LINEAR ISOTopies IN $E^2$

BY

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Abstract. This paper deals with continuous families of linear embeddings (called linear isotopies) of finite complexes in the Euclidean plane $E^2$. Suppose $f$ and $g$ are two linear embeddings of a finite complex $P$ with triangulation $T$ into a simply connected open subset $U$ of $E^2$ so that there is an orientation preserving homeomorphism $H$ of $E^2$ to itself with $H \circ f = g$. It is shown that there is a continuous family of embeddings $h_t: P \to U$ ($t \in [0, 1]$) so that $h_0 = f$, $h_1 = g$, and for each $t$, $h_t$ is linear with respect to $T$.

It is also shown that if $P$ is a PL star-like disk in $E^2$ with a triangulation $T$ which has no spanning edges and $f$ is a homeomorphism of $P$ which is the identity on $Bd P$ and is linear with respect to $T$, then there is a continuous family of homeomorphisms $h_t: P \to U$ ($t \in [0, 1]$) such that $h_0 = \text{id}$, $h_1 = f$, and for each $t$, $h_t$ is linear with respect to $T$. An example shows the necessity of the "star-like" requirement. A consequence of this last theorem is a linear isotopy version of the Alexander isotopy theorem—namely, if $f$ and $g$ are two PL embeddings of a disk $P$ into $E^2$ so that $f|Bd P = g|Bd P$, then there is a linear isotopy with respect to some triangulation of $P$ which starts at $f$, ends at $g$, and leaves the boundary fixed throughout.

1. Introduction. This paper deals with finite complexes linearly embedded in the Euclidean plane $E^2$ and continuous families of such linear embeddings which are called linear isotopies.

Suppose $(P, T)$ is a compact triangulated PL complex in a simply connected open subset $U$ of $E^2$ and $g$ is a linear homeomorphism of $P$ into $U$ that can be extended to an orientation preserving homeomorphism of $E^2$ onto itself. It is shown in Theorem 7.2 that there is a linear isotopy $h_t: (P, T) \to U$ such that $h_0 = \text{id}$ and $h_1 = g$. Each $h_t$ is linear on each simplex of $T$.

Theorem 7.2 follows from the result (Theorem 7.1) that if $(D, T)$ is a triangulated disk in $E^2$, then there is a linear isotopy $h_t: (D, T) \to E^2$ such that $h_0 = \text{id}$, $h_t(D)$ is a convex disk, and $h_t$ is decreasing in the sense that if $0 \leq s < t < 1$, $h_s(D) \subset h_t(D)$.

Suppose $(P, T)$ is a complex linearly embedded in $E^2$ and $g$ is a linear...
homeomorphism of \((P, T)\) into \(E^2\) that is fixed on a certain subcomplex \(P'\) of \(P\). Conditions are investigated under which there is a linear isotopy \(h_t\) 
\((t \in [0, 1])\) of \((P, T)\) into \(E^2\) such that
\[ h_0 = \text{id}, \text{ each } h_t \text{ is fixed on } P', \text{ and } h_1 = g. \]
It is shown in Theorem 4.1 that if \((P, T)\) is a star-like disk without spanning edges and \(P' = \text{Bd } P\), then for each \(g\) there is such an \(h_t\) 
\((t \in [0, 1])\). However, Example 4.2 shows that the result does not always hold if some 1-simplex of \(T\) is permitted to span \(P\).

§2 treats the question of extending the linear embedding into \(E^2\) of a subcomplex \(C^1\) of a finite complex \(C\) to a linear embedding of all of \(C\) into \(E^2\). Although we do not get necessary and sufficient conditions that such an embedding can be extended, we get some partial solutions (mostly in the case where \(C\) is a triangulated disk and \(C^1 = \text{Bd } D\)). Our methods give a new proof of Fary's theorem which states that the extension is possible if \(C\) is a 1-complex which can be topologically embedded in \(E^2\) and \(C^1 = \emptyset\).

A linear isotopy version of the Alexander Isotopy Theorem for PL disks in \(E^2\) appears in §5, namely, if \(P\) is a PL disk in \(E^2\) and \(f\) is a PL homeomorphism of \(P\) fixed on \(\text{Bd } P\), then there is a triangulation \(T\) of \(P\) and a linear isotopy \(h_t: (P, T) \rightarrow E^2\) 
\((t \in [0, 1])\) with \(h_0 = \text{id}, h_1 = f,\) and for each \(t, h_t|\text{Bd } P = \text{id}\).

§3 contains the definitions of a linear isotopy and the related notion of a push. It is shown that a linear isotopy of a finite complex in \(E^n\) can be extended to a linear isotopy of all of \(E^n\).

§§4 and 5 contain the theorems about linear isotopies of disks with fixed boundaries.

Linear degeneration is defined in §6. This concept is a variation of a collapse and is used to prove the theorems in §7 mentioned in the second paragraph of the introduction.

In future papers Starbird will make use of some variations of the theorems in this paper to investigate questions about linear isotopies in \(E^3\).

2. Linear embeddings. In this section we investigate the question of extending the linear embedding of a part of a complex into \(E^2\) to a linear embedding of the entire complex.

**Definition.** Let \(C\) be a complex with a triangulation \(T\). Then \(h: C \rightarrow E^n\) is a linear embedding of \(C\) into \(E^n\) if \(h\) is an embedding which is linear on each simplex in \(T\).

**Notation.** When we wish to emphasize a particular triangulation on a complex, we may write \((C, T)\) to stand for a complex \(C\) with a triangulation \(T\). We use \(\text{St}(v)\) and \(\text{Lk}(v)\) to denote the star and link respectively of a vertex \(v\) in a complex \(C\).

**Definition.** A disk \(D\) in \(E^2\) is star-like if and only if there exists a point \(s\)
in $E^2$ (necessarily in Int $D$) such that for every point $x$ in Bd $D$, the straight segment $xs$ meets Bd $D$ only at $x$. We say that such a point $s$ can see every point of Bd $D$.

Note that if $D$ is a star-like polygonal disk in $E^2$, the set of points from which all of Bd $D$ can be seen is a convex, open subset of Int $D$.

**Theorem 2.1.** Let $P$ be a polygonal disk with a triangulation $T$ which contains no spanning edge, i.e., if $v$ and $w$ are nonadjacent vertices on Bd $P$, then $vw$ is not a simplex in the triangulation $T$. Then if $h$: Bd $P$ $\rightarrow$ $E^2$ is a linear embedding of Bd $P$ such that $h$(Bd $P$) bounds a star-like disk in $E^2$, then there exists a linear embedding $H$: $P$ $\rightarrow$ $E^2$ which extends $h$.

