FILTRATIONS AND CANONICAL COORDINATES ON NILPOTENT LIE GROUPS

BY

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ABSTRACT. Let \( g \) be a finite-dimensional nilpotent Lie algebra over a field of characteristic zero. Introducing the notion of a positive, decreasing filtration \( \mathcal{F} \) on \( g \), the paper studies the multiplicative structure of the universal enveloping algebra \( U(g) \), and also transformation laws between \( \mathcal{F} \)-canonical coordinates of the first and second kind associated with the Campbell-Hausdorff group structure on \( g \). The basic technique is to exploit the duality between \( U(g) \) and \( S(g^*) \), the symmetric algebra of \( g^* \), making use of the filtration \( \mathcal{F} \). When the field is the complex numbers, the preceding results, together with the Cauchy estimates, are used to obtain estimates for the structure constants for \( U(g) \). These estimates are applied to construct a family of completions \( U(g)_{\mathcal{F}} \) of \( U(g) \), on which the corresponding simply-connected Lie group \( G \) acts by an extension of the adjoint representation.

Introduction. Let \( G \) be a connected and simply-connected Lie group with Lie algebra \( g \), and let \( \{X_i: 1 \leq i \leq d\} \) be a basis for \( g \). Then there are “canonical coordinates” \( \{\xi_i\} \) and \( \{\eta_i\} \) of the “first kind” and “second kind”, respectively, on \( G \) defined by this basis [2, III.4.3]. These functions, which are defined on a neighborhood of the identity, are related by

\[
\exp(\xi_1 X_1 + \cdots + \xi_d X_d) = \exp \eta_1 X_1 \cdots \exp \eta_d X_d,
\]

where \( \exp: g \to G \) is the exponential map. If \( U(g) \) is the universal enveloping algebra of \( g \), then the given basis for \( g \) also defines bases \( \{X(\alpha): \alpha \in \mathbb{N}^d\} \) and \( \{X^\alpha: \alpha \in \mathbb{N}^d\} \) for \( U(g) \) (where \( X^\alpha = X_1^{\alpha_1} \cdots X_d^{\alpha_d} \) and \( X(\alpha) \) is the symmetrization of \( X^\alpha \)). The goal of this paper is to study the multiplicative structure of \( U(g) \), as expressed in terms of these bases, by using the duality between \( U(g) \) and polynomials in the canonical coordinates.

We shall restrict attention to nilpotent Lie algebras \( g \), and carry out all constructions relative to a given positive, decreasing filtration \( \mathcal{F} \) on \( g \) (e.g. \( \mathcal{F} = \) descending central series). Following G. Birkhoff [1], we extend \( \mathcal{F} \) to a decreasing filtration of \( U(g) \) by ideals of finite codimension in \$1$. This gives
“two-sided” vanishing conditions on the structure constants for the multiplication on \( U(\mathfrak{g}) \), and allows us to define a “formal noncommutative power series” completion \([U(\mathfrak{g})]_{\mathbb{F}}\) of \( U(\mathfrak{g}) \).

In §2 we pass by duality to an increasing filtration on the commutative algebra \( \mathbb{P} \) of polynomial functions on \( \mathfrak{g} \). The multiplication on \( U(\mathfrak{g}) \) dualizes to a comultiplication on \( \mathbb{P} \) which preserves the filtration.

It is well known that if \( \{X_i\} \) is a Jordan-Hölder basis (i.e. the subspaces \( \mathfrak{h}_i = \text{span}\{X_k: k > i\} \) are ideals in \( \mathfrak{g} \)), then \( \{\xi_i\} \) and \( \{\eta_i\} \), defined by (0.1), give global coordinates for \( G \) and are related by a polynomial transformation \( \phi \). If we make the stronger requirement that \( \{X_i\} \) be an \( \mathbb{F} \)-basis (definition in §1), then we can say more about \( \phi \): In §3 we construct a simply-connected nilpotent Lie group \( M = M(\mathfrak{g}) \), faithfully represented as a group of locally unipotent automorphisms of \( \mathbb{P} \), and in §4 we prove that \( \phi \in M \). By duality, we are then able to study the transformation from the basis \( \{X^{\alpha}\} \) to the basis \( \{X(\alpha)\} \) of \( U(\mathfrak{g}) \).

The original motivation for this paper was the construction of algebras of “differential operators of infinite order” on \( G \), obtained by completing \( U(\mathfrak{g}) \) relative to suitable locally convex topologies (cf. [5]). We present a class of such algebras in §6, using estimates for the structure constants for \( U(\mathfrak{g}) \) obtained in §5. The construction of these algebras is considerably simpler and more general than in [5]. In future work we plan to relate the representation theory of these algebras to that of the group \( G \) and the work of Treves and collaborators on “hyperdifferential operators” (cf. [11]).

This paper continues the study of nilpotent Lie algebras and groups via filtrations started in [6] (cf. [7, Chapter I and Appendix]).

Since we are considering only nilpotent Lie algebras, the restriction to real algebras is unnecessary. In the purely algebraic part of the paper (§§1–4), the coefficient field is any field \( F \) of characteristic zero. In §§5–6, we assume for the purpose of making estimates that \( F = \mathbb{C} \) (i.e. in the original context of a real Lie group \( G \), we pass to the complexified universal enveloping algebra).

We use the customary notations of \( N \) for the nonnegative integers, \( Q \) for the rational numbers, and \( C \) for the complex numbers.

1. Filtrations on \( \mathfrak{g} \) and \( U(\mathfrak{g}) \). Let \( F \) be a field of characteristic zero, and let \( \mathfrak{g} \) be a finite-dimensional Lie algebra over \( F \). Denote by \( U(\mathfrak{g}) \) the universal enveloping algebra of \( \mathfrak{g} \).

**Definition.** A positive filtration \( \mathcal{F} \) on \( \mathfrak{g} \) is a decreasing chain of subspaces \( \{\mathfrak{g}_n\}_{n \geq 1} \), such that

\[
\mathfrak{g} = \mathfrak{g}_1 \supseteq \mathfrak{g}_2 \supseteq \cdots \supseteq \mathfrak{g}_l \supseteq \mathfrak{g}_{l+1} = 0, \quad [\mathfrak{g}_m, \mathfrak{g}_n] \subseteq \mathfrak{g}_{m+n}.
\]

The length of \( \mathcal{F} \) is the smallest integer \( l \) such that \( \mathfrak{g}_{l+1} = 0 \).

Note that there exists a positive filtration on \( \mathfrak{g} \) if and only if \( \mathfrak{g} \) is nilpotent,
with the shortest filtration being the descending central series \((g_{n+1} = [g_n, g_n])\).

