BOUNDED POINT EVALUATIONS AND
SMOOTHNESS PROPERTIES OF FUNCTIONS IN $R^p(X)$

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ABSTRACT. Let $X$ be a compact subset of the complex plane $C$. We denote by $R_0(X)$ the algebra consisting of the (restrictions to $X$ of) rational functions with poles off $X$. Let $m$ denote 2-dimensional Lebesgue measure. For $p > 1$, let $L^p(X) = L^p(X, dm)$. The closure of $R_0(X)$ in $L^p(X)$ will be denoted by $R^p(X)$. Whenever $p$ and $q$ both appear, we assume that $1/p + 1/q = 1$.

If $x$ is a point in $X$ which admits a bounded point evaluation on $R^p(X)$, then the map which sends $f$ to $f(x)$ for all $f \in R_0(X)$ extends to a continuous linear functional on $R^p(X)$. The value of this linear functional at any $f \in R^p(X)$ is denoted by $f(x)$. We examine the smoothness properties of functions in $R^p(X)$ at those points which admit bounded point evaluations. For $p > 2$ we prove in Part I a theorem that generalizes the "approximate Taylor theorem" that James Wang proved for $R(X)$.

In Part II we generalize a theorem of Hedberg about the convergence of a certain capacity series at a point which admits a bounded point evaluation. Using this result, we study the density of the set $X$ at such a point.

PART I. SMOOTHNESS PROPERTIES OF FUNCTIONS IN $R^p(X)$

Let $X$ be a compact subset of the complex plane $C$. We denote by $R_0(X)$ the algebra consisting of the (restrictions to $X$ of) rational functions with poles off $X$. Let $m$ denote 2-dimensional Lebesgue measure. For $p > 1$, let $L^p(X) = L^p(X, dm)$. The closure of $R_0(X)$ in $L^p(X)$ will be denoted by $R^p(X)$. Whenever $p$ and $q$ both appear, we will assume that $1/p + 1/q = 1$.

1. Bounded point derivations.

Definition (1.1). For $x \in X$ we say that $x$ admits a bounded point derivation of order $s$ on $R^p(X)$ if there exists a constant $C$ such that $|f^{(s)}(x)| \leq C \|f\|_p$ for all $f \in R_0(X)$.

When $x$ admits a bounded point derivation of order $s$ on $R^p(X)$, the map $f \mapsto f^{(s)}(x)/s!$ extends from $R_0(X)$ to a bounded linear functional on $R^p(X)$. 
We denote this bounded linear functional by $D_x$.

**Definition (1.2).** When $x$ admits a bounded point derivation of order 0, we say that $x$ admits a bounded point evaluation. For $f \in R^p(X)$ we define $f(x) = D^0_x f$.

**Definition (1.3).** For each $p > 2$ the inner set for $R^p(X)$ is the set of points in $X$ which admit bounded point evaluations, and we denote it by $S^p(X)$.

**Proposition (1.1).** For each $p > 2$, $S^p(X)$ is an $F_\sigma$ set.

**Proof.** Write $S^p(X) = \bigcup_{n=1}^\infty S^n_p(X)$ where

$S^n_p(X) = \{x \in X | |f(x)| \leq n\|f\|_p \text{ for all } f \in R^p(X)\}$.

We show that each set $S^n_p(X)$ is closed. Suppose that $\{x_k\} \subset S^n_p(X)$ and that $x_k \to x \in X$. Let $L_{x_k} f = f(x_k)$ and observe that the $L_{x_k}$ are a family of linear functionals bounded in norm by $n$. Since $L_{x_k} f \to f(x)$ for $f \in R_0(X)$, and $R_0(X)$ is dense in $R^p(X)$, it follows that $x \in S^n_p(X)$. Thus each $S^n_p(X)$ is closed.

2. Potentials and representing functions. In this paper $z$ will denote the identity function.

**Definition (2.1).** Let $\psi$ be a positive nondecreasing function on $(0, \infty)$. For each $g \in L^q(X)$, $q > 1$, we define the $\psi$-potential of $g$, $U^\psi_g$, by

$U^\psi_g(y) = \int \frac{|g|}{\psi(|z - y|)} \, dm$.

If $1/\psi(|z|)$ is locally summable with respect to $m$, Fubini's theorem implies that $U^\psi_g$ is locally summable; hence $U^\psi_g < \infty$ a.e. $(m)$.

**Definition (2.2).** When $\psi(r) = r$, we denote $U^\psi_g$ by $\hat{g}$.

**Definition (2.3).** When $\psi (r) = r^q$, $1 < q < 2$, we denote $U^\psi_g$ by $U^q_g$.

**Definition (2.4).** We define the Cauchy transform of $g$ to be

$\hat{g}(y) = \int (z - y)^{-1} g \, dm$ for all $y$ where $\hat{g}(y) < \infty$.

For the proof of the following lemma we refer the reader to Sinanjan [16] or Brennan [1, pp. 10–11]. Brennan's proof uses the Cauchy transform.

**Lemma (2.1).** Let $X \subset C$ be compact and have no interior. Then $R^p(X) = L^p(X)$ for $1 < p < 2$.

It follows from the Riesz representation theorem that if $x \in S^p(X)$, then there is a $g \in L^q(X)$ such that $f(x) = \int fg \, dm$ for all $f \in R^p(X)$. We call such a $g$ a representing function for $x$. If $R^p(X) \neq L^p(X)$, there is a nonzero function $g \in L^q(X)$ such that $\int fg \, dm = 0$ for all $f \in R^p(X)$. We call such a $g$ an annihilating function.
The following lemma was proved by Bishop for the sup norm case: We assume that $1 < q < 2$.

**Lemma (2.2).** Let $g \in L^q(X)$ be an annihilating function. Suppose that $\hat{g}(y)$ is defined and $\neq 0$, and that $(z - y)^{-1}g \in L^q(X)$. Then $\hat{g}(y)^{-1}(z - y)^{-1}g$ is a representing function for $y$.

**Proof.** If $f \in R_0(X)$, then $f = f(y) + (z - y)h$ for some $h \in R_0(X)$. Hence

$$
\int (z - y)^{-1}fg \, dm = f(y)\hat{g}(y) + \int hg \, dm = f(y)\hat{g}(y).
$$

**Corollary (2.1).** Let $g \in L^q(X)$ be a representing function for $x$. Let

$$
c(y) = \int (z - x)(z - y)^{-1}g \, dm = 1 + (y - x)\hat{g}(y).
$$

Then $c(y)^{-1}(z - x)(z - y)^{-1}g$ is a representing function for $y$ whenever $c(y)$ is defined and $\neq 0$.

