcal{U} \). Thus \( A \notin \sigma \Rightarrow A \notin \mathcal{U} \), and \( \mathcal{U} \subset \sigma \). 

2.14 THEOREM. Let \( \pi \) be a Lodato proximity compatible with a \( T_1 \)-space \( X \). Then \( \pi \) is an EF-proximity iff \( \kappa_\pi(\pi) \) is Hausdorff.

PROOF. \((\Rightarrow)\) Assume \( \pi \) is an EF-proximity.

(1) The dual of a maximal \( \pi \)-clan is a maximal round filter.

Let \( \sigma \) be a maximal \( \pi \)-clan. Then by Lemma 2.13, \( \sigma \) is generated by \( r_\sigma \). Let \( A \in \sigma \). We wish to find \( B \in \sigma \) such that \( B <_\pi A \). Choose \( S_1, \ldots, S_n \in \sigma \) such that \( S_i \subset T_i \) and \( \bigcap_i T_i \subset A \). Since \( \pi \) is an EF-proximity, we can choose \( U_i \) such that \( S_i <_\pi U_i <_\pi T_i \). Then \( U = \bigcap_i U_i \in \sigma \) and \( U <_\pi A \).

To see that \( \sigma \) is maximal round, suppose \( A <_\pi B \) with \( B \notin \sigma \). We need \( X \setminus A \in \sigma \). Now \( X \setminus B \in \sigma \) and \( A <_\pi B \Rightarrow A \notin \sigma \Rightarrow X \setminus A \in \sigma \).

(2) \( \kappa_\pi(\pi) \) is Hausdorff.

Let \( \sigma_1, \sigma_2 \) be \( \nu_\pi \)-clusters with \( \sigma_1 \neq \sigma_2 \). They are maximal \( \pi \)-clans, by Theorem 2.9. So each \( \sigma_i \) is a maximal round filter. Now \( \sigma_1 \neq \sigma_2 \). Let \( A \in \sigma_1 \setminus \sigma_2 \). Choose \( B \in \sigma_1 \) such that \( B <_\pi A \). Then \( X \setminus B \in \sigma_2 \). Let \( e \) denote \( e_\sigma \). Then \( \sigma_1 = e^{-1}(\mathcal{R}_\sigma) \), by Lemma 2.12. Choose \( G_i \), open sets in \( Y_\sigma \), such that \( \sigma_i \subset G_i \) and \( e^{-1}(G_i) \subset B \) and \( e^{-1}(G_2) \subset X \setminus B \). Then \( G_1 \cap G_2 = \emptyset \). For if \( \sigma \subset G_1 \cap G_2 \) then both \( B \) and \( X \setminus B \) are in \( e^{-1}(\mathcal{R}_\sigma) \).

\((\Leftarrow)\) Now assume \( \kappa_\pi(\pi) \) is Hausdorff. To show \( \pi \) is an EF-proximity it is sufficient to show that \( <_\pi \) is dense.

Suppose \( A <_\pi B \). Then by Proposition 2.5, we have \( e_\pi(A)^- \cap e_\pi(X \setminus B)^- = \emptyset \). Since every compact \( T_2 \)-space is normal, we can choose disjoint open sets \( U \) and \( V \) such that \( e_\pi(A)^- \subset U \) and \( e_\pi(X \setminus B)^- \subset V \). Again from Proposition 2.5 we have \( A <_\pi e_\pi^{-1}(U) \) and \( X \setminus B <_\pi e_\pi^{-1}(V) \). Thus \( A <_\pi e_\pi^{-1}(U) <_\pi B \).

Next we wish to obtain a characterization of the GN-compactifications. These turn out to be the principal \( T_1 \)-extensions for which the dual of every \( \pi^* \)-clan converges. Here \( \pi^* \) is the largest LO-proximity compatible with the topology, i.e., \( \{ A, B \} \in \pi^* \) iff \( A^- \cap B^- \neq \emptyset \).