Fix a positive filtration \(\mathcal{F}\), and for \(X \in g, X \neq 0\), define the \(\mathcal{F}\)-weight by
\[
\text{w}(X) = \max\{n : X \in g_n\}.
\]

We shall say that a basis \(\{X_i : 1 < i < d\}\) for \(g\) is an \(\mathcal{F}\)-basis if \(\text{w}(X_i) < \text{w}(X_{i+1})\) and
\[
g_n = \text{span}\{X_i : \text{w}(X_i) > n\},
\]
for \(n = 1, 2, \ldots, l\). The numbers \(w_i = \text{w}(X_i)\) are then independent of the choice of \(\mathcal{F}\)-basis. If \(\alpha \in \mathbb{N}^d (d = \text{dim } g)\), set
\[
|\alpha| = \sum \alpha_i, \quad w(\alpha) = \sum w_i \alpha_i.
\]

Using the basis \(\{X_i\}\) for \(g\), we can construct two bases for \(U(g)\). Namely, given \(\alpha \in \mathbb{N}^d\), set \(n = |\alpha|\) and define
\[
X^\alpha = X_1^{\alpha_1} \cdots X_d^{\alpha_d}, \quad (\alpha \in \mathbb{N}^d) \quad \text{and} \quad X(\alpha) = (n!)^{-1}(\partial/\partial t_i)^{\alpha_i}(t_1 X_1 + \cdots + t_d X_d)\text{[n]},
\]
where the products are in \(U(g)\) and
\[
(\partial/\partial t_i)^{\alpha_i} = (\partial/\partial t_1)^{\alpha_1} \cdots (\partial/\partial t_d)^{\alpha_d}.
\]

By the Poincaré-Birkhoff-Witt (PBW) theorem, \(\{X(\alpha) : \alpha \in \mathbb{N}^d\}\) and \(\{X^\alpha : \alpha \in \mathbb{N}^d\}\) are bases for \(U(g)\), which we shall call bases of the first and second kind, respectively (cf. [3, §2.1]). Note that \(X(\alpha)\) is the “symmetrization” of \(X^\alpha\).

The multiplicative structure of \(U(g)\) can then be expressed in terms of these bases by the equations
\[
(1.1) \quad (\alpha! \beta!)^{-1} X(\alpha)X(\beta) = \sum (\gamma!)^{-1} C^\gamma_{\alpha \beta} X(\gamma),
\]
\[
(1.2) \quad (\alpha! \beta!)^{-1} X^\alpha X^\beta = \sum (\gamma!)^{-1} K^\gamma_{\alpha \beta} X^\gamma.
\]

We shall call \(\{C^\gamma_{\alpha \beta}\}\) and \(\{K^\gamma_{\alpha \beta}\}\) the structure constants (of the first and second kind, respectively) for \(U(g)\).

**Proposition 1.1.** If \(\{X_i\}\) is an \(\mathcal{F}\)-basis, then the structure constants satisfy
\[
(1.3) \quad C^\gamma_{\alpha \beta} = K^\gamma_{\alpha \beta} = 0
\]
if either \(|\gamma| > |\alpha| + |\beta|\) or \(\text{w}(\gamma) < \text{w}(\alpha) + \text{w}(\beta)\).

**Remark.** If the map \(X_i \mapsto w_i X_i\) extends to a derivation of \(g\), then there is an associated action of the multiplicative group \(F^\times\) on \(U(g)\), defined by \(t \cdot X^\alpha = t^{w(\alpha)} X^\alpha\). It follows that in this case (1.3) holds whenever \(\text{w}(\gamma) \neq \text{w}(\alpha) + \text{w}(\beta)\). In general, however, there will not exist an \(\mathcal{F}\)-basis with this property, since \(g\) does not always have a one-parameter group of dilating automorphisms (cf. [4] and [7, §1.3.2]).

**Proof.** The vanishing conditions (1.3) follow from the existence of two filtrations on \(U(g)\). Namely, let
be the canonical increasing filtration of $U(g)$. The property $U_m U_n \subseteq U_{m+n}$ then implies (1.3) when $|\alpha| + |\beta| < |\gamma|$.

In the other direction, if we define (following [1])

$$J_n = \text{span}\{X^\alpha: w(\alpha) > n\},$$

then the PBW theorem and the filtration condition $[g_m, g_n] \subseteq g_{m+n}$ imply that

$$J_n = \sum g_{n_1} \cdots g_{n_k} \quad (n_1 + \cdots + n_k > n),$$

(of course, (1.4) is not a direct sum). This makes it evident that

$$J_m J_n \subseteq J_{m+n}.$$  

From the definition, we see that

$$U(g) = J_0 \supseteq J_1 \supseteq \cdots.$$  

Obviously we also have

$$\bigcap_n J_n = 0.$$  

Thus $\{J_n\}$ is a decreasing, separated filtration on $U(g)$. By (1.4) it is independent of the choice of $F$-basis, and

$$J_n = \text{span}\{X(\alpha): w(\alpha) > n\}.$$  

The PBW theorem and the filtration property (1.5) gives (1.3) when $w(\alpha) + w(\beta) > w(\gamma)$. This proves the proposition.

Using the filtration $\{J_n\}$ introduced above, we define a topology on $U(g)$ by letting the sets $\{a + J_n\}$ be a basis for the neighborhoods of $a \in U(g)$. The filtration condition (1.5) then shows that multiplication is jointly continuous in this topology, so that $U(g)$ becomes a topological algebra. Denote by $[U(g)]_\pi$ the completion of $U(g)$ in this uniformity (cf. [9, Chapter X]). The algebraic operations on $U(g)$ extend by continuity to $[U(g)]_\pi$. An easy argument using the PBW theorem establishes the following explicit realization of $[U(g)]_\pi$:

**PROPOSITION 1.2.** The algebra $[U(g)]_\pi$ is naturally isomorphic to the algebra of all formal series $\sum a_\alpha X^\alpha$, where $\{X_\alpha\}$ is an $F$-basis for $g$ and the multiplication is defined using equations (1.2).

**REMARKS.** (1) The product of the series $\sum (\alpha!)^{-1} a_\alpha X^\alpha$ and $\sum (\beta!)^{-1} b_\beta X^\beta$ is the series $\sum (\gamma!)^{-1} c_\gamma X^\gamma$, where

$$c_\gamma = \sum a_\alpha b_\beta K^\gamma_{\alpha \beta}.$$  

Note that by (1.3) the range of summation on the right side of (1.8) is finite.
(2) The analogue of Proposition 1.2, using the symmetrized basis \( \{ X(\alpha) \} \) instead of \( \{ X^\alpha \} \), also holds.

(3) The referee has pointed out that the uniform structure on \( U(\mathfrak{g}) \) introduced above is independent of the choice of positive filtration on \( \mathfrak{g} \). Indeed, if \( \mathcal{F}' \) is another positive filtration of \( \mathfrak{g} \), and \( \{ J_n' \} \) the corresponding filtration of \( U(\mathfrak{g}) \), then it is easy to verify that \( J_n \subseteq J_{n/|l|}' \) where \( l \) is the length of \( \mathcal{F} \). Hence by symmetry, we conclude that \( \{ J_n \} \) and \( \{ J_n' \} \) define the same set of neighborhoods of 0, and thus the same uniform structure, as asserted.

