TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 236, February 1978

A GLOBAL THEOREM FOR SINGULARITIES OF MAPS
BETWEEN ORIENTED 2-MANIFOLDS

BY

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ABSTRACT. Let M and N be smooth compact oriented connected 2-manifolds. Suppose f: M → N is smooth and every point p ∈ M is either a fold point, cusp point, or regular point of f i.e., f is excellent in the sense of Whitney. Let M+ be the closure of the set of regular points at which f preserves orientation and M the closure of the set of regular points at which f reverses orientation. Let p1, ..., pn be the cusp points and μ(pk) the local degree at the cusp point pk. We prove the following:

\[ \chi(M) - 2\chi(M^-) + \sum \mu(pk) = (\deg f)\chi(N) \]

where \( \chi \) is the Euler characteristic and \( \deg \) is the topological degree. We show that it is a generalization of the Riemann-Hurwitz formula of complex analysis and give some examples.

Proof of the theorem. Let \( C^\infty(M, N) \) be the set of smooth maps from M to N topologized in the usual fashion, where two functions are close if all of their derivatives are close. Let \( f \in C^\infty(M, N) \). We say p is a regular point of f if we can find local coordinates \((x, y)\) near p and \((u, v)\) near \( f(p) \) with p and \( f(p) \) having coordinates \((0, 0)\) such that \( f \) is given by \( u = x, v = y \). Say p is a fold point if the equations can be put in the form \( u = x, v = y^2 \). Say p is a cusp point if the equations can be put in the form \( u = x, v = y^3 - xy \). It is well known for these dimensions that these three types of local maps are the only stable types and that the stable types are dense. (See Whitney [17] or Callahan [2].) Say \( f \) is excellent if every point \( p \in M \) is one of the above types for \( f \). In the following discussion, we will assume that \( f \) is excellent and that the images of the cusp points are distinct.

We will find it convenient to use a different form for the equation at a cusp point. Let \( x = 3t^2 + s, y = t \). This clearly defines a \( C^\infty \) homeomorphism and the cusp equation becomes \( u = 3t^2 + s, v = -2t^3 - st \). The general fold is now along the t axis. If we want the ordered pairs of vectors \((\partial/\partial s, \partial/\partial t)\) and \((\partial/\partial u, \partial/\partial v)\) to be positively oriented in the tangent spaces \( TM \) and \( TN \)

Presented to the Society, November, 6, 1976; received by the editors May 18, 1976 and, in revised form, August 22, 1976.


Key words and phrases. Singularities, cusp points, folds, manifold, Euler characteristic, excellent map, Riemann-Hurwitz formula.

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307
respectively, we must distinguish between two equations for a cusp:

(1) \[ u = 3t^2 + s, \quad v = -2t^3 - st \]

and

(2) \[ u = 3t^2 - s, \quad v = -2t^3 + st. \]

In case (1), the local degree of \( f \) at \( p \) is +1, i.e., for a sufficiently small neighborhood \( U \) of \( f(p) \) the map \( f: V \to U \) has degree +1 where \( V \) is the component of \( f^{-1}(U) \) containing \( p \). In case (2), the local degree is −1. If \( p \) is a cusp point, we will define \( \mu(p) \) to be the local degree of \( f \) at \( p \).

Let \( p_1, \ldots, p_n \) be the cusp points of \( f \). Let \((x_k, y_k)\) be local coordinate systems defined in disjoint neighborhoods \( U_k \) of \( p_k, k = 1, \ldots, n, \) and let \((u_k, v_k)\) be local coordinate systems in disjoint neighborhoods \( V_k \) of \( f(p_k), k = 1, \ldots, n. \) Let these coordinate systems be chosen so that \( f \) takes the form (1) or (2) depending on the local degree, and such that \((\partial / \partial x_k, \partial / \partial y_k)\) and \((\partial / \partial u_k, \partial / \partial v_k)\) are positively oriented. We define a geometry on \( U_k \) by

\[
\left\{ a \frac{\partial}{\partial x_k} + b \frac{\partial}{\partial y_k}, \ c \frac{\partial}{\partial x_k} + d \frac{\partial}{\partial y_k} \right\} = ac + bd,
\]

i.e., the matrix \((g_{ij})\) defining the geometry in this coordinate system is the identity matrix. Extending \( \langle \cdot, \cdot \rangle \) to all of \( TM \) we get a Riemannian geometry on \( M \). The inner product \( \langle \cdot, \cdot \rangle \) is just the usual dot product in the given coordinate systems on \( U_k, k = 1, \ldots, n. \) Follow the same procedure on \( N. \)