**Proof.** The proof is by induction on $n$, the number of interior vertices in the triangulation $T$.

If $n = 0$, $P$ is a 2-simplex and there is nothing to do.

Assume that the theorem is true for triangulations with $k$ interior vertices where $k < n$. Suppose $T$ has $n$ interior vertices. Let $v$ be an interior vertex of $T$ so that St$(v)$ contains an edge on Bd $P$. Let $s$ be a point in $E^2$ which can see every point of $h$(Bd $P$). We will have $H$ map $v$ to $s$. For every vertex $w$ $\in$ (Bd $P$ $\cap$ St$(v)$), $H(vw)$ will be the straight segment from $h(w)$ to $s$. These segments divide the interior of $h$(Bd $P$) into star-like subdisks each of which meets the requirements of the theorem. Complete the proof by applying induction in each subdisk.

The following theorem is a corollary of Theorem 2.1.

**Theorem 2.2.** Let $P$ be a polygonal disk with a triangulation $T$ and let $h$: Bd $P$ $\rightarrow$ $E^2$ be a linear embedding such that $h$(Bd $P$) bounds a convex disk $D$ and for each spanning 1-simplex $vw$ of $P$, $h(v)$ and $h(w)$ are the ends of a straight spanning segment of $D$. Then there exists a linear embedding $H$: $P$ $\rightarrow$ $E^2$ which extends $h$.

**Figure 2.1**

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Example 2.1. It is to be noted that Theorem 2.2 is false if instead of supposing that $D$ is convex we suppose instead that it is merely star-like as can be seen by Figure 2.1.

Theorem 2.3 (Fary [4]). Let $\Gamma$ be a finite graph with no loops or multiple edges. If $\Gamma$ can be topologically embedded in $E^2$, then it can be linearly embedded in $E^2$.

Proof. Incorporate the embedded $\Gamma$ into the 1-skeleton of a topological triangulation of a disk in $E^2$ with no spanning edges. Re-embed the boundary of the disk linearly so that it bounds a star-like disk in $E^2$ and apply Theorem 2.1. Then $\Gamma$ will be embedded in the 1-skeleton of a linearly embedded disk and its edges will therefore be straight.

Question 2.1. If $C^1$ is a subcomplex of a complex $C$, under what conditions can a linear embedding of $C^1$ into $E^2$ be extended to a linear embedding of $C$? Theorem 2.3 provides an answer if $C$ is a finite 1-complex and $C^1 = \emptyset$. Theorem 2.1 only gives a sufficient condition even in the case where $C$ is a triangulated disk without spanning edges and $C^1 = \text{Bd } C$. In the case where $C$ is a triangulated disk and $C^1 = \text{Bd } C$, might a necessary and sufficient condition be that the embedding could be extended to send each acyclic 1-dimensional subcomplex of $C$ into the disk in $E^2$ bounded by image of $C^1$.

Question 2.2. There are 2-complexes which can be topologically embedded in $E^3$, but not linearly embedded. Suppose a 2-complex can be topologically embedded in $E^4$, can it then be linearly embedded in $E^4$? What higher dimensional analogs of the Fary Theorem are true?

3. Linear isotopies and the Fundamental Extension Theorem. In this section we define linear isotopies and pushes of complexes in $E^n$ and prove that a linear isotopy of a finite complex in $E^n$ can be extended to a linear isotopy of all of $E^n$. This is the only section in which theorems about $E^n$ for $n$ greater than two are proved. Whitehead proved that any finite complex linearly embedded in $E^n$ can be incorporated into a triangulation of $E^n$ without subdividing the complex. This result is given as Theorem 3.1 below and we include a proof for completeness.

Definition. Let $C$ be a complex with a triangulation $T$. A linear isotopy of $C$ is a continuous family of embeddings $h_t : C \to E^n \ (t \in [0, 1])$ so that for every $t \in [0, 1]$, $h_t$ is a linear embedding of $C$ in $E^n$. If we wish to emphasize the triangulation involved, we will write that $h_t \ (t \in [0, 1])$ is a linear isotopy of $(C, T)$. A linear isotopy $h_t : C \to E^n \ (t \in [0, 1])$ is called a simple push if it is fixed on each vertex except one. A finite sequence of such simple pushes, performed one after the preceding, is called a push. It may be noted that if there is a linear isotopy taking one embedding $h$ of a finite complex $C$ into $E^n$ onto another embedding $g$, then $h(C)$ can be pushed to $g(C)$. 


Theorem 3.1 (Whitehead [8, Theorem 5]). Suppose \((C, T_c)\) is a triangulated finite complex linearly embedded in \(E^n\). Then there is a triangulation \(T\) of \(E^n\) such that each simplex of \(T_c\) is a simplex of \(T\).

Proof. The proof is by induction on the number of simplexes in \(T_c\). The theorem is true if \(T_c\) has only one simplex. Assume \(T_c\) has \(n\) simplexes and the theorem is true for complexes with fewer. Let \(\sigma\) be a simplex of \(T_c\) that is not a face of any other simplex of \(T_c\) and let \(T'\) be a triangulation of \(E^n\) such that

1. each simplex of \(T_c\) other than \(\sigma\) is an element of \(T'\).
2. if \(t'\) is a simplex of \(T'\) that contains an element of \(T_c\) that is not a face of \(\sigma\), then \(t' \cap \text{Int} \ \sigma = \emptyset\).

If condition (2) above is not satisfied by the first \(T'\) selected, some subdivision of this \(T'\) leads to one satisfying both conditions (1) and (2).

Let \(T''\) be a triangulation of \(E^n\) such that \(\sigma\) is a simplex of \(T''\) and \(t' \cap t''\) is either empty or a face of \(\sigma\) where \(t'' \in T''\) containing \(\sigma\) as a face and \(t' \in T'\) containing as a face an element of \(T_c\) which is not a face of \(\sigma\).

Let \(S'\) be the convex cellular subdivision of \(E^n\) whose elements are the intersections of elements of \(T'\) and \(T''\). Note that each proper face of \(\sigma\) is an element of \(S'\). By subdividing \(S'\) we get a triangulation \(S\) of \(E^n\) such that each proper face of \(\sigma\) is an element of \(S\) and \(S\) is a subdivision of both \(T'\) and \(T''\).

