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REPRESENTATION THEORY OF ALGEBRAS STABLY EQUIVALENT TO AN HEREDITARY ARTIN ALGEBRA

BY

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ABSTRACT. Two artin algebras are stably equivalent if their categories of finitely generated modules modulo projectives are equivalent. The author studies the representation theory of algebras stably equivalent to hereditary algebras, using the notions of almost split sequences and irreducible morphisms. This gives a new unified approach to the theories developed for hereditary and radical square zero algebras by Gabriel, Gelfand, Bernstein, Ponomarev, Dlab, Ringel and Müller, as well as other algebras not covered previously. The techniques are purely module theoretical and do not depend on representations of diagrams. They are similar to those used by M. Auslander and the author to study hereditary algebras.

Introduction. We recall that an artin algebra is an artin ring that is a finitely generated module over its center, which is also an artin ring. Let mod Λ denote the category of finitely generated (left) Λ-modules, and $\underline{\text{mod}}\ \Lambda$ the category of finitely generated Λ-modules modulo projectives (see [8]). We also recall that two artin algebras Λ and Λ' are said to be stably equivalent if the categories of modules modulo projectives, $\underline{\text{mod}}\ \Lambda$ and $\underline{\text{mod}}\ \Lambda'$, are equivalent.

The purpose of this paper is to study the algebras that are stably equivalent to an hereditary artin algebra. This class of algebras contains the artin algebras such that the square of the radical is zero, the hereditary algebras and other algebras that are not hereditary or of radical square zero. We generalize here the results that we proved in [7] for hereditary artin algebras, using the notions of almost split sequences and irreducible maps developed by M. Auslander and I. Reiten. Hereditary artin algebras have also been studied by P. Gabriel, I. Gelfand, Nazarova and Ponomarev, V. Dlab and C. M. Ringel using techniques of representations of diagrams and K-species (see [10], [12]–[14]). These techniques apply also to artin algebras of radical square zero, also studied using different methods by W. Müller (see [15]). The treatment that we do here is quite different from the treatment of the named authors, since it does not rely on diagramatic techniques, but is module theoretical and gives a unified approach to the hereditary and radical square zero cases, as well as to other algebras not considered previously. The ideas

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and methods of proof are similar to those of [7].

We assume in all that follows that Λ is an artin algebra stably equivalent to an hereditary algebra. All the modules that we consider are finitely generated. Let Λ^{op} denote the opposite ring of Λ, and let D: mod $\Lambda \to$ mod Λ^{op} be the ordinary duality for artin algebras. We denote by Tr: $\underline{\text{mod}}\ \Lambda \to \underline{\text{mod}}\ \Lambda^{op}$ the duality given by the transpose. For a Λ-module M, let M^* denote the Λ^{op}-module $\text{Hom}_\Lambda(M, \Lambda)$. Let $\text{mod}_P\ \Lambda$ denote the full subcategory of mod Λ of the modules with no nonzero projective summands. We recall that if M is in $\text{mod}_P\ \Lambda$ and

$$P_1 \xrightarrow{f} P_0 \to M \to 0$$

is a minimal projective presentation for M, then the transpose of M, Tr M, is the cokernel of the map

$$P_0^* \xrightarrow{f^*} P_1^*.$$

Then D Tr is an equivalence between the category $\underline{\text{mod}}\ \Lambda$ of finitely generated modules modulo projectives and the category $\overline{\text{mod}}\ \Lambda$ of finitely generated Λ-modules modulo injectives. Let $\underline{\text{Hom}}_\Lambda(M, N)$ denote the set of morphisms from M to N in $\underline{\text{mod}}\ \Lambda$.

We prove that the following conditions are equivalent for an indecomposable nonprojective module M:

(a) There exists some integer $n > 0$ such that $(D\ \text{Tr})^n M$ is torsionless, i.e., submodule of a projective module.

(b) There are only a finite number of nonisomorphic indecomposable modules X such that $\underline{\text{Hom}}_\Lambda(X, M) \neq 0$.

The equivalence of (a) and (b) is proved considering chains of irreducible maps.

We study properties of modules satisfying the equivalent conditions (a) and (b). For example, we prove that if M is an indecomposable nonprojective Λ-module verifying (a) and (b) then $\underline{\text{End}}_\Lambda(M)$ is a division ring, and $\text{Ext}^1_\Lambda(M, M) = 0$. The ring has the property that all the indecomposable modules verify (b) if and only if (b) is verified for the simple Λ-modules. One can easily prove that this is equivalent to saying that for every simple Λ-module S the number of indecomposable modules X such that $\text{Hom}_\Lambda(X, S) \neq 0$ is finite. It is known that this is the case if and only if Λ is of finite representation type, i.e., the number of nonisomorphic indecomposable Λ-modules is finite. Therefore, as a consequence of the above mentioned result we obtain the following characterization of rings of finite representation type: Λ is of finite representation type if and only if for every nonprojective Λ-module M there is some $n > 0$ such that $(D\ \text{Tr})^n M$ is torsionless.

Since (a) and (b) are satisfied for all the indecomposable modules when Λ is of finite representation type, the results obtained for modules satisfying (a) or (b) hold for all the indecomposable modules when the ring is of finite representation type.

When Λ is hereditary the torsionless modules are projective and $\underline{\mathrm{Hom}}_\Lambda(M, N) = \mathrm{Hom}_\Lambda(M, N)$ for any pair of indecomposable modules M and N with no nonzero projective summands, and we obtain as a particular case the result of [7]:

The following conditions for an indecomposable Λ-module M are equivalent:

(a) There is a projective P and a chain of irreducible maps of indecomposable modules

$$P = C_k \to C_{k-1} \to \cdots \to C_0 = M.$$

(b) There is an integer $n \geqslant 0$ such that $(D\ \mathrm{Tr})^n M$ is projective.

