

CYCLIC ACTIONS ON LENS SPACES

BY

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ABSTRACT. A 3-dimensional lens space $L = L(p, q)$ is called symmetric if $q^2 \equiv \pm 1 \pmod{p}$. Let h be an orientation-preserving PL homeomorphism of even period $n (> 2)$ on L with nonempty fixed-point set. We show:

(1) If n and p are relatively prime, up to weak equivalence (PL), there exists exactly one such h if L is symmetric, and there exist exactly two such h if L is nonsymmetric.

(2) $\text{Fix}(h)$ is disconnected only if $p \equiv 0 \pmod{n}$, and there exists exactly one such h up to weak equivalence (PL).

A Z_n -action is called nonfree if $\text{Fix}(\phi) \neq \emptyset$ for some $\phi (\neq 1) \in Z_n$. We also classify all orientation-preserving nonfree Z_n -actions (PL) on all lens spaces $L(p, q)$. It follows that each of S^3 and P^3 admits exactly three orientation-preserving Z_n -actions (PL), up to conjugation.

1. Introduction. All objects in this paper are in the PL category. Let T be an orientation-preserving periodic homeomorphism of the 3-sphere S^3 with nonempty fixed-point set $\text{Fix}(T)$. It is known that $\text{Fix}(T)$ is a simple closed curve. A well-known conjecture, due to P. A. Smith, asserts that $\text{Fix}(T)$ is unknotted (see Eilenberg [2]). It follows from a result of Waldhausen [23] that the conjecture is true for all homeomorphisms of even period on S^3 .

Let $L = L(p, q)$ be a 3-dimensional lens space. Consider an orientation-preserving homeomorphism h of period n on $L(p, q)$ with $\text{Fix}(h) \neq \emptyset$, where p and n are relatively prime. It can be seen that $\text{Fix}(h)$ is a simple closed curve and, if Smith's conjecture is true and $n > 2$, then $\pi_1(L - \text{Fix}(h)) = Z$. In particular, $\pi_1(L - \text{Fix}(h)) = Z$ is always true for all homeomorphisms h of even period ($\neq 2$) (see §3). In §3, the following theorem will be proved:

THEOREM A. *Let h be an orientation-preserving homeomorphism of period n on $L = L(p, q)$ with $(p, n) = 1$, and $\pi_1(L - \text{Fix}(h)) = Z$. Then there exists exactly one such h if L is symmetric, and there exist exactly two such h if L is nonsymmetric, up to weak equivalence.*

A lens space $L(p, q)$ is called *symmetric* if $q^2 \equiv \pm 1 \pmod{p}$. Being symmetric is a topological property.

Let k be a periodic homeomorphism of a space X . The cyclic group

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generated by k shall be denoted by $\langle k \rangle$. Two actions of $\langle k \rangle$ and $\langle k' \rangle$ on X are said to be *conjugate* if there exists a homeomorphism t of X such that $\langle tkt^{-1} \rangle = \langle k' \rangle$. In this case, k and k' are called *weakly equivalent*. If $tkt^{-1} = k'$, then k and k' are said to be *equivalent*.

Moise [13] proved that if $h: S^3 \rightarrow S^3$ is periodic and preserves the orientation, and $\text{Fix}(h) (\neq \emptyset)$ is unknotted, then h is equivalent to a rotation (see P. A. Smith [19] for an alternative proof). This result is also obtained as a special type of Theorem A. The proof given here for this case seems shorter and more intuitive than the other two proofs.

In §3 we also investigate all orientation-preserving homeomorphisms h of period n on $L = L(p, q)$ with $\pi_1(L - \text{Fix}(h)) = \mathbb{Z} \oplus \mathbb{Z}$. This is the case where $\text{Fix}(h)$ is disconnected. In this case, $\pi_1(L - \text{Fix}(h)) = \mathbb{Z} \oplus \mathbb{Z}$ is always *true* for all homeomorphisms h of even period ($\neq 2$) (see §3).

THEOREM B. *Let h be an orientation-preserving homeomorphism of period n on $L = L(p, q)$, and $\pi_1(L - \text{Fix}(h)) = \mathbb{Z} \oplus \mathbb{Z}$. Then there exists such an h if and only if $p \equiv 0 \pmod n$, and h is unique up to weak equivalence.*

A Z_n -action on L shall be called *nonfree* if $\text{Fix}(\phi) \neq \emptyset$ for some $\phi (\neq 1) \in Z_n$. In §4, we classify all orientation-preserving nonfree Z_4 -actions on all lens spaces $L(p, q)$. We summarize the result of §4 as the following:

THEOREM C. *The number of nonconjugate, orientation-preserving, nonfree Z_4 -actions on $L = L(p, q)$ is given as the following:*

- (1) *For p odd, there are two if L is symmetric and four if L is nonsymmetric.*
- (2) *For $p = 2s$ (s odd), there is one if L is symmetric and two if L is nonsymmetric.*
- (3) *For $p \equiv 0 \pmod 4$, there are three if $q \equiv ad + bc \pmod p$ where $p = 2ac$ and $ad - bc = 1$, and there are four if $q \equiv ad + bc - (ab/2) \pmod p$ where $p = 2ac$ (a even) and $ad - b(c + a/2) = 1$ (the homeomorphic type of $L(p, q)$ does not depend on the choice of b, d).*

Otherwise, there exist only two nonfree Z_4 -actions.

All Z_2 -actions on the 3-sphere S^3 [9], [10], [22] and the projective 3-space P^3 [5], [7] are known. As results of §4, we have the following (see Rice [17] for free Z_4 -actions on S^3).

THEOREM D. *Each of S^3 and P^3 admits exactly three orientation-preserving Z_4 -actions, up to conjugation.*

A homeomorphism h of $L = L(p, q)$ is called *sense-preserving* if h induces the identity on $H_1(L)$.

Let h be a periodic (PL) homeomorphism of a finite simplicial complex M . It can be shown that h becomes simplicial after a suitable subdivision M_0

such that h is *regular*, i.e., $h(\sigma) = \sigma$ for a simplex $\sigma \in M_0$ only if $h|_\sigma$ is the identity.

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2. Z_n -actions on $D^2 \times S^1$. Let D^2 be the unit disk in the Gaussian plane of complex numbers and S^1 its boundary. $D^2 \times S^1$ is the solid torus whose points can be denoted by $(\rho z_1, z_2)$ where $z_1, z_2 \in S^1$ and $0 < \rho < 1$.

Let h be an orientation-preserving homeomorphism of period n on a compact n -manifold X with $\text{Fix}(h) \neq \emptyset$. Let q be a prime factor of n such that $n = mq$. Then h^m is of prime period q . It is known that $\sum_k \dim H_k(F) < \sum_k \dim H_k(X)$ over Z_q where $F = \text{Fix}(h^m)$ (for the proof, see [1]). Now, let $X = D^2 \times S^1$, and assume that $\text{Fix}(h) \subset \text{Int}(D^2 \times S^1)$. It is easy to see that $\text{Fix}(h)$ is a 1-manifold. Since $\text{Fix}(h) \subset \text{Fix}(h^m)$ and the above inequality holds, $\text{Fix}(h)$ is a simple closed curve, and $\text{Fix}(h) = \text{Fix}(h^i)$, $0 < i < n$.

DEFINITION 2.1. Let h be a homeomorphism of period n on a 3-manifold M . Let $g: M \rightarrow M/\langle h^m \rangle$ be the orbit map induced by $\langle h^m \rangle$. Then there exists a homeomorphism \bar{h} of $M/\langle h^m \rangle$, uniquely determined by h , such that $\bar{h}g = gh$. The map \bar{h} shall be called the homeomorphism on $M/\langle h^m \rangle$ induced by h .

Let h be an orientation-preserving homeomorphism of period n on $D^2 \times S^1$ with $F = \text{Fix}(h) \neq \emptyset$. Suppose that $(D^2 \times S^1, F) \approx (D^2 \times S^1, 0 \times S^1)$ (i.e., they are homeomorphic as a pair). We may assume that h is regular after a suitable subdivision. Let U be a simplicial (regular) neighborhood of F . Let $M = D^2 \times S^1/\langle h \rangle$ and $g: D^2 \times S^1 \rightarrow M$ be the orbit map. Since M is an orientable manifold and $g(U)$ is a simplicial neighborhood of $g(F)$, $g(U)$ is a solid torus. We have the following two lemmas.

LEMMA 2.2. *The orbit space $M = D^2 \times S^1/\langle h \rangle$ is a solid torus.*

PROOF. Let $U' = g(U)$. We claim that $\pi_1(M - U') = Z \oplus Z$. Since $M - U'$ has two boundary components, $H_2(M - U')$ is of rank > 1 . Since $\chi(M - U') = 0$, $H_1(M - U')$ is of rank > 2 . Therefore, since we have an obvious short exact sequence

$$0 \rightarrow Z \oplus Z \rightarrow \pi_1(M - U') \rightarrow Z_n \rightarrow 0,$$

$\pi_1(M - U')$ is an abelian group of rank 2. As $M - U'$ is covered by a contractible space and no nontrivial finite group can act freely on a finite-dimensional, contractible space (due to P. A. Smith [4]), $\pi_1(M - U')$ has no torsion subgroup, and $\pi_1(M - U') = Z \oplus Z$. Therefore, since $\text{cl}(M - U')$ is irreducible, $\text{cl}(M - U')$ may be obtained from $S^1 \times I_1 \times I_2$ by identifying each $(x, 0)$ ($x \in S^1 \times I_1$) with $(f(x), 1)$, where I_i ($i = 1, 2$) is the unit interval I and f is a homeomorphism of $S^1 \times I_1$ (see Stallings [20]). Since $\text{cl}(M - U')$ has two boundary components, f carries $S^1 \times \{i\}$ onto $S^1 \times \{i\}$

($i = 0, 1$). Since $\text{cl}(M - U')$ is orientable, f must preserve the orientation, and it can be shown that f is isotopic to the identity. Therefore, $\text{cl}(M - U') \approx S^1 \times S^1 \times I$, and M is a solid torus.

LEMMA 2.3. *There exists a disk E properly embedded in $D^2 \times S^1$ such that $h(E) = E$ and ∂E does not bound a disk in $S^1 \times S^1$.*

One may easily derive the proof of the above lemma from that of Lemma 2.8, and we omit the proof.

PROPOSITION 2.4. *Let h be an orientation-preserving homeomorphism of period n on $D^2 \times S^1$ with $F = \text{Fix}(h) \neq \emptyset$. Suppose that*

$$(D^2 \times S^1, F) \approx (D^2 \times S^1, 0 \times S^1).$$

Then there exists a unique such h , up to weak equivalence.