**Proof.** $(z - x)g$ is an annihilating function.

We now present a lemma of Brennan in [2, p. 288] which will be very useful.

**Lemma (2.3).** If $p > 2$, then $R^p(X) \neq L^p(X)$ if and only if $S^p(X)$ has positive 2-dimensional measure.

**Proof.** Suppose that $S^p(X) \neq \emptyset$ and $x \in S^p(X)$ is represented by a nonzero function $g \in L^q(X)$. Then $R^p(X) \neq L^p(X)$ because $(z - x)g \in L^q(X)$, and $\int (z - x)gf \, dm = 0$ for all $f \in R^p(X)$.

Now suppose that $R^p(X) \neq L^p(X)$ and let $g \in L^q(X)$ be a nonzero annihilating function. Then $\hat{g}$ fails to vanish on a set of positive measure in $X$. Hence there is a set $S \subset X$ of positive measure such that for $y \in S$, $\hat{g}(y) \neq 0$ and $\hat{g}(y)^{-1}(z - y)^{-1}g \in L^q(X)$. It follows from Corollary (2.1) that $S \subset S^p(X)$, and the lemma is proved.

**Remark.** If we know that there is an $x \in S^2(X)$, the difficulty in showing that there are other points in $S^2(X)$ by the above method is that $z^{-1} \not\in L^2_{\text{loc}}$.

3. Admissible functions. Fix $x \in \mathbb{C}$ and let $\Delta_n = \{y \in \mathbb{C}: |y - x| < 1/n\}$. We say that a set $E \subset \mathbb{C}$ has full area density at $x$ if \(\lim_{n \to \infty} m(E \cap \Delta_n)/m(\Delta_n) = 1\). Let $F$ be a function defined on $X, x \in X$. We say that $a$ is the approximate limit of $F$ at $x$, and write $\lim_{y \to x}^a F(y) = a$ if there exists a subset $E$ of $X$ having full area density at $x$, such that $\lim_{y \to x, y \in E}^a F(y) = a$. We say that $F$ is approximately continuous at $x$ if $\lim_{y \to x}^a F(y) = F(x)$.

If $\phi$ is a positive function on $(0, \infty)$ with $\lim_{r \to 0^+} \phi(r) = 0$, we say that $F$ admits $\phi$ as a modulus of approximate continuity at $x$ if $|F(y) - F(x)| < \phi(|y - x|)$. 

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\( \phi(|y - x|) \) for all \( y \) in a set having full area density at \( x \). We say that \( F \) satisfies an approximate Hölder condition of order \( \alpha \) at \( x \) if \( F \) admits \( C^{\alpha} \) as a modulus of approximate continuity at \( x \) for some constant \( C \).

**Definition (3.1).** We say that \( \phi \) is an admissible function if

(a) \( \phi \) is a positive, nondecreasing function defined on \((0, \infty)\), and

(b) the associated function \( \psi \), defined by \( \psi(r) = r/\phi(r) \), is nondecreasing, with \( \psi(0+) = 0 \).

**Example.** For any \( \alpha, 0 < \alpha < 1 \), \( \phi(r) = r^\alpha \) is admissible.

**Remarks.**

1. If \( \phi \) is admissible and \( 0 < \beta < 1 \), then \( \phi^\beta \) is also admissible because \( r/\phi^\beta(r) = (r/\phi(r)) \cdot \phi^{1-\beta}(r) \).

2. In using an admissible function \( \phi \) we will often refer to the triangle inequality: \( \phi(r) \leq \phi(r_1) + \phi(r_2) \) whenever \( r \leq r_1 + r_2 \). This follows from the definition of an admissible function since

\[
\phi(r) \leq \phi(r_1 + r_2) = (r_1 + r_2)/\psi(r_1 + r_2) \\
\leq r_1/\psi(r_1) + r_2/\psi(r_2) = \phi(r_1) + \phi(r_2).
\]

Wang introduced a special kind of admissible function in [17, p. 349].

**Definition (3.2).** We say that the admissible function \( \phi \) is nice if

\[
\int_0^1 \phi(r)^{-1} dr < \infty.
\]

For each \( q, 1 < q < 2 \), we will be interested in a subset of the set of nice admissible functions.

**Definition (3.3).** We say that the admissible function \( \phi \) is \( q \)-nice if

\[
\int_0^1 r^{1-q} \phi(r)^{-q} dr < \infty.
\]

Note that a nice admissible function is \( 1 \)-nice and that \( \phi(r) = r^\alpha \) is \( q \)-nice for \( \alpha < (2 - q)/q \). When \( p > 2 \), the \( q \)-nice admissible functions will be the most likely ones to be moduli of approximate continuity for functions in the unit ball of \( R^p(X) \) at points in \( S^p(X) \).

The following lemma is due to Wang [17]:

**Lemma (3.1).** Let \( g \in L^q(X), q > 1 \), and let \( x \in X. \) Then there exists a nice admissible function \( \phi \) with \( \phi(0+) = 0 \) such that \( \phi(|z - x|)^{-1}g \in L^q(X) \).

**Proof.** See Wang [17].

Our proof of the next lemma is in the spirit of Browder's result [3, p. 157]. It will be useful for studying the density of \( X \) at points in \( S^p(X) \). Let \( E \subset X \) be measurable. Define \( \rho_n \) by \( \pi \rho_n^2 = m(\Delta_n \setminus E) \). Denote \( m|\Delta_n \setminus E \) by \( m_n \).

**Lemma (3.2).** Let \( \psi \) be associated with an admissible \( \phi \). For \( q, 0 < q < 2 \), let \( \tau = \psi^q \). Then if \( g \in L^1(X) \),

\[
\lim_{n \to \infty} \frac{n^q}{\rho_n^{2-q}} \int \tau(|y - x|) U^* g(y) \, dm_n(y) = 0.
\]
Proof. Define

$$F_n(\xi) = n^q \rho_n^{q+2} \int \psi(|y - x|)^q \cdot \psi(|\xi - y|)^{-q} \, dm_n(y).$$

Then $F_n(x) < \infty$ and if $\xi \neq x$, we have for large $n$

$$|F_n(\xi)| < n^q \rho_n^q \psi(n^{-1})^q \cdot \psi(|x - \xi| - n^{-1})^{-q} \to 0 \quad \text{as} \quad n \to \infty.$$ 

Next, we will show that the $F_n$ are bounded independently of $n$. Let $D_n = \Delta(\xi, \rho_n)$. Since $\psi^q$ is increasing,

$$|F_n(\xi)| \leq n^q \rho_n^{q+2} \psi(n^{-1})^q \int_{D_n} \psi{|y - \xi|}^{-q} \, dm(y)$$

$$\leq n^q \rho_n^q \psi(n^{-1})^q \int_0^{\infty} \psi(r)^{-q} r \, dr$$

$$\leq 2\pi n^q \rho_n^q \psi(n^{-1})^q \phi(\rho_n)^q \int_0^{\rho_n} r^{-1} \, dr$$

$$= 2\pi n^q \rho_n^q \psi(n^{-1})^q \phi(\rho_n)^q \rho_n^{2-q} (2 - q)^{-1}$$

$$< 2\pi (2 - q)^{-1}.$$

Thus, the $F_n$ converge boundedly a.e. to zero. We apply the dominated convergence theorem and Fubini's theorem to obtain the lemma.