2. Filtrations and comultiplication on \( S(\mathfrak{g}^*) \). Let \( \mathfrak{g}^* \) be the vector space dual to \( \mathfrak{g} \), and \( S(\mathfrak{g}^*) \) the symmetric tensor algebra over \( \mathfrak{g}^* \). We shall identify \( S(\mathfrak{g}^*) \) with the algebra \( \mathcal{P} \) of polynomial functions on \( \mathfrak{g} \), as usual. If \( X \in \mathfrak{g} \), let \( \partial_X \) be the unique derivation of \( \mathcal{P} \) such that

\[
\partial_X (\xi) = \langle \xi, X \rangle, \quad \xi \in \mathfrak{g}^*.
\]

There is a unique bilinear pairing between \( \mathcal{P} \) and \( U(\mathfrak{g}) \) such that \( \langle f, 1 \rangle = f(0) \) and

\[
\langle f, X^n \rangle = (\partial_X)^n f(0),
\]

for \( f \in \mathcal{P} \) and \( X \in \mathfrak{g} \). Given \( \{ X_i \} \), a basis for \( \mathfrak{g} \), we set

\[
\partial_i = \partial_{X_i}, \quad \partial^\alpha = \partial_{i_1} \cdots \partial_{i_\alpha}.
\]

Then this pairing is given by

\[
\langle f, X(\alpha) \rangle = \partial^{\alpha} f(0),
\]

in terms of the corresponding basis of the "first kind" for \( U(\mathfrak{g}) \). In particular, formula (2.2) and the PBW theorem imply that this pairing is nonsingular.

Suppose now that \( \mathcal{F} \) is a positive filtration on \( \mathfrak{g} \), and let \( \{ J_n \} \) be the induced filtration on \( U(\mathfrak{g}) \), as in §1. We define

\[
\mathcal{P}_n = J_{n+1}^\perp = \{ f \in \mathcal{P} : \langle f, J_{n+1} \rangle = 0 \}.
\]

Let \( \{ X_i \} \) be an \( \mathcal{F} \)-basis for \( \mathfrak{g} \), and let \( \{ \xi_i \} \) be the dual basis for \( \mathfrak{g}^* \). Then from (2.2) we see that

\[
\mathcal{P}_n = \{ f \in \mathcal{P} : \partial^{\alpha} f(0) = 0 \text{ if } w(\alpha) > n \}
\]

(2.3)

\[
= \text{span}\{ \xi^\alpha : w(\alpha) < n \},
\]

where \( \xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_\alpha^{\alpha_\alpha} \). This makes it evident that \( \{ \mathcal{P}_n \} \) is an increasing filtration of the algebra \( \mathcal{P} \). It is clear from the definition of \( \mathcal{P}_n \) that the pairing between \( U(\mathfrak{g}) \) and \( \mathcal{P} \) extends by continuity to a pairing between \( [U(\mathfrak{g})]_{\mathfrak{g}} \) and \( \mathcal{P} \).

Let \( (U \otimes U)^* \) be the vector space dual to \( U(\mathfrak{g}) \otimes U(\mathfrak{g}) \). Via the pairing (2.1), we may consider \( \mathcal{P} \otimes \mathcal{P} \) as a subspace of \( (U \otimes U)^* \). We define a filtration on \( \mathcal{P} \otimes \mathcal{P} \) by setting
By duality, the multiplication on \(U(\frak{g})\) defines a linear map
\[
\mu: \frak{g} \to (U \otimes U)^*,
\]
such that
\[
\langle \mu(f), S \otimes T \rangle = \langle f, ST \rangle
\]
for \(f \in \frak{g}\) and \(S, T \in U(\frak{g})\).

**Proposition 2.1.** The map \(\mu\) is an algebra homomorphism from \(\frak{g}\) to \(\frak{g} \otimes \frak{g}\) and carries \(\frak{g}^n\) into \((\frak{g} \otimes \frak{g})^n\).

**Proof.** Let \(\{X_i\}\) be an \(\frak{g}\)-basis for \(\frak{g}\), and let the structure constants \(C_{\gamma \beta}^\alpha\) be defined by (1.1). It then follows from (2.2) that
\[
\mu(\xi^\gamma) = \sum_{\alpha, \beta} C_{\gamma \beta}^\alpha \xi^\alpha \otimes \xi^\beta.
\]
In view of (2.3) and Proposition 1.1, we see that the right side of (2.5) is in \((\frak{g} \otimes \frak{g})_{n(\frak{g})}\), so that \(\mu\) maps \(\frak{g}\) into \(\frak{g} \otimes \frak{g}\) and preserves the filtrations.

It remains to show that \(\mu(fg) = \mu(f) \mu(g)\). For this, we first observe that if \(f \in \frak{g}\) and \(X \in \frak{g}\), then by Taylor's formula,
\[
f(X) = \langle f, e^X \rangle,
\]
where \(e^X\) is defined by its power series as an element of \([U(\frak{g})]_{\frak{g}}\). Identifying \(\frak{g} \otimes \frak{g}\) with the polynomial functions on \(\frak{g} \times \frak{g}\) as usual, we can evaluate \(F \in \frak{g} \otimes \frak{g}\) in terms of the pairing by
\[
F(X, Y) = \langle F, e^X \otimes e^Y \rangle.
\]
When \(F = \mu(f)\), this gives the formula
\[
\mu(f)(X, Y) = \langle f, e^X e^Y \rangle,
\]
for \(f \in \frak{g}\) and \(X, Y \in \frak{g}\).

On the other hand, since \(\frak{g}\) is nilpotent, the Campbell-Hausdorff-Dynkin formula [9, Chapter X] shows that the formal series \(e^X e^Y\) can be rearranged to be expressed as \(e^Z\), where
\[
Z = X + Y + \frac{1}{2} [X, Y] + \cdots
\]
is a Lie polynomial in \(X, Y\) with coefficients in \(Q\). Writing \(Z = X \ast Y\), we can thus evaluate
\[
\mu(f)(X, Y) = f(X \ast Y),
\]
which shows that \(\mu\) is multiplicative. Q.E.D.

We shall call \(\mu\) the *comultiplication* on \(\frak{g}\). Using the multiplicative property of \(\mu\) and equation (2.5), one obtains the following formula expressing the structure constants of the first kind in terms of canonical coordinates of the
Corollary 2.2. Let the constants $C_{\alpha}^{\beta}$ be defined by (1.1) relative to an $F$-basis $\{X_i\}$ for $\mathfrak{g}$, let $\{\xi_i\}$ be the dual basis for $\mathfrak{g}^*$, and define $F_i = \mu(\xi_i)$. Then

$$C_{\alpha}^{\beta} = (\alpha! \beta!)^{-1} \partial_{\alpha}(x) \partial_{\beta}(y) F^\gamma(x, y) |_{x=y=0},$$

where $F^\gamma = F_1 \cdots F_d^\gamma$.