PROOF. We may assume that $F = 0 \times S^1$. Let h_i ($i = 1, 2$) be two homeomorphisms such as h . By Lemma 2.2, each orbit space

$$M_i = D^2 \times S^1 / \langle h_i \rangle$$

is a solid torus. Parametrize M_i in terms of $D^2 \times S^1$. Let $(1, 0)$ and $(0, 1)$ be the canonical generators of $\pi_1(S^1 \times S^1)$ (we disregard the base point as $\pi_1(S^1 \times S^1)$ is abelian). Let $g'_i: D^2 \times S^1 \rightarrow M_i = D^2 \times S^1$ be the orbit map and $g_i = g'_i|(D^2 - 0) \times S^1$. By Lemma 2.3, we may assume that $g_{i\#}[(1, 0)] = (n, 0)$. Let $g_{i\#}[(0, 1)] = (a_i, b_i)$. Since g_i is a covering projection with n sheets, $(1, 0)$ and (a_i, b_i) generate $\pi_1(S^1 \times S^1)$, and therefore we may assume that $b_i = 1$. Define t' of M_1 to M_2 by

$$t'(\rho z_1, z_2) = (\rho z_1 z_2^{a_2 - a_1}, z_2) \quad \text{for } (\rho z_1, z_2) \in D^2 \times S^1.$$

Since $(tg_1)_{\#} = g_{2\#}$, where $t = t'|(D^2 - 0) \times S^1$, it follows from the lifting theorem that there exists a homeomorphism \bar{t} of $(D^2 - 0) \times S^1$ such that $g_{2\#}\bar{t} = tg_1$. Therefore, $i\bar{h}_1\bar{t}^{-1} = \alpha(h_2)$ for some automorphism α of $\langle h_2 \rangle$. One may extend \bar{t} to $D^2 \times S^1$ such that $i\bar{h}_1\bar{t}^{-1} = \alpha(h_2)$ on $D^2 \times S^1$. This completes the proof.

(2.5). Let h be an involution on $D^2 \times S^1$. It follows from a result of [6] that h is equivalent to an involution of a form $h_1 \times h_2$ on $D^2 \times S^1$ (see also [9], [10], [22]). Especially $D^2 \times S^1$ admits exactly two (nonconjugate) obvious orientation-preserving involutions h with $\text{Fix}(h) \neq \emptyset$ (see also [21]).

COROLLARY 2.6. *There exists exactly one orientation-preserving homeomorphism h of period $2k$ ($k \neq 1$) on $D^2 \times S^1$ with $\text{Fix}(h) \neq \emptyset$, up to weak equivalence.*

PROOF. If $\text{Fix}(h^k) \subset \text{Int}(D^2 \times S^1)$, we see that $\text{Fix}(h) = \text{Fix}(h^k)$, and the result follows from (2.5) and Proposition 2.4. Indeed, we will point out that $\text{Fix}(h) \subset \text{Int}(D^2 \times S^1)$. Suppose that $\text{Fix}(h) \cap S^1 \times S^1 \neq \emptyset$. Then $\text{Fix}(h^k)$

is a disjoint union of two arcs and $D^2 \times S^1 / \langle h^k \rangle$ is homeomorphic to D^3 (see (2.5)). Since h is orientation-preserving and $\text{Fix}(h) \subset \text{Fix}(h^k)$, we see that $\text{Fix}(h) = \text{Fix}(h^k)$. Let \bar{h} be the homeomorphism on $D^2 \times S^1 / \langle h^k \rangle$ induced by h and $g: D^2 \times S^1 \rightarrow D^2 \times S^1 / \langle h^k \rangle$ be the orbit map. Since $g(\text{Fix}(h)) \subset \text{Fix}(\bar{h})$, $\text{Fix}(\bar{h})$ has at least two components. This is a contradiction since \bar{h} is a periodic map of $D^2 \times S^1 / \langle h^k \rangle \approx D^3$ ($\text{Fix}(\bar{h})$ must be an arc in this case).

Since the proof of the following lemma is easy, we omit it.

LEMMA 2.7. *Let h be a homeomorphism of period n on $D^2 \times S^1$. If $\langle h \rangle$ acts freely on $D^2 \times S^1$, then h is given by $h(\rho z_1, z_2) = (\rho z_1, \omega z_2)$ or $h(\rho z_1, z_2) = (\rho \bar{z}_1, \omega z_2)$ for $(\rho z_1, z_2) \in D^2 \times S^1$, where $\omega = e^{2\pi i/n}$, up to weak equivalence.*

Let h be an orientation-preserving homeomorphism of period 2^n ($n \neq 1$) on $D^2 \times S^1$. Let $G = \{h^i | \text{Fix}(h^i) \neq \emptyset\}$. Since G forms a group, $G = \langle h^{2^k} \rangle$ for some k , $0 < k \leq n$. One can see that $\text{Fix}(h^{2^k}) \subset \text{Int}(D^2 \times S^1)$ (use a similar argument to that in the proof of Corollary 2.6; the induced homeomorphism \bar{h} on $D^2 \times S^1 / \langle h^{2^{n-1}} \rangle$ may be free this time). Therefore, it follows from (2.5) and Corollary 2.6 that $(D^2 \times S^1, \text{Fix}(h^{2^k})) \approx (D^2 \times S^1, 0 \times S^1)$. We may assume that h is regular after a suitable subdivision of $D^2 \times S^1$. Let \bar{h} be the homeomorphism on $M_1 = D^2 \times S^1 / \langle h^{2^k} \rangle$ induced by h . By Lemma 2.2, M_1 is a solid torus. Note that \bar{h} is an orientation-preserving homeomorphism of period 2^k . Since $\langle \bar{h} \rangle$ acts freely on M_1 , it follows from Lemma 2.7 that $M_2 = M_1 / \langle \bar{h} \rangle$ is a solid torus. Let $g_1: D^2 \times S^1 \rightarrow M_1$ and $g_2: M_1 \rightarrow M_2$ be the orbit maps. Then the composite map $g = g_2 g_1$ is the orbit map of $D^2 \times S^1 \rightarrow D^2 \times S^1 / \langle h \rangle$. Let $F = \text{Fix}(h^{2^k})$. Since F is invariant under h , $g_1(F)$ is invariant under \bar{h} . We have the following lemma.

LEMMA 2.8. *There exists a disk E properly embedded in $D^2 \times S^1$ such that $h^i(E) \cap E = \emptyset$, $1 \leq i < 2^k$, and $h^{2^k}(E) = E$, and ∂E does not bound a disk in $S^1 \times S^1$.*

PROOF. It is easy to find a disk E' properly embedded in a simplicial neighborhood U of F after a suitable subdivision of $D^2 \times S^1$ such that $h^i(E') \cap E' = \emptyset$, $1 \leq i < 2^k$, and $h^{2^k}(E') = E'$, and $\partial E'$ does not bound a disk in ∂U . Then $g(E')$ is a disk properly embedded in $g(U)$. Since $\text{cl}(M_2 - g(U))$ is a boundary collar of $g(U)$ (see the proof of Lemma 2.2), there exists a disk E_2 properly embedded in M_2 such that $g(E') \subset \text{Int}(E_2)$ and ∂E_2 does not bound a disk in ∂M_2 . Since g_2 is a covering projection with 2^k sheets, $g_2^{-1}(E_2)$ is 2^k disjoint copies of disks properly embedded in M_1 . Let E_1 be the disk which is a component of $g_2^{-1}(E_2)$ and $E_1 \cap g_1(E') \neq \emptyset$. Let $E = g_1^{-1}(E_1)$. Since $g_1|_{(E' - F)}$ is a covering projection, E must be connected, and it is a disk as desired.

THEOREM 2.9. *There exist exactly three orientation-preserving Z_4 -actions on $D^2 \times S^1$, up to conjugation.*

PROOF. By Corollary 2.6 and Lemma 2.7, it is enough to show that two orientation-preserving homeomorphisms h_i ($i = 1, 2$) of period 4 on $D^2 \times S^1$, which satisfy $\text{Fix}(h_i) = \emptyset$ and $\text{Fix}(h_i^2) \neq \emptyset$, are weakly equivalent. Since $\text{Fix}(h_i^2) \subset \text{Int}(D^2 \times S^1)$, we may assume that

$$\text{Fix}(h_i^2) = 0 \times S^1 \subset D^2 \times S^1 \quad \text{for each } i$$

(see (2.5)). Let $\bar{M}_i = D^2 \times S^1 / \langle h_i^2 \rangle$ and \bar{h}_i be the homeomorphism on \bar{M}_i induced by h_i . Let $M_i = \bar{M}_i / \langle \bar{h}_i \rangle$. Let $\bar{g}_i: D^2 \times S^1 \rightarrow \bar{M}_i$ and $g_i: \bar{M}_i \rightarrow M_i$ be the orbit maps, and $f_i = g_i \bar{g}_i$. Parametrize \bar{M}_i and M_i in terms of $D^2 \times S^1$. Let $(1, 0)$ and $(0, 1)$ be the canonical generators of $\pi_1(S^1 \times S^1)$. Let

$$\bar{g}'_i = \bar{g}_i|(D^2 - 0) \times S^1, g'_i = g_i|(D^2 - 0) \times S^1, \text{ and } f'_i = f_i|(D^2 - 0) \times S^1.$$

By Lemma 2.8, we assume that $f'_{i\#}[(1, 0)] = (2, 0)$. Let $f'_{i\#}[(0, 1)] = (a_i, b_i)$. Since the index of $f'_{i\#}(\pi_1(S^1 \times S^1))$ in $\pi_1(\partial M_i)$ is 4, we may assume that $b_i = 2$. Since $f'_{i\#}[(1, 0)] = (2, 0)$ and $f'_{i\#}[(0, 1)] = (a_i, 2)$, each a_i must be odd (otherwise, the covering transformation group of the covering space $((D^2 - 0) \times S^1, f'_i)$ would be $Z_2 \oplus Z_2$). Take base points $\hat{x}_i \in S^1 \times S^1 \subset D^2 \times S^1$, $\bar{x}_i \in \partial \bar{M}_i$ and $x_i \in \partial M_i$ such that $\bar{g}_i(\hat{x}_i) = \bar{x}_i$ and $g_i(\bar{x}_i) = x_i$. Define a homeomorphism t of M_1 to M_2 such that