Let \( \omega: M \to \Lambda^2 TM \) and \( \sigma: N \to \Lambda^2 TN \) be the dual volume (area) forms in these geometries. Clearly \( \omega = \partial / \partial x_k \wedge \partial / \partial y_k \) on \( U_k \) and \( \sigma = \partial / \partial u_k \wedge \partial / \partial v_k \) on \( V_k, k = 1, \ldots, n. \) We have \( f_*: \Lambda^2 TM \to \Lambda^2 TN \) and we may define \( \phi: M \to R \) by \( f_*\omega = \phi \sigma. \) Here \( R \) is the real numbers. The following facts are clear:

(a) If \( p \in M \) and \((x, y)\) are local coordinates at \( p \) and \((u, v)\) are local coordinates at \( f(p) \) then

\[
\phi(x, y) = \psi(x, y) \partial(u, v) / \partial(x, y)
\]

where \( \partial(u, v) / \partial(x, y) \) is the Jacobian and \( \psi \) is a nonzero real-valued function.

(b) On \( U_k, \)

\[
\phi(x_k, y_k) = \partial(u_k, v_k) / \partial(x_k, y_k) \quad \text{for} \quad k = 1, \ldots, n.
\]

(c) \( \phi(p) > 0 \) iff \( p \) is a regular point of \( f \) and \( f \) preserves orientation at \( p. \)

(d) \( \phi(p) < 0 \) iff \( p \) is a regular point of \( f \) and \( f \) reverses orientation at \( p. \)

(e) \( \phi(p) = 0 \) iff \( p \) is a singular point of \( f. \)

Let \( S = \phi^{-1}(0) \), then \( S \) is the set of singular points of \( f. \) We will refer to the components of \( S \) as general folds after Whitney [17]. We claim that 0 is a regular value of \( \phi. \) To show this, we take \( p \in S. \) By (a) and (e), \( \partial \phi / \partial x = \psi \partial J / \partial x \) and \( \partial \phi / \partial y = \psi \partial J / \partial y \) at \( p, \) where \( J \) is the Jacobian. If \((x, y)\) and \((u, v)\) are chosen so that the equation of \( f \) is in normal form for a fold or cusp
at \( p \), then we can easily check that \( \partial J/\partial x \) and \( \partial J/\partial y \) are not both zero at \( p \), and \( p \) is a regular point for \( \phi \). Since 0 is a regular value of \( \phi \), we know that \( S = \phi^{-1}(0) \) has the structure of a smooth 1-submanifold of \( M \).

We now define a vector field on \( M \) which will be normal to \( S \). We define \( X = \text{grad } \phi \) by \( \langle \text{grad } \phi, W \rangle = W(\phi) \) for any vector \( W \in TM \) (cf. Milnor [10]). The zeros of \( X = \text{grad } \phi \) are at the critical points of \( \phi \). We wish the zeros of \( X \) on \( M \) to be isolated, therefore we modify \( \phi \) outside of a neighborhood of \( S \cup \bigcup_k U_k \) in such a way that \( \phi \) has only isolated critical points. This is done in such a way as to create no new zeros of \( \phi \). (We may assume that \( \phi \) is a Morse function although we will not need this.) Now we define the vector field \( Y \) on \( M \) to be normal to \( X \), i.e., \( \langle X, Y \rangle = 0 \), and such that \( ||X|| = ||Y|| \) on \( M \) and \( \langle X, Y \rangle \) has positive orientation. Now \( Y(\phi) = 0 \) and therefore \( Y \) is tangent to the level curves of \( \phi \) and provides a vector field tangent to \( S \). The vector fields \( X \) and \( Y \) are not zero on \( S \) since 0 is a regular value of \( f \). We let the tangent field \( Y|_S \) define positive orientation on \( S \). Now since we modified \( \phi \) outside of \( U_k, \ k = 1, \ldots, n \), we still have \( \phi(x_k, y_k) = \partial (u_k, v_k)/\partial (x_k, y_k) \) in \( U_k, k = 1, \ldots, n \). In these coordinate systems, \( f \) takes the form (1) or (2) and we check that \( \phi(x_k, y_k) = -x_k \). Since the geometry is given by the usual dot product, we have \( X = -\partial/\partial x_k \) and \( Y = -\partial/\partial y_k \) in \( U_k \) for \( k = 1, \ldots, n \).