2.15 DEFINITION. A clan on a \( T_1 \)-space \( X \) is a grill \( \sigma \) with the property that for \( A, B \in \sigma \) we have \( A^- \cap B^- \neq \emptyset \). A bunch is a clan \( \mu \) such that \( A \in \mu \) whenever \( A^- \in \mu \).
2.16 Remark. Let $X$ be a $T_1$-space and let $\pi^*$ be the Lodato proximity defined by

$$\{A, B\} \in \pi^* \iff A^- \cap B^- \neq \emptyset.$$ 

Then $\pi^*$ is the largest Lodato proximity compatible with the topology on $X$. A clan on $X$ is the same as a $\pi^*$-clan.

2.17 Lemma. Let $\kappa = (e, Y)$ be a $T_1$-extension of a space $X$. Let $\pi$ be the proximity on $X$ induced by $\nu_\kappa$. Then $\nu_\kappa = \nu_G(\pi)$ iff for every clan $\sigma$ on $Y$, if $eX \in \sigma$ then $\sigma$ converges.

Proof. ($\Rightarrow$) Assume that if $\sigma$ is a clan and $eX \in \sigma$ then $\sigma$ converges. Note $\nu_\kappa \subseteq \nu_G(\pi)$ since $\nu_G(\pi)$ is the largest nearness compatible with $\pi$ (Theorem 2.7).

Now suppose $\sigma$ is a $\pi$-clan. We wish to show $\sigma \in \nu_\kappa$. Note first that $c\sigma = e\sigma$. Thus $c\sigma$ is a filter, and $e\sigma$ is an Grill. Since $\sigma$ is a $\pi$-clan, clearly $e\sigma$ is a clan. Moreover, $eX \in c\sigma = e\sigma$. Thus by the assumption, $c\sigma$ converges for some $y \in Y$. We claim that $\sigma \subseteq \tau(y) \in \nu_\kappa$.

Let $S \in \sigma$. Then $e(S) \in \sigma$ and so $Y \setminus e(S)^- \not\in c\sigma$. Since $c\sigma$ converges, this says $y \in e(S)^-$. i.e. $S \subseteq \tau(y)$

($\Leftarrow$) Now suppose $\nu_\kappa = \nu_G(\pi)$. Let $\sigma$ be a clan on $Y$ such that $eX \in \sigma$. We need to show $\sigma$ converges. We will show that $e^{-1}\sigma$ is a $\pi$-clan. Since $eX \in \sigma$ we have that $S \in \sigma \Rightarrow S \cap eX \in \sigma$. Thus $A \in e^{-1}\sigma \iff e(A) \in \sigma$ and $A \in e^{-1}\sigma$ iff $e(A) \in \sigma$. From this it follows that $ce^{-1}\sigma = e^{-1}c\sigma$. Now $eX \in \sigma$ $\Rightarrow e^{-1}\sigma$ is a proper filter. Thus $e^{-1}\sigma$ is a nonempty Grill.

If $A, B \in e^{-1}\sigma$ then $e(A)$ and $e(B)$ are in $\sigma$; but $\sigma$ is a clan, so $e(A)^- \cap e(B)^- \neq \emptyset$. Thus $\{A, B\} \in \nu_\kappa$, and $e^{-1}\sigma$ is a $\pi$-clan. Now $\nu_G(\pi) = \nu_\kappa = e^{-1}\sigma \in \nu_\kappa$. Let $y \in \cap e(e^{-1}\sigma)^-$. Since $eX \in \sigma$, $e(e^{-1}\sigma) = \sigma$. Thus $y \in \cap \sigma^-$. It is easy to check that $\sigma \rightarrow y$.

2.18 Lemma. Let $\kappa = (e, Y)$ be a principal $T_1$-extension of a space $X$. The following conditions are equivalent:

(1) the dual of every clan converges;

(2) if $\sigma$ is a clan nad $eX \in \sigma$ then $\sigma$ converges.

Proof. Clearly (1) $\Rightarrow$ (2). Suppose (2) holds. Let $\sigma$ be a clan. Then $b\sigma = \{A: A^- \in \sigma\}$ is a bunch, and $cb\sigma$ is an open filter. Let $\mathcal{F}$ be the filter generated by $cb\sigma \cup \{eX\}$.

(1) $\mathcal{F}$ converges.

Note that $c\mathcal{F}$ is a grill. Moreover, since $cb\sigma \subseteq \mathcal{F}$ we have $c\mathcal{F} \subseteq b\sigma$ and so $c\mathcal{F}$ is a clan. Since $eX \in cc\mathcal{F}$ we have from (2) that $\mathcal{F}$ converges.