$$t(\rho z_1, z_2) = (\rho z_1 z_2^{(a_2 - a_1)/2}, z_2).$$

Then, by letting $t' = t|(M_1 - 0 \times S^1)$, we see that $t'_{\#} f'_{1\#} = f'_{2\#}$, and therefore there exists a lifting homeomorphism \hat{t} of $(D^2 - 0) \times S^1$ such that $f'_2 \hat{t} = t' f'_1$ and $\hat{t}(\hat{x}_1) = \hat{x}_2$. On the other hand, since each g_i is a covering projection, there exists a lifting homeomorphism \bar{t} of \bar{M}_1 to \bar{M}_2 such that $t g_1 = g_2 \bar{t}$ and $\bar{t}(\bar{x}_1) = \bar{x}_2$. We may assume that $\bar{t}(0 \times S^1) = 0 \times S^1$. Since $f_2 \bar{t} = t f_1$ on $(D^2 - 0) \times S^1$, $\bar{t} h_1 \bar{t}^{-1} = h_2$ on $(D^2 - 0) \times S^1, j = 1$ or 3 . Therefore, there exists an obvious homeomorphism t_0 of $\bar{M}_1 - 0 \times S^1$ to $\bar{M}_2 - 0 \times S^1$ induced by \bar{t} such that $\bar{g}_2 \bar{t} = t_0 \bar{g}_1$ on $(D^2 - 0) \times S^1$ and $g_2 t_0 = t g_1$ on $(D^2 - 0) \times S^1$. Since $\hat{t}(\hat{x}_1) = \hat{x}_2$, and $t_0(\bar{x}_1) = \bar{x}_2$, it follows from the unique lifting property that $t_0 = \bar{t}|(D^2 - 0) \times S^1$. Since $\bar{t}(0 \times S^1) = 0 \times S^1$, one may easily extend \hat{t} to a homeomorphism \tilde{t} of $D^2 \times S^1$ such that the following diagram commutes:

$$\begin{array}{ccccc} (D^2 \times S^1, \hat{x}_1) & \xrightarrow{\bar{g}_1} & (\bar{M}_1, \bar{x}_1) & \xrightarrow{g_1} & (M_1, x_1) \\ \downarrow \tilde{t} & & \downarrow \bar{t} & & \downarrow t \\ (D^2 \times S^1, \hat{x}_2) & \xrightarrow{\bar{g}_2} & (\bar{M}_2, \bar{x}_2) & \xrightarrow{g_2} & (M_2, x_2) \end{array}$$

Hence $\tilde{t} h_1 \tilde{t}^{-1} = h_2, j = 1$ or 3 . This completes the proof.

3. Z_n -actions on $L(p, q)$. Let h be an orientation-preserving homeomorphism of period n on $L = L(p, q)$ with $\text{Fix}(h) \neq \emptyset$. Then $\text{Fix}(h)$ is a disjoint union of at most two simple closed curves (cf. the proof of Proposition 3.1).

PROPOSITION 3.1. *If $(n, p) = 1$, then $\text{Fix}(h^m)$ ($0 < m < n$) is a simple closed curve.*

PROOF. Let $T = h^m$, $0 < m < n$. Let r be the prime period (> 1) of T^k for some k . Let $F = \text{Fix}(T^k)$. Since $\sum_i \dim H_i(F) \leq \sum_i \dim H_i(L)$ over Z_r , and $(r, p) = 1$, F is a simple closed curve and so is $\text{Fix}(T)$.

PROPOSITION 3.2. *If n is even (> 2), then $\pi_1(L - \text{Fix}(h)) = Z$ or $Z \oplus Z$.*

PROOF. See (3.3).

(3.3). Let $n > 2$. Let $h_*(a) = ka$ for $a \in \pi_1(L)$. It follows from a result of Olum [14, p. 467] that $k^2 \equiv \deg h \pmod{p}$, and h^2 is sense-preserving. Let F_0 be a component of $\text{Fix}(h)$. By the lifting theorem, there exists a lifting $T: (S^3, y) \rightarrow (S^3, y)$ of period n with $\text{Fix}(T) \neq \emptyset$ such that $gT = hg$, where $g: S^3 \rightarrow L$ is the natural projection and $g(y) \in F_0$. Since $F_0 \subset \text{Fix}(h^2)$ and h^2 is sense-preserving, we see that $g^{-1}(F_0)$ is connected (for a proof, see [8]). Let $\tilde{F} = g^{-1}(F_0)$. Then \tilde{F} is the fixed-point set of T . Assume that Smith's conjecture is true (this is true if n is even). Then $\pi_1(S^3 - \tilde{F}) = Z$ (see [15]). Since there is an obvious short exact sequence

$$0 \rightarrow Z \rightarrow \pi_1(L - F_0) \rightarrow Z_p \rightarrow 0,$$

we see that $\pi_1(L - F_0)$ is an abelian group of rank 1. Since $L - F_0$ is covered by a contractible space, $\pi_1(L - F_0)$ has no nontrivial element of finite order (see [4, p. 287]), and $\pi_1(L - F_0) = Z$. Let U be a regular (invariant) neighborhood of F_0 in L such that $U \cap \text{Fix}(h) = F_0$. Since $L - F_0$ is irreducible, it follows from a result of Stallings [20] that $\text{cl}(L - U)$ is a solid torus. Therefore, if $\text{Fix}(h)$ is not connected and n is even, it follows from (2.5) that $\pi_1(L - \text{Fix}(h)) = Z \oplus Z$.

Proposition 3.2 is not true for involutions. It is known that an involution h is sense-preserving if and only if $\pi_1(L - \text{Fix}(h)) = Z$ or $Z \oplus Z$ (see [5], [8]). Every lens space $L(p, q)$ ($p > 2$) admits an orientation-preserving involution which is not sense-preserving and has nonempty fixed-point set.

The proofs of Theorem A and Theorem B will be given in the following two parts I, II.

I. PROOF OF THEOREM A. Let $F = \text{Fix}(h)$. By Proposition 3.1, F is a simple closed curve. Let $g: S^3 \rightarrow L$ be the natural projection. Since $\pi_1(L - F) = Z$, we see that $\pi_1(S^3 - g^{-1}(F)) = Z$ and $g^{-1}(F)$ is connected. Therefore, $S^3 - g^{-1}(F)$ is irreducible, and so is $L - F$. Let U be an invariant regular neighborhood of F in L . Since $\text{cl}(L - U)$ is irreducible and $\pi_1(L - U) = Z$,

it follows that $\text{cl}(L - U)$ is a solid torus (see [20]). Hence L may be given as $L = D^2 \times S^1 \cup_k S^1 \times D^2$ for an attaching map k of $S^1 \times S^1$, where $D^2 \times S^1$ is an invariant regular neighborhood of F . Let $h_1 = h|_{D^2 \times S^1}$ and $h_2 = h|_{S^1 \times D^2}$. By Proposition 2.4 and Lemma 2.7, we may assume that

$$L = D^2 \times S^1 \cup_f S^1 \times D^2$$

for an appropriate attaching map f of $S^1 \times S^1$ and h is given by

$$e(h_1)(\rho z_1, z_2) = (\rho \omega z_1, z_2) \quad \text{on } D^2 \times S^1 \text{ and}$$

$$e'(h_2)(z_1, \rho z_2) = (\omega z_1, \rho z_2) \quad \text{on } S^1 \times D^2$$

where $\omega = e^{2\pi i/n}$, and e, e' are some automorphisms of $\langle h_1 \rangle$ and $\langle h_2 \rangle$, respectively.

Let $(1, 0)$ and $(0, 1)$ be the elements of $\pi_1(S \times S^1) = Z \oplus Z$, represented by the paths $(e^{2\pi i t}, 1)$ and $(1, e^{2\pi i t})$ ($0 < t < 1$), respectively. Suppose that $f_{\#}[(1, 0)] = (a, b)$ and $f_{\#}[(0, 1)] = (c, d)$. $f_{\#}$ may be written as

$$f_{\#} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

Then

$$\begin{vmatrix} a & c \\ b & d \end{vmatrix} = \pm 1.$$

Let $g_1: D^2 \times S^1 \rightarrow D^2 \times S^1 / \langle h_1 \rangle$ and $g_2: S^1 \times D^2 \rightarrow S^1 \times D^2 / \langle h_2 \rangle$ be the orbit maps. Let $g'_i = g_i|_{S^1 \times S^1}$ ($i = 1, 2$). Then there are obvious parametrizations of $D^2 \times S^1 / \langle h_1 \rangle$ and $S^1 \times D^2 / \langle h_2 \rangle$ in terms of $D^2 \times S^1$ and $S^1 \times D^2$, respectively, such that $g'_{i\#}[(1, 0)] = (n, 0)$ and $g'_{i\#}[(0, 1)] = (0, 1)$. We have:

LEMMA 3.4. $b \equiv 0 \pmod n$.

PROOF. Consider the following commutative diagram:

$$\begin{array}{ccc} D^2 \times S^1 \supset S^1 \times S^1 & \xrightarrow{f} & S^1 \times S^1 \subset S^1 \times D^2 \\ g'_1 \downarrow & & g'_2 \downarrow \\ D^2 \times S^1 \supset S^1 \times S^1 & \xrightarrow{f'} & S^1 \times S^1 \subset S^1 \times D^2 \end{array}$$

where f' is the induced attaching map of $S^1 \times S^1$ in $L / \langle h \rangle$. Chasing the above diagram, it can be shown that $a = a', b = nb', nc = c'$ and $d = d'$, where

$$f'_{\#} = \begin{pmatrix} a' & c' \\ b' & d' \end{pmatrix}.$$

(3.5). Let $X = D^2 \times S^1 \cup_k S^1 \times D^2$ where

$$k_{\#} = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}.$$

By Mangler [11], the isotopy classes of homeomorphisms of $S^1 \times S^1$ are precisely the automorphism classes of $\pi_1(S^1 \times S^1)$. Therefore, the integers $\alpha, \beta, \gamma,$ and δ completely determine the isotopy class of k . It can be shown that $X \approx L(\alpha, \beta)$ if $\alpha \neq 0$. We denote X by $L(\alpha, \gamma, \beta, \delta)$ if k is given by

$$k(z_1, z_2) = (z_1^{\alpha} z_2^{\gamma}, z_1^{\beta} z_2^{\delta}) \text{ for } (z_1, z_2) \in S^1 \times S^1.$$

Recall that $L(\alpha, \beta)$ is homeomorphic to $L(\alpha, \beta')$ if and only if $\beta \equiv \pm \beta'$ or $\beta\beta' \equiv \pm 1 \pmod{\alpha}$ [12], [16].

Define a homeomorphism h_0 on $L_0 = L(a, c, b, d)$ by

$$h_0(\rho z_1, z_2) = (\rho \omega z_1, z_2) \text{ on } D^2 \times S^1 \text{ and}$$

$$h_0(z_1, \rho z_2) = (\omega^a z_1, \rho z_2) \text{ on } S^1 \times D^2$$

where $\omega = e^{2\pi i/n}$. By Lemma 3.4, h_0 is well-defined. Since $a = \pm p$ (see (3.5)), and n, a are relatively prime, the period of h_0 is n . We shall denote h_0 by $h(a, c, b, d)$.