We have \( Y: M \to TM \) and \( f_*: TM \to TN \). Thus \( f_*Y: M \to TN \). The theorem will follow from an investigation of the zeros of \( f_*Y \). If \( p \) is an isolated zero of \( f_*Y \), we define \( \text{ind}_p(f_*Y) \) as follows: Choose \( (x, y) \) and \( (u, v) \), coordinate systems in neighborhoods \( U \) of \( p \) and \( V \) of \( f(p) \) respectively, such that \( (\partial/\partial x, \partial/\partial y) \) and \( (\partial/\partial u, \partial/\partial v) \) are positively oriented. Write \( f_*Y = \alpha_1 \partial/\partial u + \alpha_2 \partial/\partial v \) where \( \alpha = (\alpha_1, \alpha_2): U \to \mathbb{R}^2 \). Now choose a positively oriented coordinate circle \( C \) about \( p \) such that \( \alpha \) is not zero inside or on \( C \) except at \( p \). Then we define

\[
\text{ind}_p(f_*Y) = \text{deg} \frac{\alpha}{|\alpha|} \bigg|_C
\]

where \( \text{deg} \) is the topological degree and the image of \( \alpha/|\alpha| \) is \( S^1 \subset \mathbb{R}^2 \), with the usual counterclockwise orientation. It is easily checked that the definition is independent of the positively oriented coordinate system \( (u, v) \) chosen. This definition is analogous to the usual definition of the index of a vector field at an isolated zero (see Guillemin and Pollack [5]). Here, however, the vectors are on a different manifold from the points. In Greub [4, p. 385], this is referred to as the local dashed degree.

It is clear from the definition of fold point and cusp point that the only zeros of \( f_*Y \) on \( S \) are at cusp points. Since \( f \) is nonsingular on \( M - S \), the only other zeros of \( f_*Y \) are the zeros of \( Y \). Let us look at \( f_*Y \) in the neighborhood \( U_k \) of the cusp point \( p_k \). We have
Let \( \alpha = (-6y_k, 6y_k^2 \pm x_k) \), the sign depending on \( \mu(p_k) \). We check that

\[
\frac{\partial \alpha}{\partial y_k}
\begin{cases}
(0, 1) & \text{if } \mu(p_k) = +1,
(0, -1) & \text{if } \mu(p_k) = -1.
\end{cases}
\]

By checking the orientation of \( ((\partial \alpha/\partial x_k)|_{p_k}, (\partial \alpha/\partial y_k)|_{p_k}) \), we see that

\[
\text{ind}_{p_k}(f_* Y) = \mu(p_k).
\]

We now define \( M^+ = \phi^{-1}([0, \infty)) \) and \( M^- = \phi^{-1}((-\infty, 0]) \). Now \( M^+ \) and \( M^- \) are oriented 2-submanifolds of \( M \) with boundary \( S \). The 1-manifold \( S \) with positive orientation given by the vector field \( Y \) provides a positively oriented boundary for \( M^- \) and a negatively oriented boundary for \( M^+ \). By remarks (c) and (d) above, \( f \) preserves orientation on the interior of \( M^+ \) and reverses orientation on the interior of \( M^- \). We remark that \( M^+ \) and \( M^- \) are not usually connected and we will write