Let

\[
U = (C - \sigma) \cup (\bigcup \text{Int } t'_i) \quad \text{for } t'_i \in T' \quad \text{with } t'_i \cap (C - \sigma) \neq \emptyset,
\]

\[
V = \bigcup \text{Int } t''_i \quad \text{for } t''_i \in T'' \quad \text{having } \sigma \text{ as a face}.
\]

Then \(U, V\) are disjoint open sets and each of \(\bar{U}, \bar{V}, E^n - (U \cup V)\) is the union of elements of \(S\).

We start defining \(T\) by putting each simplex of \(S\) which is in \(E^n - (U \cup V)\) in \(T\). The proper faces of \(\sigma\) are now in \(T\).

Now consider \(\bar{U}\). It is the union of simplexes of \(T'\) and \(\text{Bd } U\) is already triangulated by \(T\). To get other simplexes of \(T\) in \(\bar{U}\) we subdivide the simplexes of \(T'\) in \(\bar{U}\) one at a time proceeding upward through dimensions. If \(t' \in T'\) in \(\bar{U}\) and triangulation \(T\) on \(\text{Bd } t'\) has already been defined, we let \(t'\) be an element of \(T\) if \(T\) did not properly divide \(\text{Bd } t'\), but if \(T\) properly divided \(\text{Bd } t'\) we obtain simplexes of \(T\) in \(t'\) by coning over \(\text{Bd } t'\) from the baricenter of \(t'\). Note that each simplex of \(T_c\) other than \(\sigma\) has now been placed in \(T\).

The simplexes of \(T\) in \(\bar{V}\) are obtained by subdividing the elements of \(T''\) in \(V\) in a fashion similar to that in which the elements of \(T'\) in \(\bar{U}\) were subdivided as described in the preceding paragraph.
In the proof of Theorem 3.1 we regarded \((C, T_c)\) as a finite complex so that we could use induction. However, once the result is proved for the finite case, it can be extended to the infinite case as follows.

**Theorem 3.2.** Suppose \((C, T_c)\) is a triangulated complex (perhaps infinite) linearly embedded as a closed subset of \(E^n\). Then there is a triangulation \(T\) of \(E^n\) such that each simplex of \(T_c\) is a simplex of \(T\).

**Proof.** The theorem is proved by enlarging \(C\) until the enlargement contains all of \(E^n\).

Let \(B_1\) be the closed \(n\)-ball with center at the origin and radius 1 while \(U_1\) is a bounded open set containing \(B_1\) such that each element of \(T_c\) that intersects \(B_1\) lies in \(U_1\). By Theorem 3.1 there is a triangulation \(T_1\) of \(E^n\) such that each simplex of \(T_c\) that intersects \(\overline{U_1}\) is a simplex in \(T_1\). We suppose that \(T_1\) is adjusted so that each simplex of \(T_1\) that intersects \(B_1\) lies in \(U_1\). Let \(C_1\) be the union of \(C\) and the simplexes of \(T_1\) that intersect \(B_1\). A simplex in the triangulation \(T_{c_1}\) of \(C_1\) is either a simplex of \(T_c\) or the face of a simplex of \(T_1\) that intersects \(B_1\).

In a similar fashion we get a sequence of triangulated complexes \((C_1, T_{c_1}), (C_2, T_{c_2}), \ldots\) such that \(T_{c_i} \subset T_{c_{i+1}}\) and \(C_i\) contains the ball with center at the origin and radius \(i\). A simplex is an element of the required triangulation \(T\) if and only if it belongs to some \(T_{c_i}\).

**Theorem 3.3 (The Fundamental Extension Theorem).** Let \((C, T_c)\) be a triangulated finite complex linearly embedded in \(E^n\) and let \(h: C \to E^n\) \((t \in [0, 1])\) be a linear isotopy of \(C\) such that \(h_0 =\) identity. Then there is a triangulation \(T\) of \(E^n\) with \(T_c \subset T\) and a linear isotopy \(H: E^n \to E^n\) \((t \in [0, 1])\) with compact support such that \(H_0 = \text{id}\) and \(H_t|C = h_t\).

**Proof.** For every \(s \in [0, 1]\), let \(T_s\) be a triangulation of \(E^n\) which contains \(h_s(C, T_c)\) as a subcomplex. Theorem 3.1 asserts that there is such a \(T_s\). For each \(T_s\) there is an \(\varepsilon_s > 0\) and a linear isotopy

\[H_s: E^n \to E^n \quad (t \in [s - \varepsilon_s, s + \varepsilon_s] \cap [0, 1])\]

that moves the vertices of \(T_s\) which are in \(h_s(C)\) according to \(h_s\), leaves all other vertices of \(T_s\) fixed, and extends linearly to the simplexes of \(T_s\).

We can find two finite collections of points in \([0, 1]\), \(\{t_i\} (i = 0, 1, \ldots, m)\) with \(0 = t_0 < t_1 < \cdots < t_m = 1\) and \(\{s_i\} (i = 1, 2, \ldots, m)\) so that \(H_{t_i}^{s_i}\) is defined for \(t \in [t_{i-1}, t_i]\), that is, \([t_{i-1}, t_i] \cap [s_i - \varepsilon_s, s_i + \varepsilon_s] = [t_{i-1}, t_i]\).

Now \(H_t: E^n \to E^n\) \((t \in [0, 1])\) is defined to be
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$H_i^{t_2} (H_i^{t_1})^{-1}$ if $t \in [t_0, t_1]$,
$H_i^{t_2} (H_i^{t_1})^{-1} H_i^{t_1} (H_i^{t_0})^{-1} H_i^{t_1}$ if $t \in [t_1, t_2]$,
\[\ldots\]
$H_i^{t_{n+1}} (H_i^{t_n})^{-1} H_i^{t_n}$ if $t \in [t_n, t_{n+1}]$.

One can obtain $T$ as follows. Let $T_0 = H_i^{t_0}(T_i)$. Then $T_0$ is a triangulation of $E^2$ with $T_0 \subseteq T_0$ and $H_i$ is linear on $T_0$ for $t \in [0, t_1]$. Also $H_i^{-1}$ is linear on $H_i(T_0)$. Let $T_1$ be a triangulation of $E^2$ which is a refinement of both $H_i(T_0)$ and $H_i^{t_2}(T_2)$ and has $H_i(T_2)$ as subcomplex and let $T_1 = H_i^{-1}(T_1)$. Also let $T_2$ be a triangulation of $E^2$ which is a refinement of each of $H_i(T_1)$ and $H_i^{t_2}(T_2)$ and has $H_i(T_2)$ as a subcomplex and let $T_2 = H_i^{-1}(T_2)$. Continuing in this fashion one gets $T_3, T_4, \ldots, T_{m-1}$. The required triangulation $T$ is $T_{m-1}$.