(c) There are only a finite number of nonisomorphic indecomposable Λ-modules X such that $\mathrm{Hom}_\Lambda(X, M) \neq 0$.

If Λ is an hereditary ring of finite representation type and M is an indecomposable Λ-module then $\mathrm{Ext}^1_\Lambda(M, M) = 0$ and $\mathrm{End}_\Lambda(M)$ is a division ring.

Let $\mathfrak{a}$ denote the two sided ideal sum of the nonprojective modules of the socle of Λ and let $\mathfrak{b}$ be the left annihilator of $\mathfrak{a}$ in Λ. For a ring Γ we denote by $\mathrm{Gr}(\Gamma)$ the Grothendieck group of Γ and by $[M]$ the element of $\mathrm{Gr}(\Gamma)$ determined by the module M. We consider the group $G = \mathrm{Gr}(\Lambda/\mathfrak{a}) \times \mathrm{Gr}(\Lambda/\mathfrak{b})$ and we associate to a Λ-module M the element $\langle M \rangle = ([M/\mathfrak{a}M], [\mathfrak{a}M])$ in G.

We define a group isomorphism $c\colon G \to G$ such that if M is an indecomposable nonprojective module then $c(\langle M \rangle) = (0, [D\ \mathrm{Tr}\ M])$ if $D\ \mathrm{Tr}\ M$ is nonprojective torsionless and $c(\langle M \rangle) = \langle D\ \mathrm{Tr}\ M \rangle$ otherwise. This isomorphism is an important tool in the study of the representation theory of the ring. For example, using it we prove that if M and N are indecomposable Λ-modules such that $\langle M \rangle = \langle N \rangle$ and there are only a finite number of indecomposable modules X such that $\underline{\mathrm{Hom}}_\Lambda(X, M) \neq 0$ then M and N are isomorphic. If the ring Λ is hereditary then $\mathfrak{a} = 0$, $G = \mathrm{Gr}(\Lambda)$ and we obtain that if M and N are two indecomposable modules with the same composition factors and $(D\ \mathrm{Tr})^n M$ is projective for some $n \geqslant 0$ then $M \cong N$. In particular, when Λ is hereditary and of finite representation type then the indecomposable modules are determined by their composition factors, results that have been proven in [7].

We also associate to the ring Λ a bilinear form B from $G \times G$ to the field of rational numbers such that the following conditions are equivalent:

(a) Λ is of finite representation type.

(b) There is some integer $m > 0$ such that $c^m = \mathrm{Id}_G$.

(c) B is positive definite.

An explicit description of the indecomposable modules can be given when the ring is of finite representation type. In this case, $c^m = \mathrm{Id}$ for some $m > 0$. Let $I_1, \ldots, I_n$ be a complete set of nonisomorphic indecomposable injective Λ-modules. Then for each $i = 1, \ldots, n$ there is a nonnegative integer $n_i < m$ such that $(D\,\mathrm{Tr})^{n_i} I_i$ is projective. Let $\mathfrak{D} = \{(D\,\mathrm{Tr})^r(I_i),\ 0 \leqslant r \leqslant n_i,\ i = 1, \ldots, n\}$. If M is an indecomposable Λ-module we say that M is D Tr-periodic if for some $k > 0$ $(D\,\mathrm{Tr})^k M \cong M$. Then if M is an indecomposable Λ-module, M is D Tr-periodic or M is in $\mathfrak{D}$. Moreover, if M is D Tr-periodic there is r such that $0 < r < m$ and $M \cong (D\,\mathrm{Tr})^r S$, for some torsionless simple nonprojective Λ-module S.

We recall from [8] that an artin algebra Λ is stably equivalent to an hereditary algebra if and only if the following conditions are satisfied:

(1) Each indecomposable submodule of an indecomposable projective Λ-module is projective or simple.

(2) If S is a nonprojective simple submodule of a projective then there is an injective module E and an epimorphism $E \to S$.

We develop the first five sections using the ideal theoretical characterization of rings stably equivalent to an hereditary ring just mentioned. We obtain then, as particular cases, results known for hereditary artin algebras and for artin algebras of radical square zero, that have been already studied separately (see [7], [13], [15]).

In the last section, instead, we use a concrete description of a functor

$$F\colon \mathrm{mod}\,\Lambda \to \mathrm{mod}\begin{pmatrix} \Lambda/\mathfrak{a} & 0 \\ \mathfrak{a} & \Lambda/\mathfrak{b} \end{pmatrix}$$

that induces a stable equivalence between the category of Λ-modules and the category of modules over the hereditary ring $\left(\begin{smallmatrix} \Lambda/\mathfrak{a} & 0 \\ \mathfrak{a} & \Lambda/\mathfrak{b} \end{smallmatrix}\right)$. Here the results are obtained using F and the fact that they are known for hereditary artin algebras (see [7]).

Most of the results of this paper can be proven by using either of the mentioned techniques. The treatment in the last section is different than the one used in the first five, to illustrate how both methods can be used.

This paper is part of my doctoral dissertation at Brandeis University (1975). I would like to take this opportunity to thank Professor Maurice Auslander, my thesis advisor, for many suggestions, ideas and helpful discussions, as well as for his constant encouragement.

1. Preliminaries and notations. We devote this section to recalling some definitions and results of [3]–[6] and [8] concerning almost split sequences, irreducible maps and stable equivalence that will be needed later.

Λ will always indicate an artin algebra; all the modules that we consider are finitely generated. We recall from [3] that a nonsplit exact sequence

$$0 \to A \to B \xrightarrow{g} C \to 0$$

in mod Λ is *almost split* if A and C are indecomposable, and given any morphism h: $X \to C$ which is not a splittable epimorphism, there is some s: $X \to B$ such that $gs = h$. It is proved in [3], for a given nonprojective indecomposable module C or for a given noninjective indecomposable module A, the existence and uniqueness of an almost split sequence

$$0 \to A \to B \to C \to 0.$$

Moreover, $A \cong D \operatorname{Tr} C$ [3, Proposition 4.3].