3. Automorphisms of $S(\mathfrak{g}^*)$. In order to pass from the basis $\{X(\alpha)\}$ to the basis $\{X^\alpha\}$ for $U(\mathfrak{g})$, it will be useful to introduce a group of locally unipotent automorphisms of the algebra $S(\mathfrak{g}^*) = \mathfrak{P}$. (Recall that a linear transformation $\phi$ on a vector space $V$ is called locally unipotent if $(\phi - 1)^n v = 0$ for all $v \in V$ and some integer $n$, which may depend on $v$.)

We first define a refinement of the filtration $\{\mathfrak{P}^n\}$ on $\mathfrak{P}$ associated with the given filtration $\mathfrak{F}$ on $\mathfrak{g}$. Set $\mathfrak{P}_0 = 0$ and

$$\mathfrak{P}_n = \mathfrak{P}_{n-1} + \sum_{0 < k < n} \mathfrak{P}_k \mathfrak{P}_{n-k}. \tag{3.1}$$

It is then immediate that

$$\mathfrak{P}_{n-1} \subseteq \mathfrak{P}_n \subseteq \mathfrak{P}_n, \tag{3.2}$$

$$\mathfrak{P}_m \mathfrak{P}_n \subseteq \mathfrak{P}_{m+n}, \tag{3.3}$$

for $m, n > 0$.

Considered as functions on the vector space $\mathfrak{g}$, the elements of $\mathfrak{P}_n$ are the nonlinear polynomials of filtration weight $< n$ (mod $\mathfrak{P}_{n-1}$). Indeed, from (2.3) we see that, modulo $\mathfrak{P}_{n-1}$, $\mathfrak{P}_n$ is spanned by monomials $\xi^\alpha$ such that $w(\alpha) = n$ and $|\alpha| > 2$. If the filtration $\mathfrak{F}$ is of length $l$, this makes it evident that

$$\mathfrak{P}_n = \mathfrak{P}_n \quad \text{when } n > l. \tag{3.4}$$

Now let $\text{Der}(\mathfrak{P})$ and $\text{Aut}(\mathfrak{P})$ denote the derivations and automorphisms, respectively, of the algebra $\mathfrak{P}$. We define

$$m = \{ T \in \text{Der}(\mathfrak{P}) : T \mathfrak{P}_n \subseteq \mathfrak{P}_n, \forall n > 0 \},$$

$$M = \{ \phi \in \text{Aut}(\mathfrak{P}) : (\phi - 1) \mathfrak{P}_n \subseteq \mathfrak{P}_n, \forall n > 0 \}. \tag{3.5}$$

It is easily verified from (3.1)--(3.3) that $m$ is a Lie subalgebra of $\text{Der}(\mathfrak{P})$, such that $m(\mathfrak{P}_n) \subseteq \mathfrak{P}_n$ and

$$m^{n+1}(\mathfrak{P}_n) = 0. \tag{3.5}$$

(Here $m^n$ denotes the linear span of all products of $n$ elements of $m$, considered as linear transformations on $\mathfrak{P}$.) It follows from (3.5) that $T \in m$ generates an automorphism $\exp(T)$ of $\mathfrak{P}$, defined by
\[ \exp(T)f = \sum (n!)^{-1} T^n f, \]
for \( f \in \mathfrak{g} \). Obviously \( \exp(\mathfrak{g}) \subseteq M \).

If \( \phi, \psi \in M \), then \((\phi - 1)\mathfrak{g} \subseteq (\phi - 1)\mathfrak{g} + (\psi - 1)\mathfrak{g} \subseteq \mathfrak{g} \). Also \( \phi \mathfrak{g} = \mathfrak{g} \), since \( \mathfrak{g} \) is finite-dimensional, and hence
\[ (\phi^{-1} - 1)\mathfrak{g} = (\phi^{-1} - 1)\phi \mathfrak{g} = (1 - \phi)\mathfrak{g} \subseteq \mathfrak{g}. \]
Thus \( M \) is a subgroup of \( \text{Aut} \mathfrak{g} \).

**Theorem 3.1** (a) \( \mathfrak{g} \) is a finite-dimensional nilpotent Lie algebra, and the map \( T \mapsto \exp(T) \) is a bijection from \( \mathfrak{g} \) onto \( M \).

(b) If \( \{ \xi_i \} \) is a basis for \( \mathfrak{g}^* \) dual to an \( \mathbb{F} \)-basis \( \{ X_i \} \), then \( M \) consists of all automorphisms of \( \mathfrak{g} \) whose action on \( \{ \xi_i \} \) is of the form
\[ (3.6) \quad \xi_i \mapsto \xi_i + q_i, \]
with \( q_i \in \mathfrak{g} \) (\( w_i = \mathbb{F} \)-weight of \( X_i \)).

**Proof.** (a) To prove that \( \exp \) is surjective, let \( \phi \in M \) and set \( S = \phi - 1 \).

Then
\[ S(\mathfrak{g}) \subseteq \mathfrak{g}, \quad S(\mathfrak{g}) \subseteq \mathfrak{g}, \quad S(\mathfrak{g}) = 0. \]
We claim that if \( 2^m > n \), then
\[ (3.7) \quad S^{m+1}(\mathfrak{g}) \subseteq \mathfrak{g}_{n-1}. \]
Indeed, this holds for \( n = 1 \), since \( \mathfrak{g}_1 = \mathfrak{g}_0 \). By the definition of \( \mathfrak{g}_n \), we see that
\[ S^m(\mathfrak{g}_n) \subseteq \sum \mathfrak{g}_{k_1} \cdots \mathfrak{g}_{k_r} + \mathfrak{g}_{n-1}, \]
where \( r = 2^m \) and the summation is over all \( k_i > 0 \) such that \( k_1 + \cdots + k_r = n \). But if \( r > n \), this range of summation is empty, which gives (3.7).

Given \( S = \phi - 1 \) as above, we define a linear transformation \( T \) on \( \mathfrak{g} \) by
\[ T^*f = \log(\phi)f = \sum \frac{(-1)^{n+1}}{n} S^nf. \]

The range of summation is finite, by (3.7), and \( T(\mathfrak{g}_n) \subseteq \mathfrak{g}_n, T(\mathfrak{g}_0) = 0 \). By the argument just given, this implies that \( T \) is locally nilpotent on \( \mathfrak{g} \). Hence \( \exp \) \( T \) is defined as a linear transformation on \( \mathfrak{g} \), and by the formal power series identity \( x = \exp(\log x) \) we have \( \exp T = \phi \). It remains to show that \( T \) is a derivation of \( \mathfrak{g} \). For this it suffices to prove that \( \exp(tT) \in \text{Aut}(\mathfrak{g}) \) for all \( t \in F \). But for fixed \( f, g \in \mathfrak{g} \), the function
\[ t \mapsto e^{tT}(fg) - (e^{tT}f)(e^{tT}g) \]
is a polynomial. Since \( \exp(nT) = \phi^n \), this polynomial vanishes when \( t \in \mathbb{N} \), and hence vanishes identically.