**Question 3.1.** Is Theorem 3.3 true if we drop the finite condition, permit $(C, T)$ to be an infinite complex linearly embedded as a closed subset of $E^2$, and do not require compact support?

**4. Linear isotopies of disks with fixed boundaries.** Let $f$ and $g$ be two linear embeddings of a triangulated disk $(P, T)$ into $E^2$ so that $f|\text{Bd} P = g|\text{Bd} P$. In this section we investigate conditions under which there is a linear isotopy or push from $f$ to $g$ which leaves the boundary fixed. Cairns proved the following theorem in this vein.

**Cairns’ Theorem** [2]. Let $(P, T)$ be a triangulated disk linearly embedded in $E^2$ where $T$ has only 3 1-simplexes in $\text{Bd} P$ and let $f$ be a linear homeomorphism of $(P, T)$ which is the identity on the boundary. Then there is a push of $(P, T)$ from the identity to $f$ which keeps $\text{Bd} P$ fixed.

In this section we prove several variations of Cairns’ Theorem. The first variation (Theorem 4.1) removes the hypothesis that $T$ has only 3 1-simplexes in the boundary and replaces it by the hypotheses that $P$ is star-like and $T$ has no spanning 1-simplex. Examples show that neither the “star-like” hypothesis nor the “no spanning 1-simplex” hypothesis can be removed.

**Theorem 4.1.** Let $(P, T)$ be a triangulated, star-like disk linearly embedded in $E^2$ where $T$ contains no spanning 1-simplex. Let $g: P \to E^2$ be a linear homeomorphism of $(P, T)$ which is the identity on $\text{Bd} P$. Then there is a push taking $(P, T)$ to $(P, g(T))$ which leaves $\text{Bd} P$ fixed throughout.

**Proof.** The proof is by induction on $n$, the number of interior vertices in the triangulation $T$. The proof bears a resemblance to the proof of Theorem 2.1. The scheme of the proof is to take an interior vertex $v$ in $T$ whose star contains an edge on $\text{Bd} P$ and push $P$, leaving $\text{Bd} P$ fixed, by $h_i^t$ ($t \in [0, 1]$)
so that \( h_1^1(v) \) can see all of \( \text{Bd} \, P \). Then we push \((P, g(T))\), leaving \( \text{Bd} \, P \) fixed, by \( h_1^2 \) \((t \in [0, 1])\) so that \( h_1^1(v) = h_1^2(g(v)) \). For every edge \( vw \) in \( T \) where \( w \in \text{Bd} \, P \), \( h_1^1(vw) = h_1^2(g(vw)) \). As in Theorem 2.1, we then apply induction in each of the subdisks into which the \( h_1^1(vw) \)'s divide \( P \) to finish the proof.

What we need to show is that there is a push of \((P, T)\) taking \( v \) to a point which can see all of \( \text{Bd} \, P \) and similarly a push of \((P, g(T))\) taking \( g(v) \) to the same point. The following two lemmas are helpful in constructing these pushes.

**Lemma 4.2.** Let \((P, T)\) be a triangulated star-like disk linearly embedded in \( E^2 \). Let \( v_0v_1v_2 \) be a simplex of \( T \) such that \( v_0 \) is an interior vertex and there is a point \( s \in \text{Int} \, v_0v_1v_2 \) which can see every point of \( \text{Bd} \, P \). Then for each neighborhood \( N \) of \( s \) there is a push of \((P, T)\) that moves \( v_0 \) into \( N \), leaves \( \text{Bd} \, P \) fixed, and is such that any vertex that is moved at all is moved in a straight line toward \( s \).

**Proof.** We first describe a continuous family of linear maps \( f_t : (P, T) \to P \) \((t \in [0, 1])\) which (except for not being homeomorphisms) satisfy the conclusion of the lemma. Then we pick a \( t_0 \) near 1 and approximate \( f_t \) \((t \in [0, t_0])\) by a suitable linear isotopy.

The family \( f_t \) \((t \in [0, 1])\) will be described by specifying each map \( f_t \) on the vertices of \( T \) and then extending linearly to \( P \).

The map \( f_t \) \((t \in [0, 1])\) is described as follows. It begins by moving \( v_0 \) directly toward \( s \) leaving all other vertices fixed. We suppose the motion is linear on \( v_0 \). We will extend this map by coning from \( v_0 \) to \( \text{Lk}(v_0) \). Either this extension is a homeomorphism for every position of \( v_0 \) as it moves to \( s \) or it is not. If it is, the lemma is proved. If not, let \( t_1 \) be the first moment at which the cone map is not a homeomorphism.

If \( t_1 < 1 \), let \( w_1 \) be a vertex which is in the way. See Figure 4.1. (If there is more than one such vertex, move each of them as described for \( w_1 \) in what follows.) That is, \( w_1 \) lies on the segment between \( f_{t_1}(v_0) \) and \( w'_1 \), a vertex in \( \text{Lk}(v_0) \). (Note that \( w_1 \) is also a vertex in \( \text{Lk}(v_0) \).) The map \( f_{t_1} \) smashes the simplex \( v_0w_1w'_1 \) and we say that \( w_1 \) is killed. From now on \( f_t \) on \( w_1 \) will be...
defined as follows: \( f_t(w_i) \) (\( t \in [t_1, 1] \)) will be the point on the intersection of the segment \( w_is \) and the segment \( f_t(v_0)f_t(w_i) \).

We continue defining \( f_t \) by moving \( v_0 \) and \( w_1 \) toward \( s \) as described and leaving all other vertices fixed and extending linearly. Let \( t_2 \in [t_1, 1] \) be the first point in \( [t_1, 1] \) at which the map \( f_{t_2}f_{t_1}^{-1} \) is not a homeomorphism of \( f_{t_1}(P) \). If \( t_2 < 1 \), we find a vertex \( w_2 \) analogous to \( w_1 \) above and a 1-simplex \( w_2'w_2'' \) so that \( w_2 \in f_{t_2}(w_2'w_2'') \). We say that \( w_2 \) is killed at time \( t_2 \). We define the map \( f_t \) (\( t \in [t_2, 1] \)) on \( w_2 \), as before, to be the point of intersection of the segment \( w_2s \) and \( f_t(w_2')f_t(w_2'') \). Continue until \( v_0 \) has been moved to \( s \).