A map f: $A \to B$ is said to be *right almost split* if it is not a splittable epimorphism, and given any morphism h: $X \to B$ which is not a splittable epimorphism, there is a morphism g: $X \to A$ such that $fg = h$. f: $A \to B$ is said to be *right minimal* if for any commutative diagram

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "g"'] & & B \arrow[dl, "f"] \\ & A & \end{tikzcd}$$

g is an isomorphism. The map f: $A \to B$ is *minimal right almost split* if it is right minimal and right almost split. There are analogous definitions by replacing right by left (see [4, §2]).

Let C in mod Λ be indecomposable. Then, if C is not projective, a map g: $B \to C$ is minimal right almost split if and only if $0 \to \operatorname{Ker}(g) \to B \to C \to 0$ is an almost split sequence. If C is projective then g: $B \to C$ is minimal right almost split if and only if g is a monomorphism and $g(B) = \mathrm{r}C$, where r denotes the radical of Λ.

We recall also that a map g: $B \to C$ is said to be *irreducible* if g is neither a split monomorphism nor a split epimorphism and for any commutative diagram

$$\begin{tikzcd} B \arrow[rr, "g"] \arrow[dr, "f"'] & & C \\ & X \arrow[ur, "h"'] & \end{tikzcd}$$

f is a splittable monomorphism or h is a splittable epimorphism (see [4]). If C in mod Λ is indecomposable then a map g: $B \to C$ where B is nonzero is irreducible if and only if there is some map g': $B' \to C$ such that (g, g'):

$B \amalg B' \to C$ is minimal right almost split [4, Theorem 2.4]. Analogous results hold for left almost split maps.

As a consequence of this we have [7, Proposition 1.1]

PROPOSITION 1.1. *If M is an indecomposable nonprojective module and there is an irreducible map f: $P \to M$ with P indecomposable projective, then D Tr M is a direct summand of τP. If there is an irreducible map $M \to P$ with P indecomposable projective then M is a direct summand of τP.*

For M in mod Λ we denote by $(\ ,M)$ the representable functor $N \to \mathrm{Hom}_\Lambda(N, M)$, for N in mod Λ. We denote by $(\underline{M}, \underline{N})$ and $(\overline{N}, \overline{M})$ the groups of morphisms from M to N in $\underline{\mathrm{mod}}\,\Lambda$ and $\overline{\mathrm{mod}}\,\Lambda$ respectively, and by $\mathrm{mod}_P\Lambda$ and $\mathrm{mod}_I\Lambda$ the full subcategories of mod Λ whose objects are the Λ modules with no nonzero projective summands and with no nonzero injective summands respectively.

We also recall that if F is a finitely presented functor from mod Λ to the category of abelian groups, then F has finite length if and only if there are only a finite number of nonisomorphic indecomposable modules X such that $F(X) \neq 0$ (see [2]).

A module M is said to be *torsionless* if it is a submodule of a projective module. We say that M is *torsion* if all the indecomposable summands of M are not torsionless. There is a characterization of the artin algebras Λ that are stably equivalent to an hereditary algebra in terms of the torsionless submodules of Λ, given in the following

PROPOSITION 1.2. *An artin algebra Λ is stably equivalent to an hereditary algebra if and only if the following conditions are satisfied*:

(1) *Each indecomposable submodule of an indecomposable projective Λ-module is projective or simple.*

(2) *If S is a nonprojective torsionless simple Λ-module then there is an injective module E and an epimorphism $E \to S$.*

If Λ satisfies the condition (1) of Proposition 1.2, then each indecomposable torsionless Λ-module is contained in an indecomposable projective Λ-module, hence is projective or simple [8, Lemma 2.2].

We recall now the following result concerning almost split sequences [3, Proposition 5.7].

Let Λ be stably equivalent to an hereditary ring. Let A be a simple noninjective module that is projective or is a factor of an injective module and let $0 \to A \to B \to C \to 0$ be the almost split sequence. Then B is a projective Λ-module.

If a ring Γ is hereditary the opposite ring Γ^{op} is also hereditary. Therefore, if Λ is stably equivalent to the hereditary ring Γ then Λ^{op} is stably equivalent

to the hereditary ring Γ^{op}. We shall say that a module M is *cotorsionless* if it is a factor of an injective module. We have then:

PROPOSITION 1.3. *Let Λ be stably equivalent to an hereditary algebra. Then*:

(a) *Every indecomposable cotorsionless module is injective or simple.*

(b) *Let S be a simple module. Then S is nonprojective torsionless if and only if S is cotorsionless noninjective.*

(c) *Let $0 \to A \to B \to C \to 0$ be an almost split sequence of Λ-modules. If A is simple torsionless then B is projective. If C is simple cotorsionless then B is injective.*

PROOF. (a) and (c) follow by duality.

(b) Let S be a noninjective cotorsionless simple module. Let $0 \to S \to B \to C \to 0$ be the almost split sequence. Then B is projective and therefore S is torsionless. Since S is cotorsionless noninjective there is a nonsplittable epimorphism $E \to S \to 0$, with E injective. Then S is not projective. So, if S is noninjective cotorsionless then S is nonprojective torsionless. The converse can be proven by duality.

Throughout the rest of this paper we will assume, unless otherwise specified, that Λ is an artin algebra stably equivalent to an hereditary artin algebra.

2. **Indecomposable modules M such that $(\ ,\underline{M})$ has finite length.** In [7] we characterized the modules M over an hereditary artin algebra such that the functor $(\ ,M)$ has finite length, as those modules such that $(D\,\mathrm{Tr})^n M$ is projective for some $n \geqslant 0$, and we proved that this is the case if and only if there is a chain of irreducible maps of indecomposable Λ-modules $C_0 \to C_1 \to \cdots \to C_k = M$ with C_i indecomposable and C_0 projective.