\[
M^+ = M^+_1 \cup \cdots \cup M^+_l \quad \text{and} \quad M^- = M^-_1 \cup \cdots \cup M^-_m
\]

where \( M^+_k, k = 1, \ldots, l \), are the connected components of \( M^+ \) and \( M^-_k, k = 1, \ldots, m \), are the connected components of \( M^- \). We recall that \( Y \) is a nonzero vector field on \( S \). Thus

\[
\sum_{p \in M^-} \text{ind}_p(Y) = \chi(M^-)
\]

where \( \chi \) is the Euler characteristic. The notion of Euler characteristic of an oriented manifold with boundary is an extension of the notion for unbounded manifolds (see Spivak [12]). If \( M^- \) is formed from a compact manifold of genus \( g \) by cutting out \( j \) discs, then \( \chi(M^-) = 2 - 2g - j \). Also \( \chi(M^-) = \sum_k \chi(M^-_k) \).

Now we have

\[
\text{ind}_p(f_* Y) = \begin{cases}
\text{ind}_p Y & \text{if } p \in M^+,
-\text{ind}_p Y & \text{if } p \in M^-.
\end{cases}
\]
Also on \( S \) we have shown that \( f_\ast Y \) is zero only at the cusps \( p_1, \ldots, p_n \) where \[
ind_{p_k}(f_\ast Y) = \mu(p_k), \quad k = 1, \ldots, n.
\]

Thus
\[
\sum_{p \in M} \ind_p(f_\ast Y) = \chi(M^+) - \chi(M^-) + \sum_{k=1}^{n} \mu(p_k).
\]

To complete the proof of the formula, we need the following (cf. Greub [4, p. 386, problem 17])

**Lemma.** Let \( \tilde{f}: M \to TN \) have only isolated zeros, and let \( f = \pi \circ \tilde{f} \) where \( \pi: TN \to N \) is the usual projection. Then
\[
\sum_{p \in M} \ind_p(\tilde{f}) = (\deg f)\chi(N).
\]

**Proof.** Let \( q \) be a regular value of \( f \) such that \( \tilde{f} \) is not zero on \( f^{-1}(q) \). Let \( W_1 \) and \( W_2 \) be vector fields on \( N \) such that \( (W_1, W_2) \) is a positively oriented basis at each point of \( N - \{q\} \) and \( W_1 \) and \( W_2 \) vanish at \( q \) with index \( \chi(N) \). Write \( \tilde{f} = \alpha_1 W_1 + \alpha_2 W_2 \) on \( M - f^{-1}(q) \) where
\[
\alpha = (\alpha_1, \alpha_2): (M - f^{-1}(q)) \to R^2.
\]
If \( p \in f^{-1}(q) \) and \( S_p \) is a small enough coordinate circle about \( q \), positively oriented, then
\[
\deg \frac{\alpha}{|\alpha|} \bigg|_{S_p} = \begin{cases} 
-\chi(N) & \text{if } f \text{ preserves orientation at } p, \\
\chi(N) & \text{if } f \text{ reverses orientation at } p.
\end{cases}
\]

Now we have
\[
\sum_{p \in N - f^{-1}(q)} \ind_p \alpha = -\sum_{p \in f^{-1}(q)} \deg \frac{\alpha}{|\alpha|} \bigg|_{S_p} = \chi(N)(\deg f).
\]

This completes the proof of the lemma.

Now by the previous lemma
\[
\sum_{p \in M} \ind_p(f_\ast Y) = (\deg f)\chi(N).
\]

Combining this with (3) above, we get
\[
\chi(M^+) - \chi(M^-) + \sum \mu(p_k) = (\deg f)\chi(N).
\]

Since \( \chi(M) = \chi(M^+) + \chi(M^-) \), we get
\[
\chi(M) - 2\chi(M^-) + \sum \mu(p_k) = (\deg f)\chi(N)
\]
and this is the desired formula. We mention that at this point we may, by continuity, drop the assumption that the images of the cusp points are distinct.
Summarizing, we have