Let \( w_1, w_2, \ldots, w_n \) be the vertices that were killed and \( t_i \) (\( i = 1, 2, \ldots, n \)) be the time at which \( w_i \) was killed. For convenience we suppose \( 0 < t_1 < t_2 < \cdots < t_n < 1 \). Let \( t_0 \) be an element of \( (t_n, 1) \) such that \( f_{t_0}(v_0) \in N \). We show how to adjust \( f_t \) (\( t \in [0, t_0] \)) so that the adjustment is a linear isotopy.

We note that \( w_n \) was killed when \( w_n'w_n'' \) was moved up to it from the non-\( s \) side. If \( w_n \) had been moved toward \( s \) slightly before \( w_n'w_n'' \) arrived, the killing would have been delayed. In fact, if \( w_n \) had been continuously moved toward \( s \) and slightly ahead of \( f_{t_i}(w_n) \), it would not have been killed in the time interval \( [0, t_0] \). (The interval \( [0, t_0] \) was used instead of \( [0, 1] \) because difficulty would have been encountered at \( t = 1 \).) Similarly \( w_{n-1} \) could have been moved continuously ahead of \( f_{t_i}(w_{n-1}) \) to prevent its being killed in the time interval \( [0, t_0] \). Moving backward through the \( w_i \)'s we find that all of the \( w_i \)'s could have been moved slightly ahead of their images under \( f_t \) so as to have a linear isotopy rather than a 1-parameter family of maps.

Note that every vertex that moved, moved directly toward \( s \), a point which could see all of \( Bd P \). Any vertex \( w_i \) which was killed was on \( f_{t_i}(w_i'w_i'') \). Therefore, \( w_i \) originally was in the triangle \( s w_i'w_i'' \) Hence \( w_i \) was an interior vertex since the triangle \( s w_i'w_i'' \) is contained in \( P \). Note that \( Bd P \) remained fixed in \( f_{t_i} \) (\( t \in [0, t_0] \)) and also remained fixed by the linear isotopy approximation. Replacing the linear isotopy by a push completes the proof of this lemma.

**Lemma 4.3.** Let \( (P, T) \) be a star-like disk with no spanning 1-simplex linearly embedded in \( E^2 \). Let \( s \in \text{Int } P \) be a point which can see \( Bd P \) and let \( N \) be a neighborhood of \( s \). Then there is a push of \( P \) such that \( Bd P \) is left fixed and each interior vertex is taken into \( N \).

**Proof.** We suppose \( N \) is the interior of a circle each of whose points can see all of \( Bd P \). Let \( v \) be an interior vertex belonging to a 2-simplex of \( T \) whose interior intersects \( N \). Use Lemma 4.2 to move \( v \) into \( N \). Iterated use of Lemma 4.2 carries the other interior vertices into \( N \).

We use Lemmas 4.2 and 4.3 to finish the proof of Theorem 4.1. Let \( s \) be a point that can see \( Bd P \). Let \( v \) be an interior vertex of \( T \) which contains an edge \( uw \) of \( Bd P \) in its star. Using Lemma 4.3 on \( (P, T) \) move \( v \) to a point \( v' \)
very close to point $s$ and using it on $(P, g(T))$ move $g(v)$ to a point $v''$ very close to $s$. They should be so close that there is a point $s' \in uwv' \cap uwv''$ so that $s'$ can see $Bd \ P$. Now apply Lemma 4.2 to move $v'$ to $s'$ (by aiming for a point beyond $s'$ on ray $v's'$) and $v''$ to $s'$. Apply induction as described in the beginning of the proof of Theorem 4.1 and the theorem is proved.

We list two corollaries of Theorem 4.1. The first is a variation of a theorem of Ho [6].

**Corollary 4.4.** Let $(P, T)$ be a triangulated, convex disk linearly embedded in $E^2$. Let $g: P \rightarrow P$ be a linear homeomorphism of $(P, T)$ which is the identity on $Bd \ P$. Then there is a push of $(P, T)$ taking $(P, T)$ to $(P, g(T))$ which leaves $Bd \ P$ fixed throughout.

**Corollary 4.5 (Cairns).** Let $(B, T)$ be a triangulated complex. Let $g_0$ and $g_1$ be two linear embeddings of $(B, T)$ into $E^2$ so that there is an orientation-preserving homeomorphism $f$ of $E^2$ with $f \circ g_0 = g_1$. Then there is a push $h_t$ $(t \in [0, 1])$ of $(B, T)$ so that $h_0 = g_0$ and $h_1 = g_1$.

**Proof.** Incorporate $g_0(B)$ and $g_1(B)$ into equivalent triangulations of a convex PL disk and apply Corollary 4.4.

In a recent paper [7], an example is presented to show that the analog in $E^3$ of Corollary 4.5 is false.

We now give an example to show that the hypothesis of star-likeness in Theorem 4.1 is necessary.

**Example 4.1.** This is an example of a polygonal disk $(P, T)$ linearly embedded in $E^2$ and a linear homeomorphism $g: P \rightarrow P$ with $g|Bd \ P$ the identity such that there is no linear isotopy of $(P, T)$ taking the identity map to $g$ while leaving $Bd \ P$ fixed.

![Figure 4.2](https://www.ams.org/journal-terms-of-use)
embed \( efg \). Since these spanning arcs in the two embeddings can be incorporated into equivalent triangulations of \( P \), the example shows that Theorem 4.1 is false without the star-like requirements.

**Example 4.2.** A slight modification of Example 4.1 shows the necessity of the “no spanning 1-simplex” hypothesis in Theorem 4.1. Just add a rectangular disk to the bottom of the disk \( P \) of Example 4.1 to make the resulting disk \( P' \) star-like but keep the bottom of \( P \) as a spanning 1-simplex of \( P' \).

**5. The Alexander Isotopy Theorem.** The standard proof \([1]\) of the Alexander Isotopy Theorem shows that for every PL homeomorphism \( f \) of a PL planar disk \( P \) to itself which fixes the boundary there is a PL isotopy \( h_t : P \to P \ (t \in [0, 1]) \) so that \( h_0 = \text{id} \), \( h_1 = f \), and for each \( t \in [0, 1] \), \( h_t|\text{Bd} \ P = \text{id} \). It is not necessarily true that for each \( t \), \( h_t \) is a linear homeomorphism of \( P \) with respect to the same triangulation of \( P \). In fact, the standard proof of Alexander’s theorem makes use of a process which squeezes badness out of existence in a limit. This type of argument cannot yield a linear isotopy. Here we show that the PL isotopy \( h_t \) can be chosen so that there is one triangulation \( T \) with respect to which each \( h_t \) is a linear homeomorphism. In a future paper Starbird will prove the analogous theorem for 3-cells in \( E^3 \).