Since \( \mathfrak{g}^* \subseteq \mathfrak{g} \) (\( L = \text{length of } \mathfrak{g} \)), and a derivation of \( \mathfrak{g} \) is determined by its
action on the linear functions, it follows that the representation of \( m \) on the finite-dimensional subspace \( \mathfrak{g}_t \) is faithful and nilpotent. Hence \( m \) is a finite-dimensional nilpotent Lie algebra. To prove that the map \( T \rightarrow \exp T \) is injective, it thus suffices, by the Campbell-Hausdorff formula, to show that \( \exp T = 1 \) implies \( T = 0 \).

If \( T \in m \) and \( f \in \mathfrak{g} \), then the function \( t \rightarrow \exp(tT)f - f \) is a polynomial in \( t \in \mathbb{F} \). If \( \exp T = 1 \), this polynomial vanishes when \( t \in \mathbb{N} \), and hence is identically zero. Differentiating at \( t = 0 \), we conclude that \( T = 0 \), and hence \( T = 0 \).

(b) By definition, the transformations in \( M \) act on \( \{ \xi_i \} \) as in formula (3.6). Conversely, given \( q_i \in \mathfrak{g}_m \), there is a unique homomorphism \( \phi: \mathfrak{g} \rightarrow \mathfrak{g} \) such that \( \phi(\xi_j) = \xi_j + q_j \). Since \( q_i \) is a polynomial function of \( \{ \xi_j: j < i \} \), it is evident that \( \phi \) is an automorphism of \( \mathfrak{g} \). Obviously \( \phi(\mathfrak{g}_n) = \mathfrak{g}_n \) and hence \( \phi(\mathfrak{g}_n) \subseteq \mathfrak{g}_n \). For \( 1 < n < l \), the linear functions \( \{ \xi_i: w_i = n \} \) are complementary basis to \( \mathfrak{g}_n \) in \( \mathfrak{g}_n \). By definition of \( \phi \), this implies that \( (\phi - 1)\mathfrak{g}_n \subseteq \mathfrak{g}_n \) when \( 1 < n < l \). For \( n > l \), this also holds, by (3.4), and hence \( \phi \in M \).

Q.E.D.

Remark. The constructions of this section are independent of the Lie algebra structure on \( \mathfrak{g} \); only the vector space structure and the filtration \( \mathcal{F} \) are involved. We shall write \( M = M(\mathcal{F}) \) when the dependence on \( \mathcal{F} \) is to be emphasized. (For a more detailed study of \( M \) and related subgroups of \( \text{Aut}(\mathfrak{g}) \), cf. [7, Appendix].)

4. Canonical coordinate transformations. Let \( \{ X_i \} \) be an \( \mathcal{F} \)-basis for \( \mathfrak{g} \). We define a homomorphism \( \phi: \mathfrak{g} \rightarrow \mathfrak{g} \) by setting

\[
(4.1) \quad \phi(f)(x) = f(p_1(x) \ast \cdots \ast p_d(x)),
\]

where \( p_i(x) = \langle \xi_i, x \rangle X_i \) is the projection onto \( F X_i \) and \( x \ast y \) is the Campbell-Hausdorff multiplication on \( \mathfrak{g} \), as in §2.

Theorem 4.1. Let \( M = M(\mathcal{F}) \) be the subgroup of \( \text{Aut}(\mathfrak{g}) \) defined in §3. Then for any \( \mathcal{F} \)-basis \( \{ X_i \} \), the transformation \( \phi \) defined by (4.1) is in \( M \).

Remarks. 1. The linear functions \( \{ \xi_i \} \) give “canonical coordinates of the first kind” for the Campbell-Hausdorff group structure on \( \mathfrak{g} \), while functions \( \eta_i = \phi^{-1}(\xi_i) \) are “canonical coordinates of the second kind,” relative to the basis \( \{ X_i \} \) (cf. [2, III.4.3]). The point of the theorem is that when \( \{ X_i \} \) is an \( \mathcal{F} \)-basis, then \( \{ \eta_i \} \) also generate \( \mathfrak{g} \), and \( \eta_i = \xi_i + q_i \), where \( q_i \in \mathfrak{g}_m \).

2. If \( \{ X_i \} \) is an arbitrary basis then \( \phi \) is not necessarily an automorphism of \( \mathfrak{g} \). For example, let \( \mathfrak{g} \) be the Heisenberg algebra with basis \( e_1, e_2, e_3 \) and commutation relations \( [e_1, e_2] = e_3, e_3 \) central. Set \( X_1 = e_1, X_2 = e_2, X_3 = e_1 + e_3 \), and let \( \{ \xi_i \} \) be the basis dual to \( \{ X_i \} \). One finds that the homomorphism \( \phi \) defined by (4.1) acts on the generators \( \{ \xi_i \} \) by
\( \phi(\xi_1) = \xi_1 - \frac{1}{2} \xi_2 (\xi_1 - \xi_3), \phi(\xi_2) = \xi_2, \) and \( \phi(\xi_3) = \xi_3 + \frac{1}{2} \xi_2 (\xi_1 - \xi_3). \) Thus \( \phi(\xi_1 - \xi_3) = (\xi_1 - \xi_3)(1 - \xi_2) \) and \( \phi(1 - \xi_2) = 1 - \xi_2. \) If \( \phi \) were an automorphism of \( F[\xi_1, \xi_2, \xi_3], \) this would give the relation
\[
(4.2) \quad \xi_1 - \xi_3 = h (1 - \xi_2),
\]
where \( h = \phi^{-1}(\xi_1 - \xi_3). \) But no polynomial \( h \) can satisfy (4.2), so \( \phi \notin \text{Aut}(\mathcal{P}) \) in this case.

**Proof of Theorem.** By the Campbell-Hausdorff formula, we can write
\[
p_1(x) * \cdots * p_d(x) = x + C(x),
\]
where \( C(x) \) is a linear combination of iterated commutators of \( \{p_i(x)\}. \) Since \( \{X_i\} \) is a \( \mathcal{P} \)-basis, any such iterated commutator is of the form \( \xi^a(x) z_a, \) where \( z_a \in g_{\omega(a)} \) and at least two of the \( \alpha_i \) are nonzero. Hence \( \xi^a \in g_{\omega(a)}. \)

Consider now the action of \( \phi \) on the linear functions \( \{\xi_i\}. \) Since \( \xi_i(\mu_\alpha) = 0 \) for \( n > w_i, \) the observations of the previous paragraph imply that \( \phi(\xi_i) = \xi_i + q_i, \) where \( q_i \in \mathcal{P}. \) Part (b) of Theorem 3.1 thus gives \( \phi \in M. \) Q.E.D.