We are going to generalize now these results to algebras that are stably equivalent to an hereditary artin algebra. In this case we will prove, for a nonprojective indecomposable module M, that the functor $(\ ,\underline{M})$ has finite length if and only if there is a positive integer n such that $(D\,\mathrm{Tr})^n M$ is torsionless, and we will also prove results about the chains of irreducible maps similar to those mentioned above for hereditary algebras. When the ring is hereditary the torsionless modules are projective and $(N, M) = (\underline{N}, \underline{M})$ for every pair of nonprojective indecomposable modules M and N; therefore the results for hereditary artin algebras proved in [7] can be obtained from these as a particular case. It is known (see [2]) that Λ is of finite representation type if and only if the functors $(\ ,S)$ have finite length for every simple Λ-module S. Since for S simple $(\ ,S)$ has finite length if and only if $(\ ,\underline{S})$ has finite length, we obtain as an application a criterion to decide when the ring is of finite representation type.

We begin by studying properties of the chains of irreducible maps $C_0 \to C_1$

$\to \cdots \to C_k = M$, with C_i indecomposable. We prove first

LEMMA 2.1. *Let M in* $\mathrm{mod}_P(\Lambda)$ *be indecomposable and assume that there is a chain of irreducible maps of indecomposable modules*

$$P = C_0 \xrightarrow{f_1} C_1 \to \cdots \xrightarrow{f_k} C_k = M$$

with P projective. Then there is some positive integer $n \leqslant k$ such that $(D\,\mathrm{Tr})^n M$ is torsionless.

PROOF. We prove the lemma by induction on the length k of the chain of maps. If $k = 1$ we have an irreducible map f_1: $P \to M$; we know by Proposition 1.1 that $D\,\mathrm{Tr}\,M$ is a summand of $\mathrm{r}\,P$ and is, therefore, torsionless. We assume that the lemma is true if $k \leqslant j$; let $P = C_0 \to^{f_1} \cdots \to C_j \to^{f_{j+1}} C_{j+1}$ be a chain of irreducible maps of indecomposable Λ-modules with P projective and C_{j+1} not projective. If C_j is projective we know by the above argument that $D\,\mathrm{Tr}(C_{j+1})$ is torsionless, so we may assume that C_j is not projective; we know then by the induction hypothesis that there is a positive integer $n \leqslant j$ such that $(D\,\mathrm{Tr})^n(C_j)$ is torsionless. We want to see that there is a positive integer $m \leqslant j+1$ such that $(D\,\mathrm{Tr})^m(C_{j+1})$ is torsionless; as $n < j+1$ we may assume that $(D\,\mathrm{Tr})^n(C_{j+1})$ is not projective. Then (see [5, Proposition 1.2]) the map

$$(D\,\mathrm{Tr})^n(f_{j+1}): (D\,\mathrm{Tr})^n(C_j) \to (D\,\mathrm{Tr})^n(C_{j+1})$$

is irreducible. But we know that $(D\,\mathrm{Tr})^n(C_j)$ is torsionless; if it is projective then $(D\,\mathrm{Tr})((D\,\mathrm{Tr})^n(C_{j+1})) = (D\,\mathrm{Tr})^{n+1}(C_{j+1})$ is a summand of $\mathrm{r}(D\,\mathrm{Tr})^n(C_j)$ and is therefore, torsionless. So we may assume that $S = (D\,\mathrm{Tr})^n(C_j)$ is a nonprojective torsionless, hence simple, Λ-module. As $S = D\,\mathrm{Tr}((D\,\mathrm{Tr})^{n-1}(C_j))$ is not injective we can consider the almost split sequence $0 \to S \to E \to C \to 0$; we know by Proposition 1.3 that the middle term E is projective. The map $(D\,\mathrm{Tr})^n(f_{j+1})$: $S \to (D\,\mathrm{Tr})^n(C_{j+1})$ is irreducible. Then $(D\,\mathrm{Tr})^n(C_{j+1})$ is isomorphic to a direct summand of E and is, therefore, projective. This ends the proof of the lemma.

We will see now that the converse of Lemma 2.1 is true. We will prove not only that if $(D\,\mathrm{Tr})^n M$ is torsionless for some $n > 0$ then there is a chain of irreducible maps as in Lemma 2.1, but also that the length of any chain of irreducible maps of indecomposable nonprojective modules $C_t \to \cdots \to C_1 \to C_0 = M$ is bounded.

We recall that if M is a Λ-module the *Loewy length of M*, that we denote $\mathrm{LL}(M)$, is the smallest positive integer s such that $\mathrm{r}^s M = 0$.

Let M be an indecomposable nonprojective module such that $(D\,\mathrm{Tr})^n M$ is torsionless for some $n > 0$. We will associate to M a pair $\alpha(M)$ of natural numbers in the following way: let n_M denote the smallest positive integer such that $(D\,\mathrm{Tr})^{n_M} M$ is torsionless; we write $\alpha(M) = (n_M, \mathrm{LL}(D\,\mathrm{Tr})^{n_M}(M))$. Let N

be the set of natural numbers. We consider $N \times N$ lexicographically ordered; this is, for (a, b), $(c, d) \in N \times N$, $(a, b) < (c, d)$ if and only if $a < c$ or $a = c$ and $b < d$.

Using the following proposition it will be easy to find a bound for the length of the chains of irreducible maps $C_0 \to C_1 \to \cdots \to C_k = M$, with C_i indecomposable and not projective, $i = 0, \ldots, k$.