**Lemma 5.1.** Let \( P \) be a PL planar disk in \( E^2 \) and \( r \) and \( s \) be two PL embeddings of an arc \( A \) into \( P \) so that \( r(A) \cap \text{Bd} \ P = s(A) \cap \text{Bd} \ P = r(\text{Bd} \ A) = s(\text{Bd} \ A) = \{p, q\} \). Then there is a triangulation \( T \) of \( A \) and a linear isotopy \( h_t : (A, T) \to E^2 \ (t \in [0, 1]) \) so that \( h_0 = r \), \( h_1 = s \), and for each \( t \), \( h_t(\text{Bd} \ A) = h_t(A) \cap \text{Bd} \ P = \{p, q\} \).

**Proof.** Find a spanning PL arc \( B \) in \( P \) so that \( B \cap \text{Bd} \ P = B \cap \text{Bd} \ P = B \cap r(A) = B \cap s(A) = \{p, q\} \). One can use the disks bounded by \( B \cup r(A) \) and \( B \cup s(A) \) respectively to guide a push of \( r(A) \) to \( B \) and a push of \( B \) to \( s(A) \).

**Theorem 5.2.** Let \( P \) be a PL disk PL embedded in \( E^2 \) and \( f \) be a PL homeomorphism of \( P \) onto itself which leaves \( \text{Bd} \ P \) fixed. Then there is a triangulation \( T \) and a linear isotopy \( h_t : (P, T) \to E^2 \ (t \in [0, 1]) \) so that \( h_0 = \text{id} \), \( h_1 = f \), and, for each \( t \in [0, 1] \), \( h_t|\text{Bd} \ P = \text{id} \).

**Proof.** Let \( T' \) be a triangulation of \( P \) with respect to which \( f \) is linear. We first find a subdivision \( T \) of \( T' \) and a linear isotopy \( g_t : (P, T) \to E^2 \ (t \in [0, 1]) \) with the properties that \( g_0 = \text{id} \), for each \( t \) in \([0, 1]\), \( g_t|\text{Bd} \ P = \text{id} \), and \( g_t|(1\text{-skeleton of } T') = f|(1\text{-skeleton of } T') \). The linear isotopy \( g_t \) is obtained as follows. Let \( \{\sigma_1, \sigma_2, \ldots, \sigma_n\} \) be a shelling of the triangulation \( T' \). Use Lemma 5.1 to push \( \text{Bd} \sigma_1 \) to \( f(\text{Bd} \sigma_1) \) leaving \( \text{Bd} \ P \) fixed and extend this push to a push of all of \( P \) using Theorem 3.3. Now \( \text{Bd} \sigma_2 \) is partly where it should be under \( f \) and partly subdivided and in the wrong place. Now push \( \text{Bd} \sigma_2 \) to...
where it should be under $f$ using Lemma 5.1 and Theorem 3.3 as above leaving $\text{Bd } P$ and all of the moved $\sigma_i$ fixed. Continue in this manner until the linear isotopy $g_i$ is produced.

Next note that for each 2-simplex $\sigma_i$ of $T'$, $g_i|\sigma_i$ and $f|\sigma_i$ are two linear homeomorphisms of $(\sigma_i, T|\sigma_i)$ such that $g_i|\text{Bd } \sigma_i = f|\text{Bd } \sigma_i$. By Corollary 4.4 there is a linear isotopy taking $g_i$ to $f$ which leaves the boundary of $\sigma_i$ fixed. Performing $g_i$ ($t \in [0, 1]$) followed by these linear isotopies on each $\sigma_i$ yields a desired linear isotopy of $P$ from $\text{id}$ to $f$.

**Question 5.1.** If $P$ is a polygonal disk in $E^2$, is there an $\epsilon > 0$ such that for any triangulation $T$ of $P$ of mesh less than $\epsilon$ and any linear homeomorphism $h$ of $(P, T)$ into $E^2$ fixed on $\text{Bd } P$, it is possible to push $(h(P), h(T))$ back to $(h, T)$ while leaving $\text{Bd } P$ fixed?

6. Linear degeneration. In §7 we improve Corollary 4.5 by restricting the linear isotopy in the conclusion to take place in a restricted part of the plane. (See Theorems 7.1, 7.2, and 7.3.) The proofs of those theorems make use of the concept of linear degeneration which is a method of transforming a triangulated complex $(C, T)$ similar to a collapse. However, we are interested in the linear structure of $C$, so each transformation will be linear with respect to the given triangulation $T$. Theorem 6.1 will be used in the next section.

A linear degeneration is shown in Illustration 6.1 and its formal definition is given below. It consists of moving the vertex $v$ to a vertex $w$ in its link so that at each moment before the end, the map is a linear homeomorphism whereas at the last moment $v$ was mapped to $w$ and the 2-simplexes $vxw$ and $vxw$ were mapped simplicially to $wx$ and $wy$ respectively. Note that if the vertices $u$ and $z$ did not exist, this movement would have resulted in a collapse. Moving $v$ to $y$ would not be a linear degeneration because the final map would not be a linear embedding of the simplex $vxw$.

**Definition.** Let $(C, T)$ be a triangulated complex linearly embedded in $E^2$ and let $vw$ be a 1-simplex of $C$. A continuous family of maps $f_t: C \rightarrow E^2$ ($t \in [0, 1]$) obtained by moving $v$ along $vw$ to $w$ and coning out to $Lk(v)$ while leaving $C - \text{St}(v)$ fixed is a simple linear degeneration provided

1. $f_t$ is monotone nonincreasing in the sense that if $s < t$, $f_t(C) \subseteq f_s(C)$,
2. for $t$ in $[0, 1), f_t$ is a linear embedding with respect to $T$, and
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(3) $f_1$ is a linear embedding of each 2-simplex of $T$ which does not have $vw$ as an edge.

Then $f_1(C)$ is regarded as a new complex linearly embedded in $E^2$ and having one fewer vertex than $C$. If a finite sequence of simple linear degenerations transforms a complex $C$ to a complex $C^1$, then $C$ linearly degenerates to $C^1$ and the composition of the simple linear degenerations is a linear degeneration.

The main tools used in this section are lemmas about the existence of low order vertices in triangulations of disks. Ho [5, Lemma 2.17] noted that the Euler characteristic formula $V - E + F = 2$ can be used to verify the following.