Lemma (3.3). Let $\psi$ be associated with an admissible $\phi$. For $0 < q < 2$, let $\tau = \psi^q$. Then if $g \in L^1(X)$, and $\delta > 0$, the set $E = \{y \in C: \tau(|y - x|) U^r_\xi(y) < \delta\}$ has full area density at $x$.

Proof. It is sufficient to prove that $\lim_{n \to \infty} m(\Delta_n \setminus E)/m(\Delta_n) = 0$ where $\Delta_n = \Delta(x, 1/n)$. We observe that since

$$m(\Delta_n \setminus E) \leq \delta^{-1} \int_{\Delta_n} \tau(|y - x|) U^r_\xi(y) \, dm(y),$$

it is sufficient to prove that

$$\lim_{n \to \infty} n^2 \int_{\Delta_n} \tau(|y - x|) U^r_\xi(y) \, dm(y) = 0.$$ 

This follows from Lemma (3.2) if we take $E$ in that lemma to be the empty set.

4. The main theorem. The following lemma in the sup norm case is due to Wilken [20]. For $x \in S^p(X)$, $p > 2$, it gives a condition for $x$ to admit a bounded point derivation of order $s$.

Lemma (4.1). Suppose there exist a representing function $g \in L^q(X)$ for
$x \in S^p(X)$, $p > 2$, and a nonnegative integer $s$ such that $(z - x)^{-s}g \in L^q(X)$. Let $c_j = \int (z - x)^{-j} g \, dm$ ($0 \leq j \leq s$) and define $G_0, \ldots, G_s$ by:

$$G_0 = g, \quad G_j = (z - x)^{-j} g - \sum_{k < j} c_{j-k} G_k.$$

Then $D^j_x$ exists, and $D^j_x f = \int f G_j \, dm$ for all $f \in R^p(X)$, $0 \leq j \leq s$.

An additional lemma will be needed in proving the theorem.

**Lemma (4.2).** Let $s$ be a nonnegative integer, and $g \in L^q(X)$, $1 < q < 2$. Suppose that $(z - x)^{-s}g \in L^q(X)$. Set $H_j = (z - x)^{-j} g$ ($0 \leq j \leq s$). For any $f \in L^p(X)$ and $y \in C$

$$\int (z - y)^{-1} f y \, dm = \sum_{j=1}^s (y - x)^{j-1} \int f H_j \, dm + (y - x)^s \int (z - y)^{-1} f H_s \, dm.$$

**Proof.** Since $H_j = (z - x)H_{j+1}$ for $0 \leq j \leq s$,

$$\int (z - y)^{-1} f H_j \, dm = \int f H_{j+1} \, dm + (y - x) \int (z - y)^{-1} f H_{j+1} \, dm$$

which implies the lemma.

Our main theorem generalizes the “approximate Taylor’s theorem” which Wang obtained for functions in $R(X)$ [17, p. 352].

**Theorem (4.1).** Let $\phi$ be an admissible function and $s$ a nonnegative integer. Suppose that $p > 2$ and that there is an $x \in S^p(X)$ represented by a $g \in L^q(X)$ such that $(z - x)^{-s}g \in L^q(X)$. Then for every $\varepsilon > 0$ there is a set $E$ in $X$ having full area density at $x$ such that for every $f \in R^p(X)$

(i) $f = \Sigma_{j=0}^s (D^j_x f)(z - x)^j + R$ where $R \in R^p(X)$ satisfies

(ii) $|R(y)| \leq \varepsilon |y - x|^s \phi(|y - x|) \|f\|_p$ for all $y \in E$, and

(iii) $\text{app lim}_{y \to x} \{ R(y)/|y - x|^s \phi(|y - x|) \} = 0$.

**Proof.** Since $(z - x)^{-s}g \in L^q(X)$, Lemma (4.1) implies that the $D^j_x$ exist for $0 \leq j \leq s$. To each $D^j_x$, $0 < j < s$, there corresponds a constant $C_j$ such that $|D^j_x f| \leq C_j \|f\|_p$ for all $f \in R^p(X)$. By Minkowski’s inequality there is another constant $C$ such that if $R$ is defined as in (i), $\|R\|_p \leq C \|f\|_p$ for all $f \in R^p(X)$.

Choose $\delta > 0$ so that $0 < C_0 (1 - \delta)^{-1} < \varepsilon / 2$. If $y \in E_1 = \{ y \in C : |y - x| \tilde{g}(y) < \delta \}$, then $c(y) = 1 + (y - x) \tilde{g}(y)$ is well defined, and $|c(y)| > 1 - \delta$. By Corollary (2.1),
\[ R(y) = c(y)^{-1} \int \left[ R(z-x)/(z-y) \right] g \, dm \]
\[ = c(y)^{-1} \int R \left[ 1 + (y-x)/(z-y) \right] g \, dm \]
\[ = c(y)^{-1} (y-x) \int \left[ R/(z-y) \right] g \, dm. \]

Next, we claim that \( R(y) = c(y)^{-1}(y-x)^{s+1}(z-x)^{-s}(z-y)^{-1} g \, dm. \)
This claim depends on Lemma (4.2). Each of the functions \((z-x)^{-j} g, 0 < j < s,\) is a linear combination of functions representing \(D^k_x, 0 < k < j,\) which implies that \(\int (z-x)^{-j} g \, dm = 0\) for \(0 < j < s,\) and the claim is proved.