The next result is the analogue of Corollary 2.2 for the structure constants of the second kind.

**Theorem 4.2.** Let the structure constants \( K_{\alpha \beta}^\gamma \) be defined by (1.2) relative to an \( \mathcal{P} \)-basis for \( \mathfrak{g}, \) and let \( \{\xi_i\} \) be the dual basis for \( \mathfrak{g}^*. \) Define \( \eta_i = \phi^{-1}(\xi_i), \) where \( \phi \) is the transformation (4.1), and set
\[
G_i = (\phi \otimes \phi) \mu(\eta_i) \in \mathcal{P} \otimes \mathcal{P}.
\]
Then
\[
(4.3) \quad K_{\gamma}^{\alpha \beta} = (\alpha! \beta!^{-1}) \partial_{(x \gamma, y \beta)} G^{\gamma}(x, y) \big|_{x = y = 0},
\]
where \( G^{\gamma} = G_{\gamma}^{a_1} \cdots G_{\gamma}^{a_d}, \) and \( \mathcal{P} \otimes \mathcal{P} \) is identified with the polynomials on \( \mathfrak{g} \times \mathfrak{g} \) as usual.

**Proof.** Let \( f \in \mathcal{P}. \) Then by the identity
\[
e^{x \gamma} = e^{x} e^{\gamma}, \quad x, y \in \mathfrak{g}
\]
in \( [U(\mathfrak{g})]_g \) and the definition of \( \phi, \) we can express
\[
\phi(f) = \sum_\alpha (\alpha!)^{-1} \xi^a \langle f, X^a \rangle.
\]
On the other hand, by Taylor's formula (2.6),
\[
\phi(f) = \sum_\alpha (\alpha!)^{-1} \xi^a \langle \phi(f), X(\alpha) \rangle.
\]
Comparing these two expansions gives the following formula for the automorphism \( \phi, \) in terms of the duality between \( \mathcal{P} \) and \( U(\mathfrak{g}): \)
\[
(4.4) \quad \langle \phi(f), X(\alpha) \rangle = \langle f, X^a \rangle.
\]
Now set \( \eta_i = \phi^{-1}(\xi_i) \) and \( \eta^a = \eta^{a_1} \cdots \eta^{a_d}, \) for \( \alpha \in \mathbb{N}^d. \) Then by (4.4) and
(2.2) we find that
\[ \langle \eta^\alpha, X^\beta \rangle = \alpha! \delta_{\alpha\beta}. \]
Hence the dual form of the defining equation for the structure constants \( K_{y}^{\gamma\alpha} \) is
\[ (4.5) \quad \mu(\eta^\gamma) = \sum_{\alpha,\beta} K_{y}^{\gamma\alpha} \eta^\alpha \otimes \eta^\beta \]
(the range of summation is finite, by Proposition 1.1). Applying the automorphism \( \phi \otimes \phi \) of \( \mathfrak{g} \otimes \mathfrak{g} \) to this equation, we thus obtain the expansion
\[ (4.6) \quad G^\gamma = \sum_{\alpha,\beta} K_{y}^{\gamma\alpha} \xi^\alpha \otimes \xi^\beta, \]
which is equivalent to (4.3). Q.E.D.

The transformation between the bases of the first and second kind for \( U(\mathfrak{g}) \) can be expressed in terms of the action of \( M \) as follows:

**THEOREM 4.3.** Let \( C_\alpha^\beta \) and \( D_\alpha^\beta \) be defined by the equations
\[ (\alpha!)^{-1} X^\alpha = \sum_{\beta} (\beta!)^{-1} C_\alpha^\beta X(\beta), \]
\[ (4.7) \quad (\alpha!)^{-1} X(\alpha) = \sum_{\beta} (\beta!)^{-1} D_\alpha^\beta X^\beta, \]
relative to an \( \mathcal{F} \)-basis \( \{X_i\} \) for \( \mathfrak{g} \). If \( \{\xi_i\} \) is the dual basis for \( \mathfrak{g}^* \), then
\[ C_\alpha^\beta = (\alpha!)^{-1} \frac{\partial \phi(\xi^\beta)}{\partial x^\alpha} \big|_{x=0}, \]
\[ D_\alpha^\beta = (\alpha!)^{-1} \frac{\partial \phi^{-1}(\xi^\beta)}{\partial x^\alpha} \big|_{x=0}, \]
where \( \phi \) is defined by (4.1). In particular,
\[ D_\alpha^\beta = C_\alpha^\beta = 0 \quad \text{if } |\alpha| < |\beta| \text{ or } w(\alpha) > w(\beta). \]

**REMARK.** By the PBW theorem, the constants \( C_\alpha^\beta \) and \( D_\alpha^\beta \) are determined by (4.7). For a recursive procedure for calculating these constants for any Lie algebra, cf. [10].

**PROOF.** Using equation (4.4), we find that
\[ C_\alpha^\beta = (\alpha!)^{-1} \langle \xi^\beta, X^\alpha \rangle = (\alpha!)^{-1} \langle \phi(\xi^\beta), X(\alpha) \rangle, \]
and similarly,
\[ D_\alpha^\beta = (\alpha!)^{-1} \langle \phi^{-1}(\xi^\beta), X(\alpha) \rangle. \]
By (2.2) this gives (4.8). Since \( \phi \in M \), we have
\[ \phi^{-1}(\xi^\beta), \phi(\xi^\beta) \in \mathfrak{g} \subset \mathcal{F}_{w(\beta)}. \]
Hence the vanishing condition (4.9) when \( w(\alpha) > w(\beta) \) follows from (4.8) and (2.3). The vanishing condition when \( |\alpha| < |\beta| \) follows by using the canonical filtration of \( U(\mathfrak{g}) \) [3, Proposition 2.4.4].

5. Estimates for structure constants. Assume now that the scalar field \( F = \mathbb{C} \). With the same hypotheses and notation as in Corollary 2.2 and Theorems 4.2 and 4.3, we have the following estimate for the order of growth of the structure constants for \( U(\mathfrak{g}) \) associated with an \( \mathcal{F} \)-basis for \( \mathfrak{g} \):

**Theorem 5.1.** There is a constant \( R > 1 \) such that

\[
(5.1) \quad \max_{\alpha, \beta} \{ |C_{\gamma}^{\alpha\beta}|, |K_{\gamma}^{\alpha\beta}|, |C_{\gamma}^{\alpha}|, |D_{\gamma}^{\alpha}| \} < R^{|\gamma|}.
\]

**Proof.** Equation (2.7) and the Cauchy estimates in a polydisc give the inequalities

\[
(5.2) \quad |C_{\gamma}^{\alpha\beta}| \leq \max \{|F_{\gamma}(x, y)| : x, y \in D \},
\]

where \( D = \{ x \in \mathfrak{g} : |\xi(x)| = 1 \text{ for } 1 < i < d \} \). The right side of (5.2) is majorized in turn by \( R^{\mathcal{F}} \), provided we take

\[
R > \max \{|F_i(x, y)| : x, y \in D, 1 < i < d \}.
\]

The same argument applies to the other structure constants, using formulas (4.3) and (4.8).