LEMMA 3.6. *h is weakly equivalent to $h_0 = h(a, c, b, d)$.*

PROOF. The orbit spaces $L/\langle h \rangle$ and $L_0/\langle h_0 \rangle$ are given as $D^2 \times S^1 \cup_f S^1 \times D^2$ and $D^2 \times S^1 \cup_{f_0} S^1 \times D^2$, respectively. By the proof of Lemma 3.4, f' and f'_0 are isotopic. Therefore, there exists a homeomorphism $t': L/\langle h \rangle \rightarrow L_0/\langle h_0 \rangle$ such that $t'(D^2 \times S^1) = D^2 \times S^1$ and $t'(0 \times S^1) = 0 \times S^1$. Since $S^1 \times D^2$ is a deformation retract of $L - \text{Fix}(h)$ (and $L_0 - \text{Fix}(h_0)$), one may obtain a lifting homeomorphism t of L to L_0 (by the lifting theorem) such that $tht^{-1} = e(h_0)$ where e is an automorphism of $\langle h_0 \rangle$.

(3.7). We observe that there exists such a map h of period n on $L = L(p, q)$ (p and n are relatively prime). Since $(p, nq) = 1$, there exist integers r, s such that $pr - (ns)q = 1$. Consider the homeomorphism $\tilde{h} = h(p, q, ns, r)$ of $L(p, q, ns, r)$. Since $L(p, q, ns, r) \approx L(p, ns)$ and $L(p, ns) \approx L(p, q)$, there exists a homeomorphism t of L to $L(p, q, ns, r)$. Then the homeomorphism $t^{-1}\tilde{h}t$ is a homeomorphism as desired.

By Lemma 3.6, our classification problem now depends on classifying the homeomorphisms $h(a, c, b, d)$ for various possible integers a, b, c and d , where $ad - bc = \pm 1$. Our information on the integers is that $a = \pm p$ and $b \equiv 0 \pmod{n}$. If $h(a, c, b, d)$ is weakly equivalent to $h(a', c', b', d')$, we shall denote the fact by $h(a, c, b, d) \sim h(a', c', b', d')$.

LEMMA 3.8. (1) $h(a, c, b, d) \sim h(-a, c, -b, d)$.

(2) $h(a, c, b, d) \sim h(a, -c, b, -d)$.

PROOF. For (1), define t of $L(a, c, b, d)$ to $L(-a, c, -b, d)$ by $t(\rho z_1, z_2) =$

$(\rho z_1^{-1}, z_2)$ on $D^2 \times S^1$ and $t(z_1, \rho z_2) = (z_1, \rho z_2)$ on $S^1 \times D^2$. It is checked that t is well-defined and $th_1 = h_2^{-1}t$, where $h_1 = h(a, c, b, d)$ and $h_2 = h(-a, c, -b, d)$. For (2), define t of $L(a, c, b, d)$ to $L(a, -c, b, -d)$ by $t(\rho z_1, z_2) = (\rho z_1, z_2^{-1})$ on $D^2 \times S^1$ and $t(z_1, \rho z_2) = (z_1, \rho z_2)$ on $S^1 \times D^2$. One may check that t is a well defined equivalence.

By the above lemma, we may assume that $a = p$ and $|\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}| = 1$.

LEMMA 3.9. For any integers b, c, d with $pd - cb = 1$,

- (1) $h(p, c, b, d) \sim h(p, c', b, d')$ for any integers c' and d' with $pd' - c'b = 1$.
- (2) $h(p, c, b, d) \sim h(p, c, b + mp, d + mc)$.
- (3) $h(p, c, b, d) \sim h(p, -c, -b, d)$.

PROOF. Let $L_1 = L(p, b, c, d)$ and L_2 is the space corresponding to the weakly equivalent homeomorphism claimed in (i), $i = 1, 2, 3$. In (1), since $pd - bc = 1 = pd' - bc'$, $c' = c + mp$, $d' = d + mb$ for some integer m . Define $t: L_1 \rightarrow L_2$ by

$$\begin{aligned} t(\rho z_1, z_2) &= (\rho z_1 z_2^{-m}, z_2) && \text{on } D^2 \times S^1, \\ t(z_1, \rho z_2) &= (z_1, \rho z_2) && \text{on } S^1 \times D^2. \end{aligned}$$

For (2), define $t: L_1 \rightarrow L_2$ by

$$\begin{aligned} t(\rho z_1, z_2) &= (\rho z_1, z_2) && \text{on } D^2 \times S^1, \\ t(z_1, \rho z_2) &= (z_1, \rho z_2 z_1^m) && \text{on } S^1 \times D^2. \end{aligned}$$

Notice that $m \equiv 0 \pmod n$ by Lemma 3.5. For (3), define $t: L_1 \rightarrow L_2$ by

$$\begin{aligned} t(\rho z_1, z_2) &= (\rho z_1, z_2^{-1}) && \text{on } D^2 \times S^1 \\ t(z_1, \rho z_2) &= (z_1, \rho z_2^{-1}) && \text{on } S^1 \times D^2. \end{aligned}$$

It is checked that those t are well-defined equivalences.

Now we are in a position to prove Theorem A.

(3.10). PROOF OF THEOREM A. By (3.7) and Lemma 3.8, we will consider two maps $h_1 = h(p, c, b, d)$ and $h_2 = h(p, c', b', d')$, where $pd - cb = 1 = pd' - c'b'$. Since $L(p, c, b, d) \approx L(p, b)$ and $L(p, c', b', d') \approx L(p, b')$, we see that $b \equiv \pm b'$ or $bb' \equiv \pm 1 \pmod p$. If $b' \equiv \pm b \pmod p$, it follows from Lemma 3.9 that $h_1 \sim h_2$.

Case 1. $b^2 \equiv \pm 1 \pmod p$ (symmetric case). Then $b \equiv \pm b' \pmod p$, and $h_1 \sim h_2$.

Case 2. $b^2 \not\equiv \pm 1 \pmod p$ (nonsymmetric case). If $b \equiv \pm b' \pmod p$, then $h_1 \sim h_2$. Now suppose that $bb' \equiv \pm 1 \pmod p$ (note that such b and b' exist in this case; see (3.7) and use the fact $(p, nb) = 1$). Then $b \not\equiv \pm b' \pmod p$. Notice that if $bb'' \equiv \pm 1 \pmod p$ for some b'' , then $b'' \equiv \pm b' \pmod p$ (since $bc \equiv \pm 1 \pmod p$). Therefore, if we show that h_1 is not weakly equivalent to h_2 ,

then we are done. Suppose that there exists a weak equivalence t between h_1 and h_2 . Since $h_2(t(D^2 \times S^1)) = t(D^2 \times S^1)$ and $t(0 \times S^1) = 0 \times S^1$, we may assume that $t(D^2 \times S^1) = D^2 \times S^1$. Consider the following commutative diagram.

$$\begin{array}{ccc}
 D^2 \times S^1 \supset S^1 \times S^1 & \xrightarrow{\begin{pmatrix} p & c \\ b & d \end{pmatrix}} & S^1 \times S^1 \subset S^1 \times D^2 \\
 \downarrow t_1 & & \downarrow t_2 \\
 D^2 \times S^1 \supset S^1 \times S^1 & \xrightarrow{\begin{pmatrix} p' & c' \\ b' & d' \end{pmatrix}} & S^1 \times S^1 \subset S^1 \times D^2
 \end{array}$$

where $t_1 = t|_{S^1 \times S^1 \subset D^2 \times S^1}$ and $t_2 = t|_{S^1 \times S^1 \subset S^1 \times D^2}$. By the way we defined the generators $(1, 0)$ and $(0, 1)$ of $\pi_1(S^1 \times S^1)$, we may assume that $t_{1\#}$ and $t_{2\#}$ are given by

$$\begin{pmatrix} \pm 1 & k \\ 0 & \pm 1 \end{pmatrix} \text{ and } \begin{pmatrix} \pm 1 & 0 \\ k' & \pm 1 \end{pmatrix}$$

for some k, k' , respectively. Therefore, by the above diagram, we see that $b \equiv \pm b' \pmod p$, which is a contradiction. This completes the proof of Theorem A.

II. PROOF OF THEOREM B. $\text{Fix}(h)$ is a disjoint union of two simple closed curves. Let $F = \text{Fix}(h)$. Let $g: S^3 \rightarrow L$ be the natural projection. Since $\pi_1(L - F) = Z \oplus Z$, and there is an obvious short exact sequence

$$0 \rightarrow \pi_1(S^3 - \tilde{F}) \rightarrow Z \oplus Z \rightarrow Z_p \rightarrow 0,$$

$\pi_1(S^3 - \tilde{F}) \cong Z \oplus Z$ where $\tilde{F} = g^{-1}(F)$. Therefore $S^3 - \tilde{F}$ is irreducible (see Theorem 2 of [15]) (of course, \tilde{F} turns out to be a disjoint union of two simple closed curves). Hence, $L - F$ is irreducible. Let U_i ($i = 1, 2$) be a regular neighborhood of each component of $\text{Fix}(h)$ such that $U_1 \cap U_2 = \emptyset$. By Stallings [20] (see the proof of Lemma 2.2), we have

LEMMA 3.11. $\text{cl}(L - U_1 - U_2)$ is homeomorphic to $\partial U_1 \times I$.

By the above lemma, L may be given as $L = D^2 \times S^1 \cup_k S^1 \times D^2$ for some attaching map k of $S^1 \times S^1$, where $D^2 \times S^1$ is an invariant regular neighborhood of a component F_1 of $\text{Fix}(h)$ such that $D^2 \times S^1 \cap \text{Fix}(h) = F_1$. Let $h_1 = h|_{D^2 \times S^1}$ and $h_2 = h|_{S^1 \times D^2}$. By Proposition 2.4, we may assume that

$$L = D^2 \times S^1 \cup_f S^1 \times D^2$$

for an appropriate attaching map f of $S^1 \times S^1$ and h is given by

$$e(h_1)(\rho z_1, z_2) = (\rho\omega z_1, z_2) \quad \text{on } D^2 \times S^1 \text{ and}$$

$$e'(h_2)(z_1, \rho z_2) = (z_1, \rho\omega z_2) \quad \text{on } S^1 \times D^2,$$

where $\omega = e^{2\pi i/n}$, and e, e' are some automorphisms of $\langle h_1 \rangle$ and $\langle h_2 \rangle$, respectively. Let $(1, 0)$ and $(0, 1)$ be the canonical generators of $\pi_1(S^1 \times S^1)$. Let $f_{\#} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$. Then $a = \pm p$ and $|\begin{vmatrix} a & c \\ b & d \end{vmatrix}| = \pm 1$.

PROPOSITION 3.12. $a \equiv 0 \pmod n$.

PROOF. The orbit space $L/\langle h \rangle$ may be given as $D^2 \times S^1 \cup_f S^1 \times D^2$. An easy computation shows that the isotopy class of f' is determined by the integers $a/n, b, c$, and nd (cf. the proof of Lemma 3.4).