PROPOSITION 2.2. *Let M be an indecomposable nonprojective module such that $(D\ \mathrm{Tr})^n M$ is torsionless for some $n > 0$. Let f: $C \to M$ be an irreducible morphism of indecomposable modules. Then, if C is not projective there is some $m > 0$ such that $(D\ \mathrm{Tr})^m C$ is torsionless and $\alpha(C) < \alpha(M)$.*

PROOF. It is enough to prove, for an irreducible morphism f: $C \to M$ with C indecomposable, that

(a) If $(D\ \mathrm{Tr})^n M$ is simple, then $(D\ \mathrm{Tr})^j C$ is projective for some j satisfying $0 \leqslant j < n$.

(b) If $(D\ \mathrm{Tr})^n M$ is not simple and $(D\ \mathrm{Tr})^j C$ is not projective for $0 \leqslant j < n$ then $(D\ \mathrm{Tr})^n C$ is a summand of $\mathrm{r}(D\ \mathrm{Tr})^n M$ and is therefore torsionless with $\mathrm{LL}((D\ \mathrm{Tr})^n C) < \mathrm{LL}((D\ \mathrm{Tr})^n M)$.

PROOF OF (a). Suppose $(D\ \mathrm{Tr})^n M$ is simple and $(D\ \mathrm{Tr})^j C$ is not projective for any j satisfying $0 \leqslant j \leqslant n - 2$. Then the map

$$(D\ \mathrm{Tr})^{n-1}(f)\colon (D\ \mathrm{Tr})^{n-1} C \to (D\ \mathrm{Tr})^{n-1} M$$

is irreducible [5, Proposition 1.2]. Let E be such that $(D\ \mathrm{Tr})^{n-1} C \amalg E \to (D\ \mathrm{Tr})^{n-1} M$ is minimal right almost split [4, Theorem 2.4]. Then

$$0 \to (D\ \mathrm{Tr})^n M \to (D\ \mathrm{Tr})^{n-1} C \coprod E \to (D\ \mathrm{Tr})^{n-1} M \to 0$$

is the almost split sequence, so there is an irreducible map $(D\ \mathrm{Tr})^n M \to (D\ \mathrm{Tr})^{n-1} C$; since $(D\ \mathrm{Tr})^n M$ is a simple torsionless module we have, by Proposition 1.3 and by [4, Theorem 2.4] that $(D\ \mathrm{Tr})^{n-1} C$ is projective.

(b) Since $(D\ \mathrm{Tr})^j C$ is not projective for $0 \leqslant j < n$, the map $(D\ \mathrm{Tr})^n(f)$: $(D\ \mathrm{Tr})^n C \to (D\ \mathrm{Tr})^n M$ is irreducible. Since $(D\ \mathrm{Tr})^n M$ is not simple and is torsionless it is projective. Hence $(D\ \mathrm{Tr})^n C$ is a summand of $\mathrm{r}(D\ \mathrm{Tr})^n M$ (see Proposition 1.1).

COROLLARY 2.3. *Let M be an indecomposable nonprojective Λ-module such that $(D\ \mathrm{Tr})^n M$ is torsionless for some integer $n > 0$. Then for any chain of irreducible maps $C_0 \to C_1 \to \cdots \to C_t = M$ of indecomposable Λ-modules of length $t \geqslant \mathrm{LL}((D\ \mathrm{Tr})^n M) + (n - 1) \cdot \mathrm{LL}(\Lambda)$ there is some integer i such that $0 \leqslant i < t$ and C_i is projective.*

PROOF. If C_i is not projective for all $0 \leqslant i < t$ then we have, by Proposition 2.2, that $\alpha(C_0) < \alpha(C_1) < \cdots < \alpha(C_t) = \alpha(M)$. For every i,

$\alpha(C_i) = (a_i, b_i)$, with $b_i \leqslant \mathrm{LL}(\Lambda)$. But, given $(a, b) \in N \times N$ and $m \in N$, the number of pairs $(c, d) \leqslant (a, b)$ such that $d \leqslant m$ is $b + (a - 1) \cdot m$. So, since $\alpha(M) = (n_M, \mathrm{LL}(D\,\mathrm{Tr})^{n_M}M)$, $\mathrm{LL}((D\,\mathrm{Tr})^{n_M}M) \leqslant \mathrm{LL}(\Lambda)$ and the C_i verify that $\alpha(C_i) \neq \alpha(C_j)$ if $i \neq j$, then the number of C_i, that is $t + 1$, is not bigger than $\mathrm{LL}(D\,\mathrm{Tr})^n M + (n - 1) \cdot \mathrm{LL}(\Lambda)$; this contradicts the hypothesis that $t \geqslant \mathrm{LL}((D\,\mathrm{Tr})^n M) + (n - 1)\mathrm{LL}(\Lambda)$. Therefore there is i such that $0 \leqslant i < t$ and C_i is projective.

We saw in [7] that if M is an indecomposable module over an hereditary ring then there is a chain of irreducible maps of indecomposable modules $C_k \to \cdots \to C_0 = M$ with C_k projective if and only if the functor $(\ ,M)$ has finite length. This is not true in general, not even for rings stably equivalent to an hereditary ring. For example, there are rings of radical square zero, hence stably equivalent to an hereditary ring, that contain a projective P such that $(\ ,P)$ has infinite length. We are going to prove that if Λ is stably equivalent to an hereditary ring there is a chain of irreducible maps as mentioned above if and only if $(\ ,\underline{M})$ has finite length, and that this is also equivalent to saying that the length of chains of irreducible maps of indecomposable nonprojective modules $C_k \to \cdots \to C_0 = M$ is bounded. Part of this is true in a more general context.