Factoring \(g = \phi(|z-x|) h\) where \(h \in L^q(X),\) we obtain by the "triangle inequality" that
\[ |g| \leq \phi(|z-y|)|h| + \phi(|y-x|)|h|. \]
Consequently,
\[ |R(y)| \leq c(y)^{-1}|y-x|^s \phi(|y-x|) \int \psi(|z-y|)^{-1}|z-x|^{-s} \phi(|y-x|) |h| \, dm \]
\[ + \int |z-y|^{-1}|z-x|^{-s} \phi(|y-x|) |h| \, dm. \]
Denote the first integral by \(I_1\) and the second by \(I_2.\) We have
\[ I_1 = c(y)^{-1}|y-x|^s \phi(|y-x|) \psi(|y-x|) \int \psi(|z-y|)^{-1}|z-x|^{-s} |h| \, dm. \]
Let \(r = \psi, k = (z-x)^{-q} \psi,\) and
\[ E_2 = \{ y \in C: \tau(|y-x|) U^q_k(y) < \delta^q \}. \]
For \(y \in E_2\) we apply Hölder's inequality to obtain
\[ I_1 \leq (1 - \delta)^{-1}|y-x|^s \phi(|y-x|) \tau(|y-x|)^{1/q} \left\{ \int |R|^p \, dm \right\}^{1/p} \{ U^q_k(y) \}^{1/q} \]
\[ \leq (1 - \delta)^{-1}|y-x|^s \phi(|y-x|) C \| f \|_p \delta \]
\[ \leq (\epsilon/2)|y-x|^s \phi(|y-x|) \| f \|_p. \]
To estimate \(I_2\) we define
\[ E_3 = \{ y \in C: |y-x|^q U^q_k(y) < \delta^q \} \text{ and let } y \in E_2 \cap E_3. \]
By Hölder's inequality,
By Lemma (3.3) the set $E = E_2 \cap E_3$ has full area density at $x$, and we have proved that for $y \in E$

$$|R(y)| \leq I_1 + I_2 \leq \epsilon |y - x|^s \phi(|y - x|) \|f\|_p$$

for any $f \in R^p(X)$. To prove (iii) let $L_y f = R(y)/|y - x|^s \phi(|y - x|)$. The above result implies that $\|L_y|| \leq \epsilon$ for $y \in E$. Let $y \to x$ in such a way that $y$ stays in $E$. Then $L_y f \to 0$ as $y \to x$ for $f \in R_0(X)$, and since $R_0(X)$ is dense in $R^p(X)$, (iii) follows.

An interesting consequence of the above theorem is that we can take the limit of Newton quotients in the set $E$ to evaluate $D_{x}f$. For $f$ a function defined on a subset of $X$, $h \in C$, we set

$$\Delta_h f = f(z + h) - f$$

so $\Delta_h f$ is a function defined on a subset of $X$. We define inductively $\Delta_0^s = \text{id}$, $\Delta_h = \Delta_h \circ \Delta_h^{-1}$ for $j > 1$. The sup norm version of the following corollary is proved in [17].

**Corollary (4.1).** If $x$ admits a bounded point derivation of order $s$ on $R^p(X)$, $p > 2$, then for all $f \in R^p(X)$

$$D_x^s f = \text{app lim}_{h \to 0} \frac{\Delta_h f(x)}{s!h^s}.$$

**Lemma (4.3).** Let $\phi$ be a $q$-nice admissible function. If $x \in S^p(X)$, $p > 2$, then $\{y \in X: \exists\ a\ function \ g_y \ that \ represents \ y \ for \ R^p(X) \ and \ satisfies \ \phi(|z - y|)^{-1}g_y \in L^q(X)\}$ has full area density at $x$.

**Proof.** Let $g \in L^q(X)$ represent $x$.

Let

$$F = \left\{ y \in C: \int |z - y|^{-q} \phi(|z - y|)^{-q} |g|^q \ dm < \infty \right\}.$$

Since $|z|^{-q} \phi(|z|)^{-q}$ is locally summable with respect to $m$, $m(C \setminus F) = 0$. Fix $\delta$, $0 < \delta < 1$, and put $E = F \cap E_1$ where $E_1 = \{ y \in C: |y - x| \tilde{g}(y) < \delta \}$. By Lemma (3.3) the set $E$ has full area density at $x$. For each $y \in E$ the function $g_y = c(y)^{-1}[(z - x)/(z - y)]g$ represents $y$. Moreover,
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$$\int \phi(|z - y|)^{-q} g_y^q \, dm = \left| c(y) \right|^{-q} \int |z - y|^{-q} \phi(|z - y|)^{-q} |z - x|^q g_y^q \, dm$$

$$< C \int |z - y|^{-q} \phi(|z - y|)^{-q} g_y^q \, dm < \infty.$$  

This proves the lemma.

**Corollary (4.2).** Suppose that $\phi$ is $q$-nice. Then at almost every point of $S^p(X)$, $p > 2$, the functions in the unit ball of $R^p(X)$ admit $\phi$ as a modulus of approximate continuity.

**Proof.** Combine Theorem (4.1) with Lemma (4.3).

In particular, it follows that at a.e. $x \in S^p(X)$, $p > 2$, the unit ball of $R^p(X)$ satisfies an approximate uniform Hölder condition of order $\alpha$ for every $\alpha < (2 - q)/q$.

**Lemma (4.4).** Let $\phi$ be admissible and $g \in L^q(X)$, $1 < q < 2$. Then if $\phi(|z - x|)^{-1} g \in L^q(X)$, $\delta > 0$, and

$$E = \left\{ y \in C : |y - x|^q \int |y - z|^{-q} |g|^q \, dm < \delta \right\},$$

it follows that $m(\Delta_n \setminus E) = o(\phi(n^{-1})^2/n^2)$.

**Proof.** We observe that

$$m(\Delta_n \setminus E) \leq \delta^{-1} \int |y - x|^q \int |z - y|^{-q} |g|^q \, dm \, dm_n(y).$$

Factor $g = \phi(|z - x|) h$ where $h \in L^q(X)$. Then

$$|g|^q \leq C \left[ \phi(|z - y|^q) |h|^q + \phi(|y - x|^q) |h|^q \right]$$

where $C$ is some constant. We have

$$m(\Delta_n \setminus E) \leq \delta^{-1} C \left[ \int |y - x|^q \int |z - y|^{-q} \phi(|z - y|)^q |h|^q \, dm \, dm_n(y) + \int |y - x|^q \int |z - y|^{-q} \phi(|y - x|)^q |h|^q \, dm \, dm_n(y) \right].$$

By substituting $|y - x|^q = \phi(|y - x|)^q \psi(|y - x|)^q$ in the first integral, and using the fact that $\phi(|y - x|)^q \leq \phi(n^{-1})^q$ for $y \in \Delta_n$, we obtain

$$m(\Delta_n \setminus E) \leq \delta^{-1} C \phi(n^{-1})^q \left[ \psi(|y - x|)^q \int \psi(|z - y|)^{-q} |h|^q \, dm \, dm_n(y) + \int |y - x|^q \int |z - y|^{-q} |h|^q \, dm \, dm_n(y) \right].$$

Let $A_n^2$ denote the sum of the two integrals on the right. Replacing $m(\Delta_n \setminus E)$ by $\pi \rho_n^2$, we obtain

$$\pi \rho_n^2 \leq \delta^{-1} C \phi(n^{-1})^q \rho_n^{2-q} n^{-q} (A_n^2).$$
where $\lim_{n \to \infty} A_n = 0$ by Lemma (3.2). Divide both sides by $\rho_n^{2-q}$ to get

$$\pi \rho_n^2 \leq \delta^{-1} C \phi(n)^q n^{-q} (A_n).$$

Now raise both sides to the power $2/q$, and the conclusion of the lemma follows.