COROLLARY 3.13. *Let T be a generator of Z_{2^n} acting on P^3 . If $\text{Fix}(T^{2^{n-1}}) \neq \emptyset$, $T^{2^{n-1}}$ is the only element of Z_{2^n} with disconnect fixed-point set.*

PROOF. Since $T^{2^{n-1}}$ is an involution, we see that if $\text{Fix}(T^{2^{n-1}}) \neq \emptyset$, it is a disjoint union of two simple closed curves [5]. If $\text{Fix}(T^i)$ is not connected for some $i \neq 2^{n-1}$, $\text{Fix}(T^{2^{n-2}})$ is not connected. Let $T' = T^{2^{n-2}}$. Then the period of T' is 4. By (3.3) and Proposition 3.12, we arrive at a contradiction.

Define a homeomorphism h_0 on $L_0 = L(a, c, b, d)$ by

$$h_0(\rho z_1, z_2) = (\rho\omega z_1, z_2) \quad \text{on } D^2 \times S^1 \text{ and}$$

$$h_0(z_1, \rho z_2) = (z_1, \rho\omega^b z_2) \quad \text{on } S^1 \times D^2,$$

where $\omega = e^{2\pi i/n}$ (see (3.5) for the definition of L_0). Denote h_0 by $h(a, c, b, d)$. By Proposition 3.12, h_0 is a well-defined homeomorphism of period n . By the proof of Proposition 3.12 and in a similar way to the proof of Lemma 3.6, we claim that h and h_0 are weakly equivalent. For, in the proof of Lemma 3.6, t' may be chosen in a way that $t'(\rho z_1, z_2) = (\rho z_1, z_2)$ on $D^2 \times S^1$ and $(D^2 - 0) \times S^1$ is a deformation retract of $L - \text{Fix}(h)$ (and $L_0 - \text{Fix}(h_0)$). Now the claim follows easily. Therefore, we reduced our classification problem to that of all homeomorphisms $h(a, c, b, d)$ for various possible a, c, b, d , where $a = \pm p$ and $ad - cb = \pm 1$.

LEMMA 3.14. *The same statements as in Lemma 3.8 and Lemma 3.9 are also true. Furthermore, $h_1 = h(p, c, b, d) \sim h_2 = h(p, -b, -c, d)$ with $pd - bc = 1$.*

PROOF. As the proofs are similar to those of Lemma 3.9 and Lemma 3.10, we omit the proofs except the last claim (for the proof of the second statement of Lemma 3.10, use the fact that $p \equiv 0 \pmod n$). For the last claim, define t of $L(a, c, b, d)$ to $L(a, -b, -c, d)$ by $t(\rho z_1, z_2) = (z_2, \rho z_1)$ on $D^2 \times S^1$ and $t(z_1, \rho z_2) = (\rho z_2, z_1)$ on $S^1 \times D^2$ such that $t(D^2 \times S^1) = S^1 \times D^2$. It is checked that t is well defined and $th_1 = h_2^b t$ (use the fact that $p \equiv 0 \pmod n$ and $pd - cb = 1$).

(3.15). **PROOF OF THEOREM B.** By Lemma 3.14, it is enough to consider two homeomorphisms $h_1 = h(p, c, b, d)$ and $h_2 = h(p, c', b', d')$ with $pd - cb = 1 = pd' - c'b'$. Since $L(p, c, b, d) \approx L(p, c', b', d')$, $b \equiv \pm b'$ or $bb' \equiv \pm 1 \pmod{p}$. If $b \equiv \pm b' \pmod{p}$, $b = \pm b' + mp$ for some m . By Lemma 3.14, $h_1 \sim h_2$. If $bb' \equiv \pm 1 \pmod{p}$, we see that $b' \equiv \pm c \pmod{p}$ (note that $pd - cb = 1$). Again by Lemma 3.14, $h_1 \sim h_2$. This completes the proof. (For existence of such h , let s, r be integers such that $ps - qr = 1$. Then since $p \equiv 0 \pmod{n}$, we can define a map $h' = h(p, r, q, s)$. Let t be a homeomorphism of $L(p, q)$ to $L(p, r, q, s)$. Then $t^{-1}h't$ is the homeomorphism as desired.)

4. Z_4 -actions on $L(p, q)$. In the present section we classify all orientation-preserving nonfree Z_4 -actions on $L = L(p, q)$.

Throughout this section, h will always denote an orientation-preserving homeomorphism of period 4 on L with $\text{Fix}(h^2) \neq \emptyset$. A subset S of L will be called h^i -invariant if S is invariant under h^i . We denote the number of components of $\text{Fix}(h^i)$ by $c(\text{Fix}(h^i))$. Notice that $c(\text{Fix}(h^i)) \leq 2$ (see §3).

It is natural to consider the following five possible cases:

- (1) $c(\text{Fix}(h)) = 0, c(\text{Fix}(h^2)) = 2.$
- (2) $c(\text{Fix}(h)) = 0, c(\text{Fix}(h^2)) = 1.$
- (3) $c(\text{Fix}(h)) = 1, c(\text{Fix}(h^2)) = 2.$
- (4) $c(\text{Fix}(h)) = 1, c(\text{Fix}(h^2)) = 1.$
- (5) $c(\text{Fix}(h)) = 2, c(\text{Fix}(h^2)) = 2.$

Since cases (4) and (5) are special types of Theorems A and B (note that Lemma 3.4 is still valid in case (4), and p is odd in this case), we just investigate cases (1), (2), and (3). In the following we divide the section into three parts, A, B, and C, according to the three cases.

A. We assume that $c(\text{Fix}(h)) = 0$ and $c(\text{Fix}(h^2)) = 2$. Let F_1 and F_2 be the components of $\text{Fix}(h^2)$. Since $\text{Fix}(h) = \emptyset$, we see that either $h(F_i) = F_i$ ($i = 1, 2$) or $h(F_1) = F_2$. The case where $h(F_i) = F_i$ and the case where $h(F_1) = F_2$ will be studied in I and II, following separately.

I. $h(F_1) = F_2$.

LEMMA 4.1. *There exists an h -invariant torus S embedded in L such that S separates L into two solid tori.*

PROOF. There exists an h^2 -invariant regular neighborhood N of F_1 in L such that $N \cap h(N) = \emptyset$. Since h^2 is a sense-preserving involution (see (3.3)), $\text{cl}(L - N)$ is homeomorphic to $S^1 \times D^2$, and $(\text{cl}(L - N), F_2) \approx (S^1 \times D^2, S^1 \times 0)$ (see (2.5)). Obviously $\text{cl}(L - N - h(N))$ is homeomorphic to $S^1 \times S^1 \times I$. Let g be the orbit map generated by h^2 . Let $U = \text{cl}(L - N - h(N))$. Then $g(U)$ is homeomorphic to $S^1 \times S^1 \times I$ (see the proof of Lemma 2.2). Let \bar{h} be the homeomorphism on $L/\langle h^2 \rangle$ induced by h . Then \bar{h} is a free

involution and \bar{h} interchanges the two boundary components of $g(U)$. Therefore, there exists an \bar{h} -invariant torus A in $\text{Int}(g(U))$ such that A is parallel to each component of $\partial g(U)$ (i.e., $(g(U), A) \approx (S^1 \times S^1 \times I, S^1 \times S^1 \times \{\frac{1}{2}\})$) (see [22]). Since $g^{-1}(A)$ is invariant under h and h^2 , we see that $g^{-1}(A)$ is connected (note that g is induced by h^2). Therefore, $g^{-1}(A)$ is a torus and it is parallel to each component of ∂U . Thus, $g^{-1}(A)$ is the torus as desired.

(4.2). By the proof of Lemma 4.1, L may be given as

$$L = D^2 \times S^1 \cup_k S^1 \times S^1 \times [-1, 1] \cup_k S^1 \times D^2,$$

where k (k' , resp.) is an attaching map of $S^1 \times S^1 \times \{1\}$ ($S^1 \times S^1 \times \{-1\}$, resp.) to $S^1 \times S^1 \subset S^1 \times D^2$ ($S^1 \times S^1 \subset D^2 \times S^1$, resp.), so that h interchanges $D^2 \times S^1$ and $S^1 \times D^2$, and $S^1 \times S^1 \times [-1, 0]$ and $S^1 \times S^1 \times [0, 1]$.

LEMMA 4.3. *Let T be an orientation-preserving homeomorphism of period 4 on $S^1 \times S^1 \times [-1, 1]$ such that $\text{Fix}(T^2) = \emptyset$ and T interchanges the two boundary components. Then T is weakly equivalent to a homeomorphism T' given by either $T'(z_1, z_2, t) = (iz_1, \bar{z}_2, -t)$ or $T'(z_1, z_2, t) = (iz_1z_2, \bar{z}_2, -t)$ for each $(z_1, z_2, t) \in S^1 \times S^1 \times [-1, 1]$.*

PROOF. By the proof of Lemma 4.1, we may assume that $T(S^1 \times S^1 \times \{0\}) = S^1 \times S^1 \times \{0\}$. Since $S^1 \times S^1 \times [0, 1]$ is h^2 -invariant, there exists a deformation of the product fibering $S^1 \times S^1 \times [0, 1]$, constant on $S^1 \times S^1 \times \{0\}$, after which $h^2(x, t) = (f(x), t)$ for each $x \in S^1 \times S^1$ and $t \in [0, 1]$ where f is an involution of $S^1 \times S^1$ (see [22]). Since the deformation is chosen constant on $S^1 \times S^1 \times \{0\}$ and h interchanges $S^1 \times S^1 \times [-1, 0]$ and $S^1 \times S^1 \times [0, 1]$, one can define a product structure on $S^1 \times S^1 \times [-1, 1]$ such that $h(x, t) = (\phi(x), -t)$ for each $x \in S^1 \times S^1$ and $t \in [-1, 1]$, where ϕ is an orientation-reversing homeomorphism of period 4 on $S^1 \times S^1$ and $\phi^2 = f$.

Now if we show that ϕ is conjugate to a map ϕ_1 or ϕ_2 where $\phi_1(z_1, z_2) = (iz_1, \bar{z}_2)$ and $\phi_2(z_1, z_2) = (iz_1z_2, \bar{z}_2)$ for each $(z_1, z_2) \in S^1 \times S^1$, then we are done. Notice that the orbit space $S^1 \times S^1 / \langle \phi \rangle$ is the Klein bottle K . Consider an exact sequence $\pi_1(K) \rightarrow Z_4 \rightarrow 0$, where $\pi_1(K) = \{x, y | xyx^{-1}y\}$ and $Z_4 = \langle a \rangle$. Since the sequence can be factored as

$$\begin{array}{ccccc} \pi_1(K) & \xrightarrow{k} & Z_4 & \longrightarrow & 0 \\ & & \nearrow & & \\ & k' \downarrow & & & \\ & H_1(K) & & & \end{array}$$

we have $k(x) = a$ or a^{-1} , and $k(y) = a^2$ or 1 . Therefore, we see that $\ker(k)$ is

given by either $A_1 = [x^4, y]$ or $A_2 = [x^4, x^2y]$, where $[x^4, y]$ is the smallest subgroup of $\pi_1(K)$ generated by $\{x^4, y\}$ (note that x^2 and y commute in $\pi_1(K)$, and A_1, A_2 are normal subgroups). Since $k'(A_1)$ has an element of order 2 but $k'(A_2)$ does not, A_1 is not automorphic to A_2 . Now it is not difficult to see that $g_{j\#}(\pi_1(S^1 \times S^1))$ ($j = 1, 2$) is automorphic to the subgroup A_j where g_j is the orbit map generated by the map ϕ_j . This completes the proof.