6. Locally convex completions of \( U(\mathfrak{g}) \). We continue to assume that the scalar field \( F = \mathbb{C} \). We shall use the estimates of §5 to construct complete, locally-convex topological algebras \( \mathcal{A} \) such that

\[
U(\mathfrak{g}) \subset \mathcal{A} \subset [U(\mathfrak{g})]_{\mathcal{F}},
\]

with \( U(\mathfrak{g}) \) dense in \( \mathcal{A} \).

Fix an \( \mathcal{F} \)-basis \( \{\mathfrak{g}_i\} \) for \( \mathfrak{g} \). Recall that the numbers \( w_i = w(X_i) \) are uniquely determined by the filtration \( \mathcal{F} \).

**Definition.** A sequence \( \mathcal{M} = \{M_\alpha : \alpha \in \mathbb{N}^d\} \) of positive numbers is an \( \mathcal{F} \)-weight sequence if \( M_0 = 1 \) and

\[
(6.1) \quad M_\gamma \leq M_\alpha M_\beta
\]

when \( w(\gamma) > w(\alpha) + w(\beta) \) (where \( w(\alpha) = \Sigma w_i \alpha_i \)).

**Example.** Take \( M_\alpha = \phi(w(\alpha)) \), where \( \phi \) is defined on \( \mathbb{N} \) and satisfies \( \phi(0) = 1 \), \( \phi(m) > 0 \), and \( \phi(m + n) \leq \phi(m)\phi(n) \). For instance, let \( \phi(m) = m^{-pm} \), with \( p \) any nonnegative real number (note that \( m^n < (m + n)^{m+n} \) by the geometric-arithmetic mean inequality).

Assume now that \( \mathcal{M} = \{M_\alpha\} \) is an \( \mathcal{F} \)-weight sequence. Define a family of seminorms on \( [U(\mathfrak{g})]_{\mathcal{F}} \) as follows:

If \( T = \Sigma_\alpha (\alpha!)^{-1}c_\alpha X^\alpha \) and \( r > 0 \), then set
(6.2) \[ \|T\|_r = \sup_{\alpha} \{ r^{\ell_\alpha} M_{\alpha} |c_\alpha| \} \]

(the value $+\infty$ is allowed for $\|T\|_r$).

**Lemma 6.1.** Let $d = \dim g$, $l = \text{length of } G$, and let $R$ be the constant in Theorem 5.1. Then for $p > 2dR \max[r, r^{1/l}]$, there is a constant $C = C(d, l, p, r, R) < \infty$ such that

\[ (6.3) \quad \|ST\|_r < C\|S\|_p\|T\|_p \]

for all $S, T \in [U(g)]_p$.

**Proof.** By Remark 1 after Proposition 1.2, $\|ST\|_r$ is majorized by

\[ (6.4) \quad \sup_{\gamma} \left\{ r^{\ell_{M_\gamma}} \sum_{\alpha, \beta} |a_\alpha b_\beta K_{\alpha\beta}| \right\}, \]

if $S = \sum_{\alpha} (\alpha!)^{-1} a_\alpha X^\alpha$ and $T = \sum_{\beta} (\beta!)^{-1} b_\beta X^\beta$. From Theorem 5.1 and the definition (6.2), we find that (6.4) is majorized by

\[ (6.5) \quad \sup_{\gamma} \left\{ \sum_{\alpha, \beta} \frac{M_{\gamma}}{M_{\alpha} M_{\beta}} (rR)^{\ell_{M_\gamma}} p^{-|\alpha| - |\beta|} \right\} \|S\|_p\|T\|_p. \]

In (6.5) the summation is over $\alpha, \beta \in \mathbb{N}^d$ such that $|\alpha| + |\beta| > |\gamma|$ and $w(\alpha) + w(\beta) < w(\gamma)$, by virtue of Proposition 1.1. Since $|\alpha| < w(\alpha) < l|\alpha|$, this range of summation is contained in the set

\[ E_\gamma = \{ \alpha, \beta \in \mathbb{N}^d : |\gamma| < |\alpha| + |\beta| < l|\gamma| \}. \]

Since $M_{\gamma} < M_{\alpha} M_{\beta}$ in (6.5), this gives (6.3), where

\[ C = \sup_{\gamma} \left\{ (rR)^{\ell_{E_\gamma}} \sum_{E_\gamma} p^{-|\alpha| - |\beta|} \right\}. \]

The finiteness of $C$ when $p > 2dR \max[r, r^{1/l}]$ follows from elementary estimates.

**Lemma 6.2.** Let $\{\xi_i\}$ be the dual basis to $\{X_i\}$. Then if $s > dR \max[r, r^{1/l}]$, there is a constant $C_1 < \infty$ such that

\[ (6.6) \quad \|T\|_r < C_1 \sup_{\alpha} \{ s^{\ell_\alpha} M_{\alpha} |\langle T, \xi_\alpha \rangle| \}. \]

Conversely, if $r > dR \max[s, s^{1/l}]$, then there is a constant $C_2 < \infty$ such that

\[ (6.7) \quad \sup_{\alpha} \{ s^{\ell_\alpha} M_{\alpha} |\langle T, \xi_\alpha \rangle| \} < C_2 \|T\|_s. \]

**Proof.** Given $s > 0$ and $T \in [U(g)]_s$, we define

\[ \|\|T\||_s = \sup_{\alpha} \{ s^{\ell_\alpha} M_{\alpha} |\langle T, \xi_\alpha \rangle| \}. \]

In order to compare the two families of seminorms $\{\|T\|_r\}$ and $\{\|\|T\||_s\}$, we
note that if $T$ has the formal series expansion

$$T = \sum_{\alpha} (\alpha!)^{-1} b_{\alpha} X(\alpha),$$

then $b_{\alpha} = \langle T, \xi^{\alpha} \rangle$. It follows that $\|T\|_r \leq D(r, s)\|T\|_s$, where

$$D(r, s) = \sum_{\alpha} (\alpha!)^{-1} s^{-|\alpha|} M_\alpha^{-1} \|X(\alpha)\|_s.$$

Now from the definition, we have $(\beta!)^{-1}\|X^\beta\|_r = r^{l\beta} M_\beta$. Using Theorems 4.3 and 5.1, we can thus estimate

$$(\alpha!)^{-1}\|X(\alpha)\|_r \leq \sum_{\beta} (\beta!)^{-1} |D^\beta_a| \|X\|_r \leq R^{|\alpha|} \sum r^{|\beta|} M_\beta,$$

with the last summation over $\beta \in \mathbb{N}^d$ such that $|\beta| < |\alpha|$ and $w(\beta) > w(\alpha)$. In this range $M_\beta < M_0 M_\alpha = M_\alpha$, and $|\alpha|/l < |\beta| < |\alpha|$, so this gives the bound

$$D(r, s) \leq \sum (R/s)^{|\alpha|} r^{|\beta|}$$

(sum over $\alpha, \beta$ with $|\alpha|/l < |\beta| < |\alpha|$). Elementary estimates show that $D(r, s) < \infty$, provided that $s > dR \max [r, r^{1/l}]$. This proves (6.6).