In the next corollary we consider functions $f \in R^p(X)$ to be defined on $C$ by setting $f(x) = 0$ for $x \notin X$.

**Corollary (4.3).** Let $\varepsilon > 0$. If $x \in S^p(X)$, $p > 2$, is represented by $g \in L^q(X)$, and $(z - x)^{-q} g \in L^q(X)$ for some $\alpha > q - 1$, then there is an integer $N_x$ depending on $x$ such that for $n > N_x$

$$m(\Delta_n)^{-1} \int_{\Delta_n} |f - f(x)| \, dm \leq \varepsilon \|f\|_p \quad \text{for all } f \in R^p(X).$$

**Proof.** Let $E$ be the set in the conclusion of Theorem (4.1) when $\varepsilon/2$ and $x \in S^p(X)$ are given and $\phi(r) \equiv 1$.

Let $Y_{\Delta_n \setminus E}$ be the characteristic function of $\Delta_n \setminus E$. Then by Hölder’s inequality,

$$\pi^{-1} n^2 \int_{\Delta_n \setminus E} |f - f(x)| \, dm = \pi^{-1} n^2 \int Y_{\Delta_n \setminus E} |f - f(x)| \, dm$$

$$\leq C n^2 \left[ m(\Delta_n \setminus E) \right]^{1/q} \|f\|_{L^p(\Delta_n \setminus E)}$$

where $C$ is a constant. By Lemma (4.4)

$$\left[ m(\Delta_n \setminus E) \right]^{1/q} = o(n^{-(2/q)-(2\alpha/q)}).$$

Thus if $\alpha > q - 1$, we can choose an integer $N_x$ so that $n > N_x$ implies that $C n^2 \left[ m(\Delta_n \setminus E) \right]^{1/q} < \varepsilon/2$. Hence,

$$m(\Delta_n)^{-1} \int_{\Delta_n} |f - f(x)| \, dm \leq (\varepsilon/2)\|f\|_p + (\varepsilon/2)\|f\|_{L^p(\Delta_n \setminus E)}$$

$$\leq \varepsilon \|f\|_p.$$ 

This completes the proof.
Corollary (4.4). If \( p > 2 + \sqrt{2} \), then for a.e. \( x \in S^p(X) \),
\[
\lim_{n \to \infty} m(\Delta_n)^{-1} \int_{\Delta_n} |f - f(x)| \, dm = 0 \quad \text{for any} \ f \in R^p(X).
\]

Proof. This follows from Lemma (4.3) and Corollary (4.3).

Given \( f \in L^1(dm) \), the set of points \( x \in C \) such that
\[
\lim_{n \to \infty} m(\Delta_n)^{-1} \int_{\Delta_n} |f - f(x)| \, dm = 0
\]
is called the Lebesgue set of \( f \). For an arbitrary \( f \in L^1(dm) \), a.e. (\( m \)) point \( x \in C \) belongs to the Lebesgue set of \( f \) (see [5, p. 156]). The above corollary identifies points belonging to the Lebesgue sets of all \( f \in R^p(X) \). It would be interesting to know whether the corollary holds for \( p > 2 \).

Part II. Capacity and bounded point evaluations

1. Capacity theorems. Before proving a capacity result about bounded point evaluations, we will need two lemmas of Hedberg [9]. Let \( \Omega \) denote the complex plane when \( p > 2 \) and the unit disk when \( p = 2 \).

Definition (1.1). Let \( A' \subset \beta \) be a compact set. Then
\[
\gamma_q(A') = \inf_{\omega} \int |\nabla \omega|^q \, dm
\]
where the inf is taken over Lipschitz functions \( \omega \) with compact support contained in \( \Omega \) such that \( \omega(z) > 1 \) on \( X \).

For noncompact sets \( F \), \( q \)-capacity is defined by \( \gamma_q(F) = \sup_{K \subset F} \gamma_q(K) \), \( K \) compact.

Let \( U \) be an open set (bounded if \( p = 2 \)) in the complex plane and denote by \( L^p_q(U) \) the space of analytic functions in \( L^p(U) \). If \( f \) is analytic in \( \Omega \setminus X \) where \( X \subset \Omega \) is compact, we write \( \alpha(f) = (2\pi i)^{-1} \int_C f(z) \, dz \) where \( C \) is any Jordan curve in \( \Omega \) enclosing \( X \).

Lemma (1.1). Let \( X \subset \Omega \) be compact. Then there are positive constants \( C_1 \) and \( C_2 \), depending only on \( p \), such that
\[
C_1 \gamma_q(X)^{1/q} \leq \sup_f |\alpha(f)| \leq C_2 \gamma_q(X)^{1/q}
\]
where the sup is taken over functions \( f \) in \( L^p_q(\Omega) \), \( 2 < p < \infty \), with \( \int_{\Omega \setminus X} |f(z)|^p \, dm \leq 1 \).

We denote the annulus \( \{ z : 2^{n-1} \leq |z - x| \leq 2^n \} \) by \( A_n(X) \). We write \( A_n = A_n(0) \).

Lemma (1.2). Let \( X \subset \Omega \) be compact. There is a constant \( C \), depending only on \( p \), such that for \( z \not\in A_{n-1} \cup A_n \cup A_{n+1} \)
\[ |f(z)| < \frac{C \Gamma_q(A_n \setminus X)^{1/q}}{|z| - 2^{-n}} \|f\|_{\Omega \setminus X, p} \]

for \( f \) analytic outside \( A_n \setminus X \), \( f(\infty) = 0 \) and \( \int_{\Omega \setminus X} |f(z)|^p \, dm < \infty \).

The following theorem was proved in the sup norm case by Wang [18, p. 223]. Wang essentially followed O'Farrell [13], who elaborated on a method of Gamelin [7, p. 206]. We assume that \( x = 0 \) and that \( 0 \in \partial X \).