We recall first some definitions and results of [9] about the category mod (mod Λ) of finitely presented contravariant functors from mod Λ to the category of abelian groups. If F is in mod (mod (Λ)), $\mathrm{r}F$ is defined to be the intersection of all the maximal subfunctors of F (see [9]). Using the fact that simple functors are finitely presented, it is shown in [9] that $\mathrm{r}F$ is finitely presented and $F/\mathrm{r}F$ is a finite sum of simple functors. We write $\mathrm{r}^iF = \mathrm{r}(\mathrm{r}^{i-1}F)$, if $i > 1$. The *Loewy length* of F is defined to be the smallest positive integer n with $\mathrm{r}^nF = 0$, if such an n exists, and ∞ otherwise. Then F has finite length if and only if it has finite Loewy length. We will denote the length of F by $l(F)$.

We also recall from [1] and [2] that the functors $(\ ,M)$ with M in mod Λ are projective objects in mod (mod Λ) and that, if M is indecomposable, then $(\ ,M)/\mathrm{r}(\ ,M)$ is a simple functor.

We will need the following result of [6].

PROPOSITION 2.4. *Let G be in* mod (mod (Λ)), f: $(\ ,C) \to G$ *a projective cover and* $F = (\ ,A)/\mathrm{r}(\ ,A)$, *where A is indecomposable in* mod Λ, *a simple object that is a direct summand of* $\mathrm{r}^iG/\mathrm{r}^{i+1}G$ *for some* $i \geqslant 1$. *Then there is some chain*

$$A = C_0 \xrightarrow{f_1} C_1 \to \cdots \to C_{i-1} \xrightarrow{f_i} C_i$$

of irreducible maps between indecomposable modules and a splittable monomorphism $C_i \to C$ such that the image of the composition morphism

$(\ ,C_0) \to \cdots \to (\ ,C_i) \to (\ ,C) \to G$ *is contained in* $\mathfrak{r}^i G$ *but not in* $\mathfrak{r}^{i+1}G$.

We will use also the following result about almost split sequences. (See [2], [9].)

LEMMA 2.5. *Let* Λ *be an arbitrary artin algebra,* C *an indecomposable* Λ*-module and* $f\colon B \to C$ *a right almost split map. Then*

(a) *The cokernel of the map* $(\ ,f)\colon (\ ,B) \to (\ ,C)$ *is a simple functor in* mod (mod Λ).

(b) *The cokernel of the map* $(\ ,\underline{f})\colon (\ ,\underline{B}) \to (\ ,\underline{C})$ *is simple or zero.*

PROOF. (a) is proven in [2, Corollary 2.6]. (b) is a trivial consequence of (a).

LEMMA 2.6. *Let* Λ *be an artin algebra (not necessarily stably equivalent to an hereditary algebra). Let* M *be an indecomposable nonprojective* Λ*-module such that the length of the chains of irreducible maps of indecomposable nonprojective* Λ*-modules* $C_k \xrightarrow{f_k} \cdots \xrightarrow{f_1} C_0 = M$ *is bounded. Then*

(a) $(\ ,\underline{M})$ *has finite length and for every irreducible map of indecomposable modules* $f\colon C \to M$ *the length of* $(\ ,\underline{C})$ *is finite.*

(b) *Let* $g\colon P \to M$ *be the projective cover of* M. *Then* g *can be written as a sum of compositions of irreducible maps between indecomposable modules. In particular, there is a chain of irreducible maps*

$$P_1 = D_s \xrightarrow{f_s} \cdots \xrightarrow{f_1} D_0 = M$$

with the D_i *indecomposables,* P_1 *projective and* $f_1 \cdots f_s \neq 0$.

PROOF. Let K be the maximum of the lengths of chains of irreducible maps of indecomposable nonprojective modules of the form

$$C_k \to C_{k-1} \to \cdots \to C_0 = M.$$

We prove (a) by induction on K. If $K = 0$ then for any irreducible map $C_1 \to M$, C_1 is projective; therefore, if $E \to M \to 0$ is minimal right almost split then E is projective; if X is an indecomposable module not isomorphic to M and there is a nonzero map $f\colon X \to M$, then f can be factored through E, that is projective. So $\underline{f} = 0$ and therefore $(\underline{X}, \underline{M}) = 0$ if X is not isomorphic to M; thus $(\ ,\underline{M})$ has finite length.

We assume now that the theorem is true if $K \leqslant r$, and consider $K = r + 1$. Let $E \xrightarrow{f} M \to 0$ be minimal right almost split, let $E = \coprod_{i=1}^{t} E_i$, with E_i indecomposable for $i = 1, \ldots, t$. Then the map $f|E_i\colon E_i \to M$ is irreducible and therefore the length of the chains of irreducible maps of indecomposable nonprojective Λ-modules $D_s \to \cdots \to D_0 = E_i$ is smaller than $r + 1$, so by the induction hypothesis we know that $(\ ,\underline{E_i})$ has finite length, $i = 1, \ldots, t$. Then $(\ ,\underline{E})$ has finite length. On the other hand, we know by Lemma 2.5 that the cokernel of the map $(\ ,\underline{f})\colon (\ ,\underline{E}) \to (\ ,\underline{M})$ is simple or zero. Hence, as $(\ ,\underline{E})$ has finite length, $(\ ,\underline{M})$ has also finite length.

(b) Let g: $P \to M$ be the projective cover of M. We want to prove that g can be written as a sum of compositions of irreducible maps between indecomposable modules. We proceed again by induction on the maximum K of the lengths of chains of irreducible maps of indecomposable nonprojective modules of the form $C_k \to C_{k-1} \to \cdots \to C_0 = M$.

If $K = 0$ and f: $E \to M$ is minimal right almost split then E is projective, and if we write $E = \coprod_{i=1}^{t} E_i$, with E_i indecomposable then $f = \coprod_{i=1}^{t} f|E_i$ is a sum of irreducible maps.