6-Lemma. Let $T$ be a triangulation of a disk $D$, $V_I$ be the number of interior vertices of $T$, $V_B$ be the number of boundary vertices, and for each vertex $v$, $o(v)$ be the order of $v$ (number of 1-simplexes ending at $v$). Then

$$6V_I - \sum_{v_i \in \text{Int } D} o(v_i) + 4V_B - \sum_{v_j \in \text{Bd } D} o(v_j) = 6.$$ 

The 6-Lemma implies the following.

5-3 Lemma. If $T$ is a triangulation of a disk, then either some interior vertex of $T$ has order less than or equal to 5 or there are three boundary vertices each of which has order less than or equal to 3.

We find the 5-3 Lemma useful when it is used in conjunction with the following fact about 3, 4, and 5 sided polygons.

Star-like Lemma. Let $J$ be a polygonal simple closed curve in $E^2$ with 3, 4, or 5 sides. Then $J$ bounds a star-like disk $D$ and has a vertex $v$ such that for each point $w$ of $J$ not on an edge of $J$ containing $v$, $vw$ spans $D$. Also, if a point $s$ of Int $D$ can see every point of $J$, then every point on the open segment $(sv)$ can see all of $J$.

We use the 5-3 Lemma and the Star-like Lemma to prove a theorem similar to a shelling theorem.

Theorem 6.1. Let $P$ be a polygonal disk in $E^2$ having a triangulation $T$. Then $(P, T)$ linearly degenerates to a 2-simplex in such a way that at each stage the image of $P$ is a disk. In fact, if $vw$ is a boundary 1-simplex of $T$, we can pick the degeneration so that $vw$ remains fixed during the degeneration and is an edge of the final 2-simplex.

Proof. The proof is by induction on the number $n$ of 2-simplexes in $T$. If $n = 1$, there is nothing to do. Assume the theorem for $k$ less than $n$ and let $T$ have $n$ 2-simplexes.
If $T$ has a spanning 1-simplex $st$, then $st$ divides $P$ into two disks $D_1$ and $D_2$ where $vw \subset D_2$. Induction implies that $D_1$ degenerates to a 2-simplex having $st$ as a side and a move of the opposite vertex of this 2-simplex produces a degeneration to $D_2$. Induction implies that $D_2$ in turn degenerates to a 2-simplex having $vw$ as an edge. Hence, we suppose that $T$ has no spanning 1-simplex. Hence no boundary vertex is of order 2 (unless $P$ is a 2-simplex).

Case 1. Suppose there is an interior vertex $s$ of order less than 6. The Star-like Lemma implies that there is a vertex $t$ in $\text{Lk}(s)$ so that $s$ can be moved to $t$ to produce a simple linear degeneration. The image of $\text{Bd} P$ is left fixed under this degeneration and induction implies that the linear degeneration of $P$ to a 2-simplex can be completed.

Case 2. Suppose there is no interior vertex of order less than 6. It follows from the 5-3 Lemma that there is a boundary vertex $s$ other than $v$ or $w$ which is of order 3. One can obtain a decreasing linear degeneration by moving $s$. Note that $vw$ remained fixed during this move.

Example 6.1. While we could strengthen Theorem 6.1 so as to leave any given 1-simplex of $T$ fixed or any 2-simplex that intersects $\text{Bd} P$ fixed, the example illustrated by Figure 6.1 shows that we could not hope to keep a given interior 2-simplex fixed. No linear degeneration of $P$ to a 2-simplex leaves the center 2-simplex fixed.

We next investigate the possibilities of degenerating a triangulated disk $P$ to complexes other than a 2-simplex. A necessary condition for linearly degenerating a complex $K$ to a complex $L$ is that there exist a linear map of $K$ onto $L$. In the next theorem we investigate the possibility of linearly degenerating a disk to an arc. We make use of the Shortest Arc Property defined below.

Definition. Let $P$ be a disk with triangulation $T$ and $A$ be an arc which is the union of 1-simplexes of $T$. Then $A$ has the Shortest Arc Property if and only if no edge path in $T$ that joins the ends of $A$ contains fewer 1-simplexes of $T$ than $A$ does.

Note that if there is a linear retraction from a triangulated disk $(P, T)$ to a boundary arc $A$, then $A$ must have the Shortest Arc Property.
Theorem 6.2. Suppose \((P, T)\) is a triangulated disk and \(A\) is an arc in the 1-skeleton of \(T\) with its ends on \(\text{Bd} \ P\). Then the following are equivalent:

1. The disk \((P, T)\) linearly degenerates to \(A\) leaving \(A\) fixed.
2. There is a linear retraction of \(T\) to \(A\).
3. The arc \(A\) has the Shortest Arc Property.

Proof. Our plan is to show that

\[
\text{LR} \Rightarrow \text{SAP} \Rightarrow \text{LD} \Rightarrow \text{LR}
\]

where LR means \(T\) linearly retracts to \(A\), SAP means that \(A\) has the Shortest Arc Property in \(P\), and LD means that \(T\) linearly degenerates to \(A\). Since the first and third of these implications are trivial, we concentrate on the second. This implication is demonstrated by using induction on the number \(n\) of 2-simplexes in \(T\). In the cases where \(T\) has only one or two 2-simplexes, SAP \(\Rightarrow\) LD. We suppose \(T\) has \(n\) 2-simplexes and that the implication follows for triangulations with fewer.

Our plan is to linearly degenerate the closure of each component of \(P - A\) onto the part of \(A\) on its boundary so we suppose with no loss of generality that \(A \subset \text{Bd} \ P\).

Case 1. Suppose there is an edge path \(E\) joining two vertices \(v_0\) and \(v_1\) of \(A\) such that \(E \subset \text{Bd} \ P\) and \(E\) has the same number of 1-simplexes as the number in \(A\) between \(v_0\) and \(v_1\). Then there is an edge path \(B\) between vertices \(v_0\) and \(v_1\) of \(A\) with the above properties and such that \(B \cap A = \{v_0, v_1\}\). Then \(B\) separates \(P\). Let \(C\) be the closure of a component of \(P - B\) missing the part of \(A\) between \(v_0\) and \(v_1\). We show in the next paragraph that \((B \cup A) \cap C\) has the Shortest Arc Property with respect to \(C\).