**Theorem (1.1).** Let \( \phi \) be an admissible function and \( s \) a nonnegative integer. Suppose that there is a function \( v \in L^q(X) \) which represents 0 for \( R^q(X) \) such that \( |z|^{-s} \phi(|z|)^{-1} v \in L^q(X) \). Then

\[
\sum_{1}^{\infty} 2^{q(s+1)\phi(2^{-n})^{-q} \Gamma_q(A_n \setminus X)} < \infty.
\]

**Proof.** Suppose that

\[
\sum_{1}^{\infty} 2^{q(s+1)\phi(2^{-n})^{-q} \Gamma_q(A_n \setminus X)} = \infty.
\]

We will show that this leads to a contradiction. We may assume that for each \( n \)

\[ 2^{q(s+1)\phi(2^{-n})^{-q} \Gamma_q(A_n \setminus X)} < 1. \]

If not, choose \( Y_n \) compact, \( Y_n \subseteq A_n \) such that

\[ \frac{1}{2} < 2^{q(s+1)\phi(2^{-n})^{-q} \Gamma_q(A_n \setminus X \cup Y_n)} < 1, \]

and set \( Y = \bigcup Y_n \cup X \). Then define \( v^*(z) = v(z) \) for \( z \in X \) and \( v^*(z) = 0 \) for \( z \in Y \setminus X \). Clearly, \( |z|^{-s} \phi(|z|)^{-1} v^* \in L^q(Y) \) and \( v^* \) represents 0 for \( R^q(Y) \).

Now choose integers \( M_1 < N_1 < M_2 < N_2 < \cdots \) so that

\[ 1 < \sum_{n=M_j}^{N_j} 2^{q(s+1)\phi(2^{-n})^{-q} \Gamma_q(A_n \setminus X)} < 2. \]

For each \( n \) we choose by Lemma (1.1) compact sets \( K_n \subseteq A_n \setminus X \) and functions \( f_n \in L^p_0(\Omega \setminus K_n) \) so that:

(i) \[ |\alpha(f_n)| > C_1 2^{-1} \Gamma_q(A_n \setminus X)^{1/q} \left( \int_{\Omega \setminus K_n} |f_n(z)|^p \, dm \right)^{1/p} \]

(ii) \[ f_n = 0 \] on \( K_n \) and

(iii) \[ \|f_n\|_{\rho, p} = 2^{q(s+1)\phi(2^{-n})^{-q} \Gamma_q(A_n \setminus X)^{1/p}}. \]
Let \( g_j(z) = \phi(|z|)z^{s+1} \sum_{n=M_j}^{N_j} f_n(z) \). We will show that \( \|g_j\|_{X,p} < C \) for all \( j \).

In the following discussion \( C \) will denote any constant that is independent of \( n \) and \( j \). Lemma (II.1.2) implies that for \( z \in A_k, k < n - 1 \),

\[
|f_n(z)| < C 2^{q(s+1)n+k} \phi(2^{-n})^{-q} \Gamma_q(A_n \setminus X),
\]

and for \( z \in A_k, k > n + 1 \),

\[
|f_n(z)| < C 2^{q(s+1)n+n} \phi(2^{-n})^{-q} \Gamma_q(A_n \setminus X).
\]

We may assume that \( X \subset \{|z| < 1\} \). Then for \( z \in A_k \cap X, k < n - 1 \),

\[
\phi(|z|)|z|^{s+1}|f_n(z)| < C 2^{q(s+1)n+n-(s+1)n} \phi(2^{-n})^{-q} \Gamma_q(A_n \setminus X).
\]

For \( z \in A_k, k > n + 1 \),

\[
\phi(|z|)|z|^{s+1}|f_n(z)| < C 2^{q(s+1)n+n-(s+1)n} \phi(2^{-n})^{-q} \Gamma_q(A_n \setminus X)
\]

\[
< C 2^{q(s+1)n} \phi(2^{-n}) \Gamma_q(A_n \setminus X).
\]

Now

\[
\int_X |g_j(z)|^p \, dm = \sum_{k=0}^{\infty} \int_{A_k \cap X} \left| \sum_{n=M_j}^{N_j} \phi(|z|)z^{s+1} f_n(z) \right|^p \, dm
\]

\[
< C \sum_{k=0}^{\infty} \int_{A_k \cap X} \left[ \left( \sum_{n=M_j}^{N_j} \phi(|z|)|z|^{s+1}|f_n(z)| \right)^p + \sum_{n=M_j}^{n+k+1} \phi(|z|)|z|^{s+1}|f_n(z)|\right] \, dm.
\]

By the above estimates and the choice of \( M_j, N_j \), we have for \( z \in A_k \)

\[
\sum_{n=\max(k+2,M_j)}^{N_j} \phi(|z|)|z|^{s+1}|f_n(z)| < C \sum_{n=M_j}^{N_j} 2^{q(s+1)n} \phi(2^{-n})^{-q} \Gamma_q(A_n \setminus X) < C.
\]

Similarly,

\[
\sum_{n=M_j}^{\min(k-2,N_j)} \phi(|z|)|z|^{s+1}|f_n(z)| < C \sum_{n=M_j}^{N_j} 2^{q(s+1)n} \phi(2^{-n})^{-q} \Gamma_q(A_n \setminus X) < C.
\]

Thus

\[
\sum_{k=0}^{\infty} \int_{A_k \cap X} \left( \sum_{n=M_j; n \neq k-1,k,k+1}^{N_j} \phi(|z|)|z|^{s+1}|f_n(z)| \right)^p \, dm < C.
\]

Next, we estimate
\[
\sum_{k=0}^{\infty} \int_{A_k \cap X} \left( \sum_{n=k-1}^{k+1} \phi(|z|)|z|^{s+1}|f_n(z)| \right)^p \, dm.
\]

For each \(k\),
\[
\int_{A_k \cap X} \left( \sum_{n=k-1}^{k+1} \phi(|z|)|z|^{s+1}|f_n(z)| \right)^p \, dm
\leq C \phi(2^{-k+1})^p 2^{-p(k-1)} \|f_{k-1}\|_p
\leq C \phi(2^{-k+1})^p 2^{-(k-1)(-p+pq(s+1))} \Gamma_q(A_{k-1} \setminus X)
\leq C 2^{q(s+1)(k-1)-q} \Gamma_q(A_{k-1} \setminus X)
\]

and similarly for \(f_k\) and \(f_{k+1}\). Thus
\[
\sum_{k=0}^{\infty} \int_{A_k \cap X} \left( \sum_{n=k-1}^{k+1} \phi(|z|)|z|^{s+1}|f_n(z)| \right)^p \, dm
\leq C \sum_{k=M_j}^{N_j} 2^{q(s+1)k} \phi(2^{-k})^{-q} \Gamma_q(A_k \setminus X)
\leq C \text{ by choice of } M_j \text{ and } N_j.
\]