So we assume that (b) is true when $K \leqslant r$, and we suppose $K = r + 1$. Let f: $E \to M$ be minimal right almost split. Then $E = \coprod_{i=1}^{t} E_i$, with E_i indecomposable. Since each $f|E_i$: $E_i \to M$ is irreducible, the lengths of the chains of irreducible maps $D_s \to \cdots \to D_0 = E_i$ are smaller than $r + 1$, and by the induction hypothesis we know that there are projectives P_i and epimorphisms g_i: $P_i \to E_i$ that can be written as a sum of compositions of irreducible maps between indecomposable Λ-modules. Then the composition

$$\coprod_{i=1}^{t} P_i \xrightarrow{\coprod g_i} \coprod_{i=1}^{t} E_i \xrightarrow{\Sigma f|E_i} M \to 0$$

is an epimorphism that can be written as a sum of compositions of irreducible maps. This proves (b).

PROPOSITION 2.7. *Assume that Λ is stably equivalent to an hereditary artin algebra and let M be an indecomposable nonprojective Λ-module. Then the following conditions are equivalent*:

(a) *There is a projective module P and a chain of irreducible maps of indecomposable Λ-modules*

$$P = C_k \xrightarrow{f_k} \cdots \xrightarrow{f_1} C_0 = M.$$

(b) *There exists some integer $n > 0$ such that $(D\,\mathrm{Tr})^n M$ is torsionless.*

(c) *There is an integer K such that the length of any chain of irreducible maps of indecomposable nonprojective modules $C_t \to \cdots \to C_0 = M$ is smaller than K.*

(d) *The projective cover f: $P \to M$ can be written as a sum of compositions of irreducible maps between indecomposable modules.*

(e) $(\ ,\underline{M})$ *has finite length.*

PROOF. Lemma 2.1 proves that (a) $\Rightarrow$ (b), Corollary 2.3 proves that (b) $\Rightarrow$ (c); (c) $\Rightarrow$ (d) is a consequence of Proposition 2.6. Obviously (d) $\Rightarrow$ (a). Proposition 2.6 also proves that (c) $\Rightarrow$ (e), so it is enough to prove that (e) $\Rightarrow$ (b). Assume that $(\ ,\underline{M})$ has finite length. We want to prove that there exists some integer $n > 0$ such that $(D\,\mathrm{Tr})^n M$ is torsionless. If N is an indecomposable nonprojective module then $(\underline{N}, \underline{M}) \cong (\overline{D\mathrm{Tr}N}, \overline{D\mathrm{Tr}M})$. All

the indecomposable noninjective Λ-modules N_1 can be written as $N_1 = D\,\mathrm{Tr}\,N$, with $N = \mathrm{Tr}\,DN_1$, and there are only a finite number of injective indecomposable modules. Combining this with the fact that the length of $(\ ,\underline{M})$ is finite, we have that there are only a finite number of indecomposable modules N_1 such that $(\overline{N_1}, \overline{D\mathrm{Tr}M}) \neq 0$, i.e., $(\ ,\overline{D\mathrm{Tr}M})$ has finite length.

Let θ: $P \to D\,\mathrm{Tr}\,M$ be an indecomposable summand of the projective cover of $D\,\mathrm{Tr}\,M$. We call $(\overline{\ ,\theta})$ the composition $(\ ,P) \to (\ ,D\,\mathrm{Tr}\,M) \to (\ ,\overline{D\mathrm{Tr}M})$. Since (b) is verified if $D\,\mathrm{Tr}\,M$ is torsionless to prove that (e) $\Rightarrow$ (b) we may assume that $D\,\mathrm{Tr}\,M$ is not torsionless; we shall prove that in this case the map $(\overline{\ ,\theta})$ is nonzero.

So we assume that $D\,\mathrm{Tr}\,M$ is not torsionless and that $(\overline{\ ,\theta})(\mathrm{id}) = 0$; this means that θ factors through an injective module I, i.e., that we have a commutative diagram

$$\begin{tikzcd} P \arrow[rr, "\theta"] \arrow[dr, "\alpha"'] & & D\,\mathrm{Tr}\,M \\ & I \arrow[ur, "\beta"'] & \end{tikzcd}$$

with I injective. $\mathrm{Im}(\beta)$ cannot contain an injective summand because $D\,\mathrm{Tr}\,M$ is in $\mathrm{mod}_I(\Lambda)$. Therefore $\mathrm{Im}(\beta)$ is torsionless semisimple and the image of θ, that is contained in the image of β, is contained in $\mathrm{soc}(D\,\mathrm{Tr}\,M)$. On the other hand, $P \to^{\theta} D\,\mathrm{Tr}\,M$ is a summand of the projective cover of $D\,\mathrm{Tr}\,M$, so $\mathrm{Im}(\theta) \not\subseteq \mathrm{r}\,D\,\mathrm{Tr}\,M$. As $\mathrm{Im}(\theta) \subseteq \mathrm{soc}(D\,\mathrm{Tr}\,M)$ we have that $D\,\mathrm{Tr}\,M = \mathrm{soc}(D\,\mathrm{Tr}\,M)$. So the indecomposable module $D\,\mathrm{Tr}\,M$ must be simple and therefore $\mathrm{Im}(\beta) = D\,\mathrm{Tr}\,M$, contradiction, because $\mathrm{Im}(\beta)$ is torsionless and we assumed that $D\,\mathrm{Tr}\,M$ is not torsionless.

We have then a nonzero map $(\overline{\ ,\theta})$: $(\ ,P) \to (\ ,\overline{D\mathrm{Tr}M})$; the functor $(\ ,\overline{D\mathrm{Tr}M})$ has finite length, so there is some i such that $(\ ,P)/\mathrm{r}(\ ,P)$ is isomorphic to a direct summand of $\mathrm{r}^i(\ ,\overline{D\mathrm{Tr}M})/\mathrm{r}^{i+1}(\ ,\overline{D\mathrm{Tr}M})$. Since $(\ ,D\,\mathrm{Tr}\,M) \to (\ ,\overline{D\mathrm{Tr}M})$ is a projective cover we know by Proposition 2.4 that there is a chain of irreducible maps of indecomposable Λ-modules $P = C_i \to^{f_i} \cdots \to^{f_1} C_0 = D\,\mathrm{Tr}\,M$.