To obtain an estimate in the opposite direction, we write

$$T = \sum_{\beta} (\beta!)^{-1} a_{\beta} X^\beta = \sum_{\alpha, \beta} (\alpha!)^{-1} a_{\beta} C_{\beta}^\alpha X(\alpha),$$

where $\{C_{\beta}^\alpha\}$ are the constants defined by equations (4.7). Thus

$$\langle T, \xi^{\alpha} \rangle = \sum_{\beta} C_{\beta}^\alpha a_{\beta},$$

so that

$$|\langle T, \xi^{\alpha} \rangle| \leq C(\alpha, r)\|T\|_r,$$

where

$$C(\alpha, r) = \sum |C_{\beta}^\alpha| r^{|\beta|} M_\beta^{-1}. \tag{6.8}$$

By Theorem 4.3 we can restrict $\beta$ to the range $|\alpha| < |\beta|$ and $w(\beta) < w(\alpha)$ in the summation (6.8). Hence $M_\alpha < M_0 M_\beta = M_\beta$, so that

$$C(\alpha, r) < M_\alpha^{-1} \sum (R/r)^{|\beta|}$$

(sum over $|\alpha| < |\beta| < l|\alpha|$). From this estimate (6.7) follows easily, completing the proof.

We can now state the main result of this section.

**Theorem 6.3.** Let $M$ be an $\mathcal{F}$-weight sequence, and let the seminorms $\|T\|_r$ be defined by (6.2) relative to some $\mathcal{F}$-basis for $g$. Define
\(A = \{ T \in \left[ U(g) \right]_g : \| T \|_r < \infty \text{ for all } r > 0 \}\).

Then

1. \(A\) is a subalgebra of \([U(g)]_g\);
2. With the locally convex topology defined by the family of seminorms \(\{ \| \cdot \|_r \}_{r > 0}\), \(A\) is a Fréchet space containing \(U(g)\) as a dense subspace, and multiplication is jointly continuous;
3. \(A\) and its topology are independent of the choice of \(\mathfrak{g}\)-basis used to define the seminorms \(\| \cdot \|_r\).

**Definition.** If \(M\) is an \(\mathfrak{g}\)-weight sequence, then the \(M\)-completion \([U(g)]_M\) of \(U(g)\) is the topological algebra \(A\) of Theorem 6.3.

**Proof of Theorem.** Statements (1) and (2) are easy consequences of Lemma 6.1 and straightforward estimates. For example, to show that \(X(g)\) is dense in \(A\), let \(T \in A\) have the formal expansion

\[T = \sum_{\alpha} (\alpha!)^{-1} b_{\alpha} X^\alpha,\]

and define \(T_n\) by the same series, but with \(|\alpha| < n\). Now \(|b_{\alpha}| < s^{-|\alpha|} M_\alpha^{-1} \|T\|_r\), and \(\|X^\alpha\|_r = \alpha! M_\alpha^{-|\alpha|}\), for any \(r, s > 0\). Thus if \(s > r\), we can estimate \(\|T - T_n\|_r < C_n \|T\|_s\), where

\[C_n = \sum_{|\alpha| > n} (r/s)^{|\alpha|}.\]

Since \(C_n \to 0\) as \(n \to \infty\), this shows that \(T_n \to T\) in the \(A\)-topology.

To prove (3), let \(\{X_i\}\) and \(\{Y_i\}\) be two \(\mathfrak{g}\)-bases for \(g\), with corresponding dual bases \(\{\xi_i\}\) and \(\{\eta_i\}\). Then \(w(X_i) = w(Y_i) = w_i\) and \(X_i = \sum c_{ij} Y_j\), with \(c_{ij} = 0\) if \(w_j < w_i\). It follows that for \(\alpha \in \mathbb{N}^d\) we can express

\[\eta^\alpha = \sum g_{\alpha \beta} \xi^\beta,\]

where the summation is over \(\beta \in \mathbb{N}^d\) such that \(w(\beta) < w(\alpha)\) and \(|\beta| = |\alpha|\). Furthermore, \(g_{\alpha \beta}\) is a product of \(|\alpha|\) factors \(c_{ij}\), so that \(|g_{\alpha \beta}| < R^{|\alpha|}\), for some constant \(R\). Thus we have

\[|\langle T, \eta^\alpha \rangle| < (Rd)^{|\alpha|} \max_{w(\beta) < w(\alpha)} |\langle T, \xi^\beta \rangle|.\]

Using (6.10) and Lemma 6.2, it is then easy to verify that the two \(\mathfrak{g}\)-bases give equivalent families of seminorms on \([U(g)]_M\). Q.E.D.

**Remarks.** (1) It follows by Lemma 6.2 that the canonical symmetrization map [3, §2.4.5]:

\[\omega : S(g) \to U(g)\]

extends to a linear isomorphism between \([S(g)]_M\) and \([U(g)]_M\). Here we
define \([S(\mathfrak{g})]_{\mathfrak{g}} = [U(V)]_{\mathfrak{g}_0}\), where \(V\) is the vector space \(\mathfrak{g}\) but with trivial Lie bracket.

(2) By the same argument as in the proof of Theorem 6.3, one sees that if \(\phi \in \text{Aut}(\mathfrak{g})\) and \(\phi(\mathfrak{g}_n) \subseteq \mathfrak{g}_n\), then \(\phi\) extends by continuity from \(U(\mathfrak{g})\) to \([U(\mathfrak{g})]_{\mathfrak{g}_0}\).

(3) Combining Remarks 1 and 2, we conclude that the adjoint representation of \(G\) on \(U(\mathfrak{g})\) extends by continuity to a representation on \([U(\mathfrak{g})]_{\mathfrak{g}_0}\), which is equivalent to the representation on \([S(\mathfrak{g})]_{\mathfrak{g}_0}\). (Here \(G = \exp \mathfrak{g}\).) In this respect the algebras \([U(\mathfrak{g})]_{\mathfrak{g}_0}\) are more satisfactory than the completions \(\mathfrak{g}_\lambda, \lambda < 1\), constructed in [5].

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