Combining the above estimates, we obtain
\[
\int_X |g_j|^p \, dm \leq C \text{ for all } j.
\]

Next we pass to a subsequence of the \(\{g_j\}\) that converges weakly to \(g \in L^p(X)\). Denote the subsequence also by \(\{g_j\}\). We form \(h_j(z) = z \phi(|z|)^{-1}g_j(z)\) and \(F_j(z) = z^{-s-1}h_j(z)\), which are analytic in \(\mathbb{C} \setminus \Delta(0, 2^{-M_j})\). By the above estimates the functions \(h_j\) and \(F_j\) are uniformly bounded on compact subsets of \(\mathbb{C} \setminus \{0\}\). Hence, there are subsequences that converge uniformly on compact subsets of \(\mathbb{C} \setminus \{0\}\) to \(h(z) = z \phi(|z|)^{-1}g(z)\) and \(F(z) = z^{-s-1}h(z)\) respectively.

We claim that \(h\) is a polynomial of degree \(s + 1\) with \(h(0) = 0\). The above estimates show that there is a number \(M > 0\) that bounds the \(h_j\) in the following sense: to any \(z \in \Delta(0, 1) \setminus \{0\}\) there corresponds an integer \(J\) such that for \(j > J\) and \(|\xi| > |z|, \, |h_j(\xi)| < M\). This implies that \(h\) is bounded near \(0\), so \(h\) is entire and \(\lim_{z \to 0} h(z) = 0\). To show that \(h\) is a polynomial we consider
\[
\lim_{z \to \infty} z^{-s-1}h(z) = F(\infty) = \lim_{j \to \infty} F_j(\infty).
\]

For all \(j\), \(F_j(\infty) = \sum_{n=M_j}^{N_j} f_n(\infty)\) lies in \([C_1/2, 3C_2]\) where \(C_1\) and \(C_2\) are the constants of Lemma (1.1). Therefore, we have that \(\lim_{j \to \infty} F_j(\infty) = \beta \in [C_1, 2C_2]\), and
functions in \( R^p(X) \)

\[
h(z) = \beta z^{s+1} + \sum_{i=1}^{s} \beta_i z^i \quad \text{where } \beta_i \text{ is a constant for each } i.
\]

Thus

\[
g_j = \phi(|z|)z^{-1}h_j \rightarrow \phi(|z|)z^{-1}h = \beta \phi(|z|)z^s + \sum_{i=1}^{s} \beta_i \phi(|z|)z^{s-i-1}
\]

weakly and pointwise on each bounded subset of \( C \setminus \{0\} \).

This means that if \( u \in L^q(X) \), then

\[
\int g_j u \, dm \rightarrow \int \beta \phi(|z|)z^s u \, dm + \sum_{i=1}^{s} \beta_i \int \phi(|z|)z^{s-i-1} u \, dm.
\]

Wilkin’s lemma (Lemma (1.4.1)) and the original hypothesis imply that there is a function \( v_j \in L^q(X) \) which is a linear combination of the functions \( z^{-j}v \), \( 0 < j < s \), such that

\[
\int f v_j \, dm = \frac{f^{(j)}(0)}{j!}
\]

for all \( f \in R_0(X) \). Taking \( u = \phi(|z|)^{-1}v_j \), we get a contradiction.

The next theorem may be proved in a similar way, and we omit many of the details.

**Theorem (1.2).** Let \( \phi \) be an admissible function and \( s \) a nonnegative integer. Suppose that there is a function \( v \in L^q(X) \) representing \( 0 \) for \( R^p(X) \) such that

\[
|z|^{-s} \phi(|z|)^{-1}v \in L^q(X).
\]

Then

\[
\lim_{r \to 0} \frac{r^{-q-s} \phi(r)^{-q} T_q(\Delta(0, r) \setminus X)}{r} = 0.
\]

**Proof.** Suppose that there is a sequence \( r_n \to 0 \) and a \( b > 0 \) such that

\[
r_n^{-q-s} \phi(r_n)^{-q} T_q(\Delta(0, r_n) \setminus X) > b \quad \text{for all } r_n.
\]

We may assume as before that

\[
2^{q(s+1)} \phi(2^{-n})^{-q} T_q(A_n \setminus X) < 1 \quad \text{for all } n.
\]

Note that if \( 2^{-k} > r_n \) and \( |2^{-k} - r_n| < 2^{-k-1} \),

\[
2^{q(s+1)} \sum_{n=k}^{\infty} 2^{q(s+1)} \phi(2^{-n})^{-q} T_q(A_n \setminus X) > b.
\]

Thus there is a sequence of integers \( M_1 < N_1 < M_2 < N_2 < \cdots \) such that

\[
2 > \sum_{n=M_j}^{N_j} 2^{q(s+1)} \phi(2^{-n})^{-q} T_q(A_n \setminus X) > 2^{-q(s+1)} b
\]

for all \( j \). The proof then proceeds as before.

2. **Density at bounded point evaluations.** We will get an estimate for \( T_q \) capacity in terms of the measure \( m \). The following lemma is in [4].
Lemma (2.1). Let \( \mu \) be a measure of total mass 1 (i.e. \( \int \, d\mu = 1 \)). If \( 1 < q < 2 \) and \( p = q/(q - 1) \), then
\[
\int \left\{ \int |\xi - z|^{-1} \, d\mu(\xi) \right\}^p \, dm \leq C \left\{ \sup_{z \in \mathbb{C}} \left| \int |\xi - z|^{q-2} \, d\mu(\xi) \right| \right\}^{p-1}
\]
where \( C \) is some constant depending only on \( p \).

Lemma (2.2). For each \( q, 1 < q < 2 \), there is a positive constant \( C \) such that
\[
\Gamma_q(X) \geq Cm(X)^{(2-q)/2}
\]
for all compact sets \( X \subset \mathbb{C} \).