(2) If $\mathcal{F} \rightarrow y$ then $\sigma \rightarrow y$.

Suppose $\mathcal{F} \rightarrow y$. We claim that since $\kappa$ is principal, we have that $cb\sigma \rightarrow y$. For let $G$ be open such that $y \in G$. Let $U$ be open such that $y \in U^+ \subseteq G$.
(See Definition 1.12.) Then $U^+ \subseteq \mathcal{G}$ and so we can choose $T \in c\sigma$ such that $T \cap eX \subseteq U^+$. Since $c\sigma$ is open, $T^i \subseteq c\sigma$. We claim $T^i \subseteq U^+ \subseteq G$.

Let $z \in T^i$. Then $T \in \mathcal{N}_z$ and $e^{-1}(T) \in e^{-1}(\mathcal{N}_z)$. But $e^{-1}(T) \subseteq e^{-1}(U)$ and so $z \in U^+$.

Now since $\sigma \subseteq b\sigma$ we have $c\sigma \subseteq c\sigma$. Since $c\sigma \to y$, clearly $c\sigma \to y$. □

**2.19 Definition.** A topological space is **clan-complete** iff the dual of every clan converges.

**2.20 Theorem.** Let $\kappa = (e, Y)$ be a principal $T_1$-extension of $X$. Then $\kappa$ is a **GN-compactification** iff $Y$ is clan-complete.

**Proof.** Let $\kappa$ be a principal $T_1$-extension of $X$. Suppose $\kappa$ is equivalent to $\text{Ext}(\nu_G(\pi)) = \kappa_G$, where $\pi$ is a Lodato proximity compatible with $X$. Note that $\kappa$ and $\kappa_G$ induce the same nearness (Lemma 1.9). Since $\nu_G$ is cluster-generated, we have that $\nu_G$ is the nearness induced by $\kappa_G$ (Theorem 1.18). Thus $\nu_G = \nu_\kappa$, and also $\nu_\kappa$ is compatible with $\pi$. Thus Lemma 2.17 applies, and every clan $\sigma$ with $eX \in c\sigma$ has a convergent dual. Since $\kappa$ is principal, this is equivalent to $Y$ being clan-complete (Lemma 2.18).

Conversely, suppose $Y$ is clan-complete. Let $\pi$ be the proximity induced by $\nu_\kappa$. Then by Lemma 2.17, $\nu_\kappa = \nu_G(\pi)$. Since $\kappa$ is principal, $\kappa$ is equivalent to $\text{Ext}(\nu_G(\pi))$ (Theorem 1.19). Thus $\kappa$ is a **GN-compactification** of $X$. □

**2.21 Corollary.** Let $X$ be a fixed $T_1$-space. The map $\pi \to \kappa_G(\pi)$ is a bijection from the Lodato proximities compatible with $X$ to the clan-complete principal $T_1$-extensions of $X$.

**Proof.** If $\pi$ is a Lodato proximity compatible with $X$ then $\kappa_G(\pi)$ is clan-complete, by the preceding theorem. Suppose now that $\kappa_G(\pi_1)$ and $\kappa_G(\pi_2)$ are equivalent. Then they induce the same nearness under $\text{Tr}$ (Lemma 1.9). Since $\nu_G(\pi_1)$ is cluster-generated, it must be the nearness induced by $\kappa_G(\pi_1)$ (Theorem 1.18). Thus $\nu_G(\pi_1) = \nu_\kappa(\pi_2)$, and hence $\pi_1 = \pi_2$.

Finally, if $\kappa$ is a clan-complete principal $T_1$-extension of $X$, then by the preceding theorem, $\kappa$ is equivalent to $\kappa_G(\pi)$ for some Lodato proximity $\pi$ compatible with $X$. □

The final result is a form of the correspondence obtained by Smirnov [7] between $EF$-proximities and $T_2$-compactifications.

**2.22 Corollary.** Let $X$ be a fixed completely regular $T_2$-space. Then the map $\pi \to \kappa_G(\pi)$ is a bijection from the compatible $EF$-proximities to the $T_2$-compactifications of $X$.

**Proof.** Theorem 2.14 and Corollary 2.21. □
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