By Proposition 2.4, (4.2) and Lemma 4.3, we may assume that

$$L = D^2 \times S^1 \cup_{\alpha} S^1 \times S^1 \times [-1, 1] \cup_{\beta} S^1 \times D^2$$

for appropriate attaching maps α and β (we shall denote this L by $L[\alpha, \beta]$), and $S^1 \times S^1 \times [-1, 1]$ is h -invariant and h interchanges $S^1 \times D^2$ and $D^2 \times S^1$ such that

$$h^2(\rho z_1, z_2) = (-\rho z_1, z_2) \quad \text{on } D^2 \times S^1,$$

$$h^2(z_1, \rho z_2) = (z_1, -\rho z_2) \quad \text{on } S^1 \times D^2, \quad \text{and}$$

$$h(z_1, z_2, t) = (iz_1, \bar{z}_2, -t) \quad \text{on } S^1 \times S^1 \times [-1, 1]$$

$$(\text{or } h(z_1, z_2, t) = (iz_1 z_2, \bar{z}_2, -t)).$$

If h is defined on $L[\alpha, \beta]$ in the above manner and $h(z_1, z_2, t) = (iz_1, \bar{z}_2, -t)$ on $S^1 \times S^1 \times [-1, 1]$ ($h(z_1, z_2, t) = (iz_1 z_2, \bar{z}_2, -t)$, resp.), this h will be said to be of type (α, β) (type $(\alpha, \beta)^*$, resp.). Any two homeomorphisms of types (α, β) and $(\alpha', \beta')^*$ are not conjugate (see the proof of Lemma 4.3). In fact, in this case $L[\alpha, \beta]$ is not even homeomorphic to $L[\alpha', \beta']$ (see Lemma 4.11).

Let $(1, 0)$ and $(0, 1)$ be the canonical generators of $\pi_1(S^1 \times S^1)$. Let $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ be the matrix of β . Then

$$\begin{vmatrix} a & c \\ b & d \end{vmatrix} = \pm 1.$$

The proof of Lemma 4.5 is based on homeomorphisms of type (α, β) . A similar proof goes through for the case of type $(\alpha, \beta)^*$ (in this case one can observe that the homomorphism $f_{i\#}$ in the proof may be given by $f_{i\#}[(1, 0)] = (2, 0)$ and $f_{i\#}[(0, 1)] = (1, 1)$, and the matrix of β'_i may be given by

$$\begin{pmatrix} a/2 & c - a/2 \\ b & 2d - b \end{pmatrix}.$$

LEMMA 4.4. a is even.

PROOF. See the proof of Lemma 4.5.

LEMMA 4.5. Suppose that h_i ($i = 1, 2$) is of type (α_i, β_i) (or $(\alpha_i, \beta_i)^*$). If the

matrices of β_1 and β_2 are the same, then $L[\alpha_1, \beta_1] \approx L[\alpha_2, \beta_2]$, and h_1 is weakly equivalent to h_2 .

PROOF. Let g_i be the orbit map generated by $\langle h_i^2 \rangle$. We may assume that $L[\alpha_i, \beta_i] / \langle h_i^2 \rangle$ is given as $L[\alpha'_i, \beta'_i]$ for some α'_i, β'_i such that

$$g_i(S^1 \times S^1 \times [-1, 1]) = S^1 \times S^1 \times [-1, 1]$$

and

$$g_i(S^1 \times D^2) = S^1 \times D^2.$$

(and $g_i(S^1 \times 0) = S^1 \times 0$). Let

$$f_i = g_i|_{S^1 \times S^1 \times \{1\}} \quad \text{and} \quad f'_i = g_i|_{S^1 \times S^1 \subset S^1 \times D^2}.$$

Then we may further assume that $f_{i\#}[(1, 0)] = (2, 0)$ and $f_{i\#}[(0, 1)] = (0, 1)$, and $f'_{i\#}[(1, 0)] = (1, 0)$ and $f'_{i\#}[(0, 1)] = (0, 2)$. Let $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ be the matrix of β_i . A simple computation shows that the matrix of β'_i may be given by $\begin{pmatrix} a/2 & c \\ b & 2d \end{pmatrix}$. Let \bar{h}_i be the free involution on $L[\alpha'_i, \beta'_i]$ induced by h_i . We may assume that

$$\bar{h}_i(S^1 \times D^2) = D^2 \times S^1 \quad (\text{and } \bar{h}_i(S^1 \times 0) = 0 \times S^1),$$

$$\bar{h}_i(S^1 \times S^1 \times [0, 1]) = S^1 \times S^1 \times [-1, 0] \quad \text{and}$$

$$\bar{h}_i|_{S^1 \times S^1 \times [-1, 1]} = \bar{h}_2|_{S^1 \times S^1 \times [-1, 1]}$$

(see the definition of h_i).

We will define a homeomorphism t of $L[\alpha'_1, \beta'_1]$ to $L[\alpha'_2, \beta'_2]$ such that $t\bar{h}_1 = \bar{h}_2t$, and $t(0 \times S^1) = 0 \times S^1$ and $t(S^1 \times 0) = S^1 \times 0$, where $0 \times S^1 \subset D^2 \times S^1$ and $S^1 \times 0 \subset S^1 \times D^2$. Since β'_1 and β'_2 are isotopic (see (3.5)), there exists t_1 of $S^1 \times S^1 \times [0, 1] \cup_{\beta'_1} S^1 \times D^2$ to $S^1 \times S^1 \times [0, 1] \cup_{\beta'_2} S^1 \times D^2$ such that t_1 sends $S^1 \times S^1 \times [0, 1]$ to $S^1 \times S^1 \times [0, 1]$ by $t_1(z_1, z_2, t) = (z_1, z_2, t)$, and $t_1(0 \times S^1) = 0 \times S^1$. Let t be the homeomorphism of $L[\alpha'_1, \beta'_1]$ to $L[\alpha'_2, \beta'_2]$ defined by $t = t_1$ on $S^1 \times S^1 \times [0, 1] \cup_{\beta'_1} S^1 \times D^2$ and $t = \bar{h}_2 t_1 \bar{h}_1$ on $D^2 \times S^1 \cup_{\alpha'_1} S^1 \times S^1 \times [-1, 0]$. This t is well defined (recall that $\bar{h}_1|_{S^1 \times S^1 \times \{0\}} = \bar{h}_2|_{S^1 \times S^1 \times \{0\}}$), and it is the map as desired.

Let $t' = t|_{S^1 \times S^1 \times \{1\}}$. Then we see that $t'_\# f_{1\#} = f_{2\#}$. Since $S^1 \times S^1 \times \{1\}$ is a deformation retract of

$$(D^2 - 0) \times S^1 \cup_{\alpha_1} S^1 \times S^1 \times [-1, 1] \cup_{\beta_1} S^1 \times (D^2 - 0)$$

(we denote this by $L'[\alpha_i, \beta_i]$), it follows from the lifting theorem that there exists a homeomorphism \bar{t} of $L'[\alpha_1, \beta_1]$ to $L'[\alpha_2, \beta_2]$ such that $g_2 \bar{t} = \bar{t} g_1$ on $L'[\alpha_1, \beta_1]$. Since $\text{Fix}(h^2) = 0 \times S^1 \cup S^1 \times 0 \subset L[\alpha_i, \beta_i]$, and $t(0 \times S^1) = 0 \times S^1$ and $t(S^1 \times 0) = S^1 \times 0$ in $L[\alpha'_i, \beta'_i]$, one can extend \bar{t} to a homeomorphism \bar{t}' of $L[\alpha_1, \beta_1]$ to $L[\alpha_2, \beta_2]$ such that $g_2 \bar{t}' = \bar{t}' g_1$. Therefore, we see that h_1 and h_2 are weakly equivalent. This completes the proof.

(4.6). According to Lemma 4.5, if h is of type (α, β) (or $(\alpha, \beta)^*$), then h is

completely determined (up to weak equivalence) by the matrix of β . Therefore, we may assume that

$$L = D^2 \times S^1 \cup_{\alpha} S^1 \times S^1 \times [-1, 1] \cup_{\beta} S^1 \times D^2$$

where $\beta(z_1, z_2, 1) = (z_1^a z_2^c, z_1^b z_2^d)$ and $\alpha(z_1, z_2, -1) = (z_1^b z_2^{-d}, z_1^a z_2^{-c})$ (or $\alpha(z_1, z_2, -1) = (z_1^b z_2^{b-d}, z_1^a z_2^{a-c})$ for $(\alpha, \beta)^*$), and h is defined by

$$h(\rho z_1, z_2) = (i^a z_2, \rho i^b z_1) \text{ on } D^2 \times S^1,$$

$$h(z_1, z_2, t) = (iz_1, z_2^{-1}, -t) \text{ on } S^1 \times S^1 \times [-1, 1]$$

(or $h(z_1, z_2, t) = (iz_1 z_2, z_2^{-1}, -t)$ for $(\alpha, \beta)^*$), and

$$h(z_1, \rho z_2) = (\rho i^b z_2, i^a z_1) \text{ on } S^1 \times D^2,$$

where $h(D^2 \times S^1) = S^1 \times D^2$ and $S^1 \times S^1 \times [-1, 1]$ is h -invariant. It is checked that this h is well-defined and of period 4 (recall that a is even (Lemma 4.4), and therefore b is odd). The differences between the two new homeomorphisms h of types (α, β) and $(\alpha, \beta)^*$ have been indicated by inserting parentheses. We shall denote this homeomorphism h of type (α, β) (type $(\alpha, \beta)^*$, resp.) by $h^\dagger[a, c, b, d]$ ($h^*[a, c, b, d]$, resp.) and the corresponding L by $L^\dagger[a, c, b, d]$ ($L^*[a, c, b, d]$, resp.). We let $h[a, c, b, d]$ denote either $h^\dagger[a, c, b, d]$ or $h^*[a, c, b, d]$. In what follows $h[a, c, b, d] \sim h[a', c', b', d']$ means $h^\dagger[a, c, b, d] \sim h^\dagger[a', c', b', d']$ and $h^*[a, c, b, d] \sim h^*[a', c', b', d']$.

LEMMA 4.7. (1) $h[a, c, b, d] \sim h[-a, -c, b, d]$.

(2) $h[a, c, b, d] \sim h[a, c, -b, -d]$.