**Theorem 1.** Let $M$ and $N$ be smooth, compact, oriented, connected 2-manifolds and let $f: M \to N$ be excellent. Let $M^+$ be the closure of the set of regular points where $f$ preserves orientation and $M^-$ the closure of the set of regular points at which $f$ reverses orientation. Let $p_1, \ldots, p_n$ be the cusp points and let $\mu(p_k)$ be the local degree at the cusp point $p_k$, then

\[(4) \quad \chi(M) - 2\chi(M^-) + \sum \mu(p_k) = (\deg f)\chi(N)\]

where $\chi$ is the Euler characteristic. The conclusion may also be stated

\[\chi(M^+) - \chi(M^-) + \sum \mu(p_k) = (\deg f)\chi(N).\]

The following corollary is a result of A. Haefliger [6, p. 53].

**Corollary 1.** Let $M$ be a smooth, compact, oriented, connected 2-manifold and let $f: M \to R^2$ be excellent. Let $M^+, M^-, p_k$ be defined as in the theorem. Then the number of cusp points $n$ is less than or equal to $|\chi(M^+) - \chi(M^-)|$.

**Proof.** We may consider $R^2$ as $S^2$ minus a point. Letting $N = S^2$ in the theorem, we have $\deg f = 0$ and $\sum \mu(p_k) = \chi(M^-) - \chi(M^+)$. Now $n < |\sum \mu(p_k)|$ and the result follows.

We also have the following corollary (cf. Levine [8], [9], and Thom [14]).

**Corollary 2.** Under the hypotheses of the theorem, the number of cusp points is even.

**Proof.** We note that that $\chi(M)$ and $\chi(N)$ are even and $\chi(p_k)$ has the same parity as $n$.

For another formula related to (4), see Tucker [16]. Also, in combinatorially classifying stable maps from bordered surfaces to the sphere, Francis and Troyer [3], [18] have proved a formula related to (4).

We now show how the standard Riemann-Hurwitz formula (see Hille [7, p. 110] or Bliss [1, p. 89]) follows from (4). Suppose $f: M \to N$ is orientation-preserving except for branch points at $p_1, \ldots, p_n$ of orders $v_1, \ldots, v_n$, respectively. (Such a map is called sense preserving or SP by Titus [15].) Now $f$ is not excellent but $f$ may be perturbed slightly so that instead of a branch point of order $v_k$ at $p_k$, we have $v_k + 1$ cusps of positive type on a simple closed general fold. This general fold bounds a component of $M^-$ in its interior (see Callahan [2] for a demonstration of this for $v_k = 2$). Thus $\chi(M^-) = n$ and $\sum \mu(p_k) = n + \sum v_k$. Thus (4) gives

\[(5) \quad \chi(M) + \sum (v_k - 1) = (\deg f)\chi(N).\]

If $N$ is the sphere, $\chi(N) = 2$ and (5) is the Riemann-Hurwitz formula.

**Some Examples.** We give two examples of formula (4).
1. Let $M$ be a torus considered as a sphere $N$ with one handle. Let $f$ be obtained by projecting $M$ onto $N$. The configuration of cusps and general folds is shown in Figure 1. The four cusps are of positive type. We have $\sum \mu(p_k) = 4$, $\chi(M) = 0$, $\chi(N) = 2$, $\chi(M^-) = 1$, and $\deg f = 1$.

2. Let $M$ be a torus and $N$ a sphere (see Figure 2). Construct $f$ by sending $S_k$ to $T_k$, $k = 1, \ldots, 4$, preserving the orientations indicated, and sending the indicated points on $S_1$ and $S_3$ to the cusps on $T_1$ and $T_3$. All cusps are of positive type and $\sum \mu(p_k) = 4$, $\chi(M^-) = 0$, $\chi(M) = 0$, $\chi(N) = 2$, and $\deg f = 2$. We remark that $f$ is a perturbation of the usual branched covering of a torus over a sphere given by the algebraic function

$$w^2 - (1 - z^2)(1 - k^2z^2) = 0.$$


References

3. G. K. Francis and S. F. Troyer, Excellent maps with given folds and cusps (preprint).

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