If \((B \cup A) \cap C\) did not have the Shortest Arc Property with respect to \(C\), there would be a too-short arc \(B'\) in \(C\) from \(w_0\) to \(w_1\) such that \(B' \cap [(B \cup A) \cap C] = \{w_0, w_1\}\). This is impossible because there is no place for \(w_0\) and \(w_1\) to lie. They cannot both be on \(A\) or else \(A\) does not have the Shortest Arc Property. They cannot be both on \(B\) or else we could replace a part of \(B\) by \(B'\) and get a shorter arc than \(B\) between \(v_0\) and \(v_1\). One cannot belong to \(A\) and the other to \(B\) for if \(w_0\) were in \(A\), \(w_1\) in \(B\), and \(v_0\) between \(w_0\) and \(w_1\) on \((B \cup A) \cap C\), the union of \(B'\) and the part of \(B\) from \(w_1\) to \(v_1\) would be shorter than the arc on \(A\) from \(w_0\) to \(v_1\). Hence there is no place for \(w_0\) and \(w_1\) and \((B \cup A) \cap C\) has the Shortest Arc Property in \(C\).

Since \(C\) has fewer than \(n\) 2-simplexes, it follows by induction that \(C\) linearly degenerates to \((B \cup A) \cap C\). Note that \((B \cap C)\) is a spanning arc of \(P\) and the closure of \(P - C\) is a disk \(D\) which has fewer than \(n\) 2-simplexes. Induction implies that there is a linear degeneration of \(D\) onto \(D \cap A\). The composition of these two linear degenerations is the required one for Case 1.

Case 2. Suppose there is no edge path \(E\) as supposed in Case 1. The 5-3
Lemma implies that there is a vertex \( v_0 \) of \( T \) such that \( v_0 \) is an interior vertex of order less than 6 or \( v_0 \) is a boundary point of order less than 4 which is not an end of \( A \).

We note that \( v_0 \) does not lie on \( A \). If it did, it could not be of order 2 since \( A \) has the Shortest Arc Property and it could not be of order 3 or else Case 1 would apply.

If \( v_0 \in \text{Bd} \, P - A \), the linear degeneration can be started by moving \( v_0 \). Induction can be used to show that it can be completed.

If \( v_0 \) is an interior vertex, it follows from the Star-like Lemma that we can start the linear degeneration by moving \( v_0 \) to a vertex \( w_0 \) of \( \text{Lk}(v_0) \). This degeneration reduced the number of 2-simplexes and induction implies that the linear degeneration can be completed if \( A \) has the Shortest Arc Property with respect to the new triangulation. However, it does have this property since if there were a too-short arc \( B' \) in the new triangulation, it would pass through \( w_0 \) and the inverse of \( B' \) would contain an arc \( B \) through \( v_0 \) showing that Case 1 applied.

One can easily construct an example showing the necessity in Theorem 6.2 of requiring that the ends of \( A \) be on \( \text{Bd} \, P \). See Figure 6.2.

7. Linear isotopies in open subsets of \( E^2 \). In this section we show that only a part of \( E^2 \) need be used in pushing a complex from one linear embedding to another. The following theorems are variations of Cairn's result listed above as Corollary 4.5.

**Theorem 7.1.** Let \( (P, T) \) be a triangulated disk linearly embedded in \( E^2 \). Then there is a linear isotopy \( h_t: P \to P \) \((t \in [0, 1])\) of \( (P, T) \) such that \( h_0 = \text{identity} \), \( h_1(P) \) is convex, and \( h_t(P) \) \((t \in [0, 1])\) is monotonically decreasing.

**Proof.** By Theorem 6.1, \( P \) can be linearly degenerated to a 2-simplex of \( T \). This degeneration can be approximated by a monotonically nonincreasing push of \( (P, T) \) which pushes \( P \) to a convex set. This fact can be proved by induction on \( n \) where the inductive statement is that if \( (P, T) \) is a triangulated disk in \( E^2 \) with no three collinear vertices, then an \( n \) step linear degeneration of \( P \) followed by a nonincreasing push taking \( P \) to a convex disk can be replaced by a nonincreasing push of \( P \) to a convex disk.
The push so obtained can be replaced by a monotonically decreasing linear isotopy to prove Theorem 7.1.

**Theorem 7.2.** Let \((P, T)\) be a triangulated disk, \(U\) be a connected open subset of \(E^2\), and \(f_0\) and \(f_1\) be linear embeddings of \(P\) in \(U\) such that there is an orientation preserving homeomorphism \(G\) of \(E^2\) so that \(G \circ f_0 = f_1\). Then there is a push \(h_t : P \to U\ (t \in [0, 1])\) of \((P, T)\) into \(U\) so that \(h_0 = f_0\) and \(h_1 = f_1\).

**Proof.** Using Theorem 7.1, \(f_0(P)\) can be linearly isotoped to a convex set which can in turn be linearly isotoped to a small convex set. Likewise \(f_1(P)\) can be linearly isotoped to a small set. These linear isotopies can be replaced by pushes. We can then push the shrunk \(f_0(P)\) into a convex open set which also contains the shrunk \(f_1(P)\). Now Corollary 4.4 can be applied to finish the proof as was done in Corollary 4.5.

**Theorem 7.3.** Let \((C, T)\) be a triangulated complex, \(D\) be an open disk in \(E^2\), and \(f_0\) and \(f_1\) be two linear embeddings of \(C\) into \(D\) such that there is an orientation preserving homeomorphism \(G\) of \(E^2\) so that \(G \circ f_0 = f_1\). Then there is a push \(h_t : C \to D\ (t \in [0, 1])\) of \((C, T)\) into \(D\) so that \(h_0 = f_0\) and \(h_1 = f_1\).

**Proof.** Incorporate \(f_0(C)\) and \(f_1(C)\) into equivalently triangulated disks contained in \(D\) and apply Theorem 7.2.

**Example 7.1.** It should be noted that the hypothesis that \(D\) be a disk in Theorem 7.3 is necessary as shown by Figure 7.1.

![Figure 7.1](image-url)

**Question 7.1.** In §4 and this section, Cairns' Theorems were modified by keeping part of a complex fixed during a push and by restricting the part of the plane in which the push takes place. Another possible avenue of inquiry is suggested in the following question: Let \(\{A_i\}_{i=1}^n\) be a collection of arcs. Let \(f_0, f_1: \bigcup A_i \to E^2\) be maps of the disjoint union of the \(A_i\)'s into \(E^2\) so that \(f_0, f_1|A_i\) is linear for each \(i\) and so that there is an orientation preserving homeomorphism \(G\) of \(E^2\) so that \(G \circ f_0 = f_1\). Then is there an isotopy \(h_t : E^2 \to E^2\ (t \in [0, 1])\) so that \(h_0 = id\), for each \(i\), \(h_t(f_0(A_i)) = f_1(A_i)\), and for each \(t \in [0, 1]\) and each \(i\), \(h_t|f_0(A_i)\) is a linear homeomorphism?

**References**


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