PROOF. For (1), define t of $L[a, c, b, d]$ to $L[-a, c, -b, d]$ by $t(\rho z_1, z_2) = (\rho z_1, z_2^{-1})$ on $D^2 \times S^1$, $t(z_1, z_2, t') = (z_1, z_2, t')$ on $S^1 \times S^1 \times [-1, 1]$, and $t(z_1, \rho z_2) = (z_1^{-1}, \rho z_2)$ on $S^1 \times D^2$. For (2), define t of $L[a, c, b, d]$ to $L[a, -c, b, -d]$ by $t(\rho z_1, z_2) = (\rho z_1^{-1}, z_2)$ on $D^2 \times S^1$, $t(z_1, z_2, t') = (z_1, z_2, t')$ on $S^1 \times S^1 \times [-1, 1]$, and $t(z_1, \rho z_2) = (z_1, \rho z_2^{-1})$ on $S^1 \times D^2$. It is checked that these t 's are well-defined equivalences.

LEMMA 4.8. (1) $h^\dagger[a, c, b, d] \sim h^\dagger[a, -c, -b, d]$ and (2) $h[a, c, b, d] \sim h[a, c, b', d']$, where $ad - cb = ad' - cb' = 1$.

PROOF. For (1), define t of $L[a, c, b, d]$ to $L[a, -c, -b, d]$ by $t(\rho z_1, z_2) = (\rho z_1^{-1}, z_2)$ on $D^2 \times S^1$, $t(z_1, z_2, t') = (z_1, z_2^{-1}, t')$ on $S^1 \times S^1 \times [-1, 1]$, and $t(z_1, \rho z_2) = (z_1, \rho z_2^{-1})$ on $S^1 \times D^2$. In (2), since $ad - cb = ad' - cb'$ we see that $d' = d - mc$ and $b' = b - ma$ for some m . Define t of $L[a, c, b, d]$ to $L[a, c, b', d']$ by $t(\rho z_1, z_2) = (\rho z_1 z_2^{-m}, z_2)$ on $D^2 \times S^1$, $t(z_1, z_2, t') = (z_1, z_2, t')$ on $S^1 \times S^1 \times [-1, 1]$, and $t(z_1, \rho z_2) = (z_1, \rho z_2 z_1^{-m})$ on $S^1 \times D^2$. It is checked that these t 's are well-defined equivalences.

LEMMA 4.9. (1) $L^\dagger = L^\dagger[a, c, b, d]$ is homeomorphic to $L(2ac, ad + bc)$. (2) $L^* = L^*[a, c, b, d]$ is homeomorphic to $L(2ac - a^2, ad + bc - ab)$. (3) L^\dagger and L^* are symmetric.

PROOF. Let α, β be the maps in (4.6). Then L may be viewed as $D^2 \times S^1 \cup_{\beta\alpha^{-1}} S^1 \times D^2$ in the obvious way. The result follows from the matrix of $\beta\alpha^{-1}$ and (3.5).

One can observe the following by an elementary argument, and we omit the proof.

LEMMA 4.10. Let t, s, t', s' be integers such that $ts = \pm t's'$ and $(t, s) = 1 = (t', s')$. If $tw + vs \equiv \pm(t'w' + v's') \pmod{2ts}$ where $tw - vs = 1 = t'w' - v's'$, then either $t = \pm t'$ or $t = \pm s'$.

LEMMA 4.11. Let a and a' be even integers. Then $L^\dagger[a, c, b, d]$ and $L^*[a', c', b', d']$ are not homeomorphic for any possible integers.

PROOF. By the proof of Lemma 4.7, we may assume that a and a' are positive even integers and $ad - bc = 1 = a'd' - b'c'$. Suppose the contrary that they were homeomorphic. Then, by Lemma 4.9, we see that

$$\pm ac = a'(c' - a'/2)$$

and

$$\pm(ad + bc) \equiv a'(2d' - b')/2 + b'(c' - a'/2) \pmod{2ac}$$

(since L^* is symmetric). Therefore, it follows from Lemma 4.10 that either $c = \pm a'/2$ and $a = \pm 2(c' - a'/2)$ or $c = \pm(c' - a'/2)$ and $a = a'$ (let $t = a/2, t' = a'/2, s = c,$ and $s' = c' - a'/2$). But, in this case an elementary argument shows that these two possibilities cannot occur (using the above relations and the fact that

$$ad - bc = 1 = a'(2d' - b')/2 - b'(c' - a'/2),$$

one can see that

$$\pm(2ad - 1) \equiv 1 \pm b'a \quad \text{or} \quad a(2d' - b') - 1 \pmod{2ac};$$

therefore, $a = 2$ or b' is even, which leads to a contradiction; note that a is divisible by 4 in this case).

LEMMA 4.12. Let a and a' be positive even integers and $ad - bc = 1 = a'd' - b'c'$.

(1) $L_1 = L^\dagger[a, c, b, d]$ is homeomorphic to $L_2 = L^\dagger[a', c', b', d']$ if and only if $a' = a$ and $c' = \pm c$.

(2) $L_1 = L^*[a, c, b, d]$ is homeomorphic to $L_2 = L^*[a', c', b', d']$ if and only if one of the following holds: (i) $a' = a$ and either $c' = c$ or $c' = a - c$. (ii) If $2c - a > 0$, then $a' = 2c - a$ and either $c' = c$ or $c' = c - a$ (if $a - 2c > 0$, then $a' = a - 2c$ and either $c' = -c$ or $c' = a - c$).

PROOF. The following argument works for either case of the above two claims. Suppose that L_1 is homeomorphic to L_2 . Then the results follow from Lemmas 4.9 and 4.10 (see the proof of Lemma 4.11). For the converse statements, it follows from the proofs of Lemmas 4.8 and 4.9.

Now we are in a position to state our conclusion.

THEOREM 4.13. *If $c(\text{Fix}(h)) = 0$ and $c(\text{Fix}(h^2)) = 2$, and h interchanges the two components of $\text{Fix}(h^2)$, then $p \equiv 0 \pmod 4$ and q is one of the following forms;*

- (1) $q \equiv ad + bc \pmod p$, where $p = 2ac$ (a even) and $\begin{vmatrix} a & c \\ b & d \end{vmatrix} = 1$.
- (2) $q \equiv ad + bc - (ab/2) \pmod p$, where $p = 2ac$ (a even) and

$$\begin{vmatrix} a & c + a/2 \\ b & d \end{vmatrix} = 1.$$

Up to weak equivalence, there exists exactly one such h on $L(p, q)$ for p, q of form (1), and there exists exactly two such h on $L(p, q)$ for p, q of form (2).

REMARK. The homeomorphic type of $L(p, q)$ does not depend on the choice of b and d (see the proof of Lemma 4.8). For simplicity, in the proof we let the integers p, q of form (1) (form (2), resp.) denote p_1, q_1 (p_2, q_2 , resp.) Notice that $p_2 \equiv 0 \pmod 8$. It follows from Lemmas 4.9 and 4.11 that $L(p_1, q_1)$ is not homeomorphic to $L(p_2, q_2)$ for any possible p_1, q_1, p_2, q_2 .

PROOF. The first part of the theorem follows from Lemma 4.4, (4.6), and Lemma 4.9. For existence, let $h = th't^{-1}$, where $h' = h^\dagger[a, c, b, d]$ (or $h^*[a, c + a/2, b, d]$) and t is a homeomorphism of $L^\dagger[a, c, b, d]$ (or $L^*[a, c + a/2, b, d]$) to $L(p, q)$. For the classification of such h 's, since $L(p_1, q_1)$ and $L(p_2, q_2)$ are not homeomorphic (see the above note), we consider them separately.

Case 1. $L \approx L(p_1, q_1)$. It follows from (4.6) and Lemmas 4.7 and 4.12 that it is enough to show that $h^\dagger[a, c, b, d] \sim h^\dagger[a, c', b', d']$, where $c' = \pm c$ and $ad - bc = 1 = ad' - b'c'$. One can see that this follows from Lemma 4.8.

Case 2. $L \approx L(p_2, q_2)$. By (4.6) and Lemma 4.7, we compare $h_1 = h^*[a, c, b, d]$ with $h_2 = h^*[a', c', b', d']$, where $ad - bc = 1 = a'd' - b'c'$ and a, a' are even. According to Lemma 4.12, we consider the following subcases.

Subcase (i). $a' = a$ and either $c' = c$ or $c' = a - c$. If $c' = c$, then $h_1 \sim h_2$ by Lemma 4.8. Now suppose that $c' = a - c$. By Lemma 4.8, we may assume that $b' = -b$ and $d' = d - b$. Define t of $L^*[a, c, b, d]$ to $L^*[a, a - c, -b, d - b]$ by

$$\begin{aligned} t(\rho z_1, z_2) &= (\rho z_1^{-1}, z_2) && \text{on } D^2 \times S^1, \\ t(z_1, z_2, t') &= (z_1 z_2, z_2^{-1}, t') && \text{on } S^1 \times S^1 \times [-1, 1], \text{ and} \\ t(z_1, \rho z_2) &= (z_1, \rho z_2^{-1}) && \text{on } S^1 \times D^2. \end{aligned}$$

It is checked that this t is a well-defined equivalence between h_1 and h_2 .

Subcase (ii). $a' = 2c - a$ and either $c' = c$ or $c' = c - a$. Since the following argument also works for the case where $a - 2c > 0$, we simply assume that $2c - a > 0$. Suppose the contrary that h_1 were weakly equivalent to h_2 . Then there exists a weak equivalence t of $L_1 = L^*[a, c, b, d]$ to $L_2 = L^*[a', c', b', d']$. Note that $t(0 \times S^1) = 0 \times S^1$ or $S^1 \times 0$ where $0 \times S^1 \subset D^2 \times S^1$ and $S^1 \times 0 \subset S^1 \times D^2$. We may assume that $t(0 \times S^1) = 0 \times S^1$ and $t(S^1 \times 0) = S^1 \times 0$ (since h_2 interchanges $S^1 \times 0$ and $0 \times S^1$). Since $h_2 t(D^2 \times S^1) = t(S^1 \times D^2)$ and $h_2 t(S^1 \times D^2) = t(D^2 \times S^1)$, we may assume that $t(D^2 \times S^1) = D^2 \times S^1$ and $t(S^1 \times D^2) = S^1 \times D^2$. Note that the matrices of $h_i|_{S^1 \times S^1 \times [-1, 1]}$ are given by $\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$. Since the matrix M_1 of $t|_{S^1 \times S^1 \times [-1, 1]}$ must commute with $\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$, M_1 is of the form

$$\begin{pmatrix} \varepsilon_1 & \gamma \\ 0 & \varepsilon_2 \end{pmatrix}$$

where each $\varepsilon_i = \pm 1$, and $\gamma = 0$ or ± 1 (note that $|M_{11}| = \pm 1$). Let M_2 be the matrix of $t|_{S^1 \times S^1 \subset S^1 \times D^2}$. Then M_2 may be given by

$$\begin{pmatrix} \varepsilon'_1 & 0 \\ \gamma' & \varepsilon'_2 \end{pmatrix}$$

where each $\varepsilon'_i = \pm 1$ and γ' is some integer (see how $(0, 1)$ is defined). Since we require that

$$M_2 \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a' & c' \\ b' & d' \end{pmatrix} M_1,$$

we see that $a' = a$, which is a contradiction to the fact that $a' = 2c - a$ and $(a, c) = 1$. Therefore, in this case h_1 and h_2 are not weakly equivalent.