Proof. Define \( f = m(X)^{-1} \int_X (z - \xi)^{-1} \, dm(\xi) \). Then \( f \) is analytic in \( \mathbb{C} \setminus X \) and \( f'(\infty) = 1 \). To estimate \( \|f\|_{p,\mathbb{C} \setminus X} \) we apply Lemma (II.2.1) with \( \mu = m(X)^{-1} \chi_X \) where \( \chi_X \) is the characteristic function of \( X \). We get
\[
\|f\|_{p,\mathbb{C} \setminus X, \mathbb{C}} < C \left\{ \sup_{z \in \mathbb{C}} m(X)^{-1} \int_X |z - \xi|^{q-2} \, dm(\xi) \right\}^{1/q}
\]
We will use \( C \) to denote any constant depending only on \( p \). Choose \( R > 0 \) so that \( R^2 = m(X) \), and let \( D = \Delta(\xi, R) \). Then since \( r^{q-2} \) is a decreasing function of \( r \),
\[
m(X)^{-1} \int_X |z - \xi|^{q-2} \, dm(\xi) \leq \pi^{-1} R^{-2} \int_0^R r^{q-2} \, dr \, d\theta
\]
\[
= \pi^{-1} R^{-2} \int_0^2 \int_0^R r^{q-2} \, dr \, d\theta
\]
\[
= 2 R^{-2} \int_0^R r^{q-1} \, dr
\]
\[
= 2(q - 1)^{-1} R^{-2} R^q = 2(q - 1)^{-1} R^{q-2}.
\]
Applying the above inequality for \( \|f\|_{p,\mathbb{C} \setminus X, \mathbb{C}} \), we have
\[
\|f\|_{p,\mathbb{C} \setminus X, \mathbb{C}} \leq CR^{(q-2)/q}.
\]
Define \( g = f/\|f\|_{p,\mathbb{C} \setminus X, \mathbb{C}} \). Then \( g \) is analytic in \( \mathbb{C} \setminus X \) and \( \|g\|_{p,\mathbb{C} \setminus X, \mathbb{C}} = 1 \). Moreover,
\[
g'(\infty) = f'(\infty)/\|f\|_{p,\mathbb{C} \setminus X, \mathbb{C}} \geq CR^{(2-q)/q} > Cm(X)^{(2-q)/2q}.
\]
By Lemma (II.1.1) we conclude that
\[
\Gamma_q(X) \geq Cm(X)^{(2-q)/2},
\]
and the proof is complete.

Corollary (2.1). Let \( \phi \) be an admissible function and \( s \) a nonnegative integer. Suppose that there is a function \( v \in L^q(X) \) representing 0 for \( R^s(X) \),
functions in $\mathbb{R}^p(X)$

$p > 2$, such that $|z|^{-1} \phi(|z|) \in L^q(X)$. Then

$$m(\Delta(0, n^{-1}) \setminus X) = o\left(\phi(n^{-1})^{2t} (n^{-1})^{2t(z+1)}\right),$$

where $t = q/(2 - q)$.

**Proof.** This follows from Theorem (II.1.2) and Lemma (II.2.2).

3. **An example.** In this section we use Hedberg’s capacity theorems to construct a Swiss cheese $Y$ such that $\cap_{p > 2} S^p(Y) = \{0\}$. Let $X_0$ be the closure of a set having positive measure whose boundary consists of finitely many analytic curves. The first step is to show that for a given $\epsilon > 0$ and $p > 2$ one can construct a Swiss cheese $X = X_0 \setminus \bigcup_{i=1}^\infty D_i$ such that:

1. $2^{-n} r_i^2 < \epsilon$, where $r_i$ is the radius of $D_i$; and
2. for some $p', p > p' > 2$, $S^p(X) = \varnothing$. For $n = 1, 2, \ldots$ we define $X_n$ inductively by letting $X_n = X_{n-1} \setminus G_n$, where $G_n = \cup \{\Delta((2^{-n}, (e2^{-n})^{3/(2-q)}),

where the summation is taken over all Gaussian integers $t$ such that $|t|^n < 1$. Then set $X = \cap_{n=0}^\infty X_n$. Since each $G_n$ consists of $< 22^n$ disks

$$\sum_{i=1}^\infty r_i^2 < 22^n \left[ (e2^{-n})^{3/(2-q)} \right]^{2-q} = \epsilon.$$

Now choose $q', q < q' < 2$, so that $3(2 - q')/(2 - q) < q'$. Let $x \in X$. We claim that $x \notin S^{q'}(X)$ where $1/p' + 1/q' = 1$. Within any disk centered at $x$ and having radius $2^{-n}$, there is a disk in $\mathbb{C} \setminus X$ having radius at least $4^{-1}(e2^{-n})^{3/(2-q)}$. Hence

$$\lim_{n \to \infty} 2^{nq'} \Gamma_{q'}(\Delta(x, 2^{-n}) \setminus X)$$

$$> 4q' \lim_{n \to \infty} 2^{nq'} (e2^{-n})^{3(2-q')/(2-q)} > 0.$$

Thus by Theorem (II.1.2), $x \notin S^{q'}(X)$, and $X$ is the desired set.

Given $\epsilon, 0 < 1/2$, it is possible by the above construction to remove open disks $D_k$ of radius $r_k$ from $A_j(0)$ to obtain a Swiss cheese $Y_j$ such that $\sum_{k=1}^\infty r_k^2 < \epsilon_j$ (1/p' + 1/q' = 1), and $S^{q'}(Y_j) = \varnothing$ for some $p_j, p_j > p_j > 2$. Choose the $\epsilon_j$ so that $\sum_{j=1}^\infty 2\epsilon_j < \infty$, and define $Y = \cup_{j=0}^\infty Y_j \cup \{0\}$.

We will use Hedberg’s theorem [9] to prove that for any $p > 2$, $0 \in S^p(Y)$. Let $p > 2$. There is an integer $J$ such that $p > p_j > 2$ for $j > J$. Hence,

$$\sum_{j=1}^\infty 2^{j\epsilon} (A_j(0) \setminus X) < C \sum_{j=1}^\infty 2^{j\epsilon} \sum_{k=1}^\infty r_k^2 < C \sum_{j=1}^\infty 2^{j\epsilon} < \infty.$$

By Hedberg’s theorem $0 \in S^p(Y)$, and since $p > 2$ was arbitrary, $0 \in \cap_{p > 2} S^p(Y)$. That $0$ is the only point in $\cap_{p > 2} S^p(Y)$ follows from the construction of $Y$ and the fact that $x \in S^p(Y)$ if and only if $x \in S^p(Y \setminus \Delta(x, r))$ for any $r > 0$.

Given any compact set $X$ it would be interesting to find necessary and sufficient conditions for $\cap_{p > 2} S^p(X)$ to have positive measure. Lemma (I.2.3)
implies that a sufficient condition is that there exist a single g which represents 0 for $R^p(X)$ for all $p > 2$.

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