Now if we show that $h^*[2c - a, c, b', d']$ is weakly equivalent to $h^*[2c - a, c - a, b'', d'']$ where $(2c - a)d' - cb' = 1 = (2c - a)d'' - (c - a)b''$, then we are done. In fact, it follows from Lemma 4.8 and the above subcase (i) that

$$\begin{aligned} h^*[2c - a, c, b', d'] &\sim h^*[2c - a, c, b - 2d, -d] \\ &\sim h^*[2c - a, c - a, 2d - b, d - b] \\ &\sim h^*[2c - a, c - a, b'', d'']. \end{aligned}$$

This completes the proof.

II. $h_i(F_i) = F_i$. Now consider the case where $h_i(F_i) = F_i$ ($i = 1, 2$). Let N be an h -invariant regular neighborhood of F_1 such that $N \cap F_2 = \emptyset$. Then $\text{cl}(L - N) \approx S^1 \times D^2$ (see (3.3)). By Theorem 2.9, we may assume that $L = D^2 \times S^1 \cup_f S^1 \times D^2$ for an appropriate attaching map f of $S^1 \times S^1$ and h is given by $h_1^j(\rho z_1, z_2) = (\rho iz_1, -z_2)$ on $D^2 \times S^1$ and $h_2^k(z_1, \rho z_2) = (-z_1, \rho iz_2)$ on $S^1 \times D^2$ where $h_1 = h|_{D^2 \times S^1}$ and $h_2 = h|_{S^1 \times D^2}$, and

$j, k = 1$ or 3 . Let $(1, 0)$ and $(0, 1)$ be the canonical generators of $\pi_1(S^1 \times S^1)$. Let g_i be the orbit maps generated by $\langle h_i \rangle$ ($i = 1, 2$). Parametrize the orbit spaces of $\langle h_1 \rangle$ and $\langle h_2 \rangle$ in terms of $D^2 \times S^1$ and $S^1 \times D^2$, respectively. Let $g'_i = g_i|_{S^1 \times S^1}$. By the proof of Theorem 2.9, we may assume that $g'_{1\#}[(1, 0)] = (2, 0)$, $g'_{2\#}[(0, 1)] = (0, 2)$, $g'_{1\#}[(0, 1)] = (r', 2)$, and $g'_{2\#}[(1, 0)] = (2, r)$ for some odd integers r', r . Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the matrix of f .

LEMMA 4.14. $a \equiv 0 \pmod{4}$.

PROOF. $L/\langle h \rangle$ may be given as

$$L/\langle h \rangle = D^2 \times S^1 \cup_{f'} S^1 \times D^2$$

where f' is induced by f . A simple computation shows that

$$\begin{aligned} a' &= a, & b' &= (2b + ar)/2, & c' &= (2c - r'a)/2, \\ d' &= (4d + 2cr - 2r'b - ar'r)/4, \end{aligned}$$

where the matrix of f' is

$$\begin{pmatrix} a' & c' \\ b' & d' \end{pmatrix}.$$

From b' and d' , we see that $a \equiv 0 \pmod{4}$.

Define a homeomorphism h_0 on $L(a, c, b, d)$ (see (3.5) for the notation) by

$$\begin{aligned} h_0(\rho z_1, z_2) &= (\rho z_1, -z_2) && \text{on } D^2 \times S^1 & \text{ and} \\ h_0(z_1, \rho z_2) &= (-z_1, \rho i^{2d+b} z_2) && \text{on } S^1 \times D^2. \end{aligned}$$

It is checked that h_0 is well defined and h is of period 4 as $ad - bc = \pm 1$.

LEMMA 4.15. h is weakly equivalent to h_0 .

PROOF. $L/\langle h \rangle$ and $L_0/\langle h_0 \rangle$ may be given as

$$D^2 \times S^1 \cup_{f'} S^1 \times D^2 \quad \text{and} \quad D^2 \times S^1 \cup_{f'_0} S^1 \times D^2,$$

respectively. Since f' and f'_0 are isotopic, there exists a homeomorphism t of $L/\langle h \rangle$ to $L_0/\langle h_0 \rangle$ such that $t(D^2 \times S^1) = D^2 \times S^1$, and $t(z_1, \rho z_2) = (z_1, \rho z_2)$ on $S^1 \times D^2$. Let g and g_0 be the orbit maps generated by $\langle h|_{S^1 \times (D^2 - 0)} \rangle$ and $\langle h_0|_{S^1 \times (D^2 - 0)} \rangle$. We may assume that $g_{\#}[(1, 0)] = (2, r)$ and $g_{0\#}[(1, 0)] = (2, r)$ for some odd r (compare $h|_{S^1 \times D^2}$ with $h_0|_{S^1 \times D^2}$). Since $t(z_1, \rho z_2) = (z_1, \rho z_2)$ on $S^1 \times D^2$, we have $t'_{\#} g_{\#} = g_{0\#}$ where $t' = t|_{S^1 \times (D^2 - 0)}$. Notice that $S^1 \times (D^2 - 0)$ is a deformation retract of $L - \text{Fix}(h^2)$ (and $L_0 - \text{Fix}(h_0^2)$). Now one may complete the proof in the same manner as in Theorem 2.9.

We now state our claim.

THEOREM 4.16. If $c(\text{Fix}(h)) = 0$ and $c(\text{Fix}(h^2)) = 2$, and h fixes each component of $\text{Fix}(h^2)$, then $p \equiv 0 \pmod{4}$. Up to weak equivalence, there exists

exactly one such h on each lens space $L(4m, q)$.

PROOF. We denote h_0 by $h(a, c, b, d)$. By Lemma 4.15, it is enough to consider all possible homeomorphisms $h(a, c, b, d)$. It is checked that the same statements as in Lemma 3.14 is also true in this case (use the same homeomorphisms t as in the lemma). Therefore, one may complete the proof by the same way as done in (3.15) (for the existence of such h , see also (3.15)).

B. We assume that $c(\text{Fix}(h)) = 0$ and $c(\text{Fix}(h^2)) = 1$. L may be given as $D^2 \times S^1 \cup_f S^1 \times D^2$ where $D^2 \times S^1$ is an h -invariant regular neighborhood of $\text{Fix}(h^2)$. Let $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ be the matrix of f .

LEMMA 4.17. $b = 2s$ for some odd s , and therefore a is odd.

PROOF. Using Theorem 2.9, one can show that b is even and $2d - br \equiv 0 \pmod{4}$ for some odd r (see the proofs of Lemmas 3.4 and 4.14). Therefore, $b \not\equiv 0 \pmod{4}$.

By proceeding as in the second part of A, we may assume that $L = L(a, c, b, d)$ and h is given by

$$\begin{aligned} h(\rho z_1, z_2) &= (\rho i z_1, z_2) && \text{on } D^2 \times S^1 \text{ and} \\ h(z_1, \rho z_2) &= (i^a z_1, -\rho z_2) && \text{on } S^1 \times D^2. \end{aligned}$$

Note that a is odd. Let us denote this h by $h(a, c, b, d)$. It is checked that the same statements as in Lemmas 3.8 and 3.9 are also true in this case (use the same homeomorphisms t as in the lemmas; for the proof of (2) of Lemma 3.9, note that $m \equiv 0 \pmod{4}$ since both b and $b + mp$ are even but not divisible by 4). Therefore, one can prove the following theorem in the same way as in (3.10) (for the existence of such h , note that there exist some integers r, s such that $pr - 2sq = 1$ since p is odd and $(p, q) = 1$; here we may assume that s odd, and use the method in (3.10)).

THEOREM 4.18. *If $c(\text{Fix}(h)) = 0$ and $c(\text{Fix}(h^2)) = 1$, then p is odd. Up to weak equivalence, there exists exactly one such h on $L = L(p, q)$ (p odd) if L is symmetric, and there exist exactly two such h if L is nonsymmetric.*

Rice [17] showed that every free Z_4 -action on S^3 is conjugate to the orthogonal action. This, along with Theorem A and the above theorem, shows that:

THEOREM 4.19. S^3 admits exactly three orientation-preserving Z_4 -actions, up to conjugation.

C. We assume that $c(\text{Fix}(h)) = 1$ and $c(\text{Fix}(h^2)) = 2$. Again L may be given as $D^2 \times S^1 \cup_f S^1 \times D^2$ where $D^2 \times S^1$ is an h -invariant regular neighborhood of $\text{Fix}(h)$. Let $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ be the matrix of f . Using Theorem 2.9, it can be seen that a is even and $2b + ar \equiv 0 \pmod{4}$ (see the proofs of Lemmas 3.4

and 4.14). Therefore, we see that $a = 2s$ for some odd s . By proceeding in the same spirit as in the previous cases, we may assume that $L = L(a, c, b, d)$ and h is given by

$$\begin{aligned} h(\rho z_1, z_2) &= (\rho i z_1, z_2) && \text{on } D^2 \times S^1 \quad \text{and} \\ h(z_1, \rho z_2) &= (-z_1, \rho i^b z_2) && \text{on } S^1 \times D^2. \end{aligned}$$

Note that b is odd. We denote this h by $h(a, c, b, d)$. It is checked that the same statements as in Lemmas 3.8 and 3.9 are also true in this case. Therefore, one may conclude the following result (see (3.10)).

THEOREM 4.20. *If $c(\text{Fix}(h)) = 1$ and $c(\text{Fix}(h^2)) = 2$, then $p = 2s$ for some odd s . Up to weak equivalence, there exists exactly one such h on each $L = L(2s, q)$ (s odd) if L is symmetric, and there exist exactly two if L is nonsymmetric.*

It follows from results of this section that there exists exactly one orientation-preserving nonfree Z_4 -action on P^3 , up to conjugation (see Theorem 4.20). Now suppose that Z_4 acts freely on P^3 . Let $M = P^3/Z_4$. Then we see that $\pi_1(M)$ is an abelian group of order 8. Therefore, it follows from a result of Epstein [3] that $\pi_1(M) = Z_8$. We may assume that $\pi_1(M)$ acts freely on S^3 . Therefore, we have $M \approx L(8, q)$ for some q (see Ritter [18]). Thus the possible orbit types are $L(8, 1)$ and $L(8, 3)$. Since there exist free Z_4 -actions whose orbit spaces are $L(8, 1)$ and $L(8, 3)$, we have the following.

THEOREM 4.21. *P^3 admits exactly three orientation-preserving Z_4 -actions, up to conjugation.*

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