We know that (a) $\Rightarrow$ (b), so there is an integer $m > 0$ such that $(D\,\mathrm{Tr})^m(D\,\mathrm{Tr}\,M)$ is torsionless, i.e., $(D\,\mathrm{Tr})^{m+1}M$ is torsionless. This ends the proof of (e) $\Rightarrow$ (b).

Let S be a simple Λ-module, X in $\mathrm{mod}_P(\Lambda)$ and f: $X \to S$ be a nonzero map that factors through a projective P, i.e., such that there is a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\alpha"'] & & S \\ & P \arrow[ur, "\beta"'] & \end{tikzcd}$$

Then $\alpha(X)$ cannot contain a projective summand and is, therefore, a sum of nonprojective torsionless simples. Let $S_1 \subseteq \alpha(X)$ be a simple such that $\beta(S_1) \neq 0$. Then $\beta|S_1\colon S_1 \to S$ is an isomorphism. The map $f\colon X \to S$ is an epimorphism, so there is a map $\rho\colon P \to X$ such that $\beta = f\rho$. Then

$$f\cdot\left((\rho|S_1)\cdot(\beta|S_1)^{-1}\right) = (\beta|S_1)(\beta|S_1)^{-1} = \mathrm{Id}_S,$$

so S is a direct summand of X. We have proven

LEMMA 2.8. *Let S be a simple Λ-module and X be an indecomposable Λ-module not projective and not isomorphic to S. Then $(X, S) = (\underline{X}, \underline{S})$ and therefore the length of $(\ ,\underline{S})$ is finite if and only if $(\ ,S)$ has finite length.*

We end this section with the following summary of the previous results.

THEOREM 2.9. *For an artin algebra Λ stably equivalent to an hereditary ring the following conditions are equivalent*:

(a) *Λ is of finite representation type.*

(b) *For every indecomposable nonprojective Λ-module M there exists an integer $n > 0$ such that $(D\ \mathrm{Tr})^n M$ is torsionless.*

(c) *For every nonprojective simple Λ-module S there exists an integer $n > 0$ such that $(D\ \mathrm{Tr})^n S$ is torsionless.*

(d) *For every indecomposable nonprojective Λ-module M there is a projective Λ-module P and a chain of irreducible maps of indecomposable Λ-modules*

$$P = C_k \xrightarrow{f_k} C_{k-1} \to \cdots \xrightarrow{f_1} C_0 = M.$$

(e) *For every indecomposable module M there is a positive integer K such that the length of any chain of irreducible maps of indecomposable nonprojective Λ-modules $C_k \to^{f_k} \cdots \to^{f_1} C_0 = M$ is smaller than K. In particular there is a chain*

$$P = D_s \xrightarrow{f_s} \cdots \xrightarrow{f_1} D_0 = M,$$

with P projective, f_i irreducible and D_i indecomposable, $i = 0, \ldots, s$ such that $f_s \cdots f_1 \neq 0$.

(f) *For every simple nonprojective Λ-module S there is a projective P and a chain of irreducible maps of indecomposable modules $P = C_k \to^{f_k} \cdots \to^{f_1} C_0 = S$.*

PROOF. (a) $\Rightarrow$ (b). If Λ is of finite representation type and M is an indecomposable nonprojective module then $(\ ,M)$ has finite length; therefore, $(\ ,\underline{M})$ has finite length and we know by Proposition 2.7 that there is $n > 0$ such that $(D\ \mathrm{Tr})^n M$ is projective.

It is obvious that (b) $\Rightarrow$ (c), (e) $\Rightarrow$ (d) and (d) $\Rightarrow$ (f). Propositions 2.6 and 2.7 show that (a) $\Rightarrow$ (b), (b) $\Rightarrow$ (e) and (c) $\Rightarrow$ (f). So it is enough to see that (f) $\Rightarrow$ (a). To see that Λ is of finite representation type it is enough to see that

the length of $(\ ,S)$ is finite for every simple Λ-module S. This is true if S is projective. If S is not projective, as (f) holds, there is a chain of irreducible maps of indecomposable modules $P = C_k \to \cdots \to C_0 = S$, so we have by Proposition 2.7 that $l(\ ,\underline{S}) < \infty$, and therefore, by Lemma 2.8, $l(\ ,S) < \infty$.

3. **Some properties of $(\overline{M}, \overline{N})$ and $\mathrm{Ext}^1_\Lambda(M, N)$.** We devote this section to studying some properties of $(\overline{M}, \overline{N})$, $(\underline{M}, \underline{N})$ and $\mathrm{Ext}^1_\Lambda(M, N)$, for indecomposable modules M and N such that the length of the functor $(\ ,\underline{M})$ is finite. In particular we will prove that if Λ is of finite representation type then $\overline{\mathrm{End}}_\Lambda(M)$ is a division ring for every indecomposable noninjective Λ-module M, and that $\mathrm{Ext}^1_\Lambda(M, M) = 0$ if M is indecomposable.

When Λ is an hereditary artin algebra, then $(M, N) = (\underline{M}, \underline{N})$, for M, N in $\mathrm{mod}_P\Lambda$ and $(M, N) = (\overline{M}, \overline{N})$ if M, N are in $\mathrm{mod}_I\Lambda$. So, if M and N are nonprojective noninjective indecomposable modules then $(\underline{M}, \underline{N}) = (\overline{M}, \overline{N})$. A similar result can be proven when Λ is stably equivalent to an hereditary algebra.

LEMMA 3.1. *Let M,N in* $\mathrm{mod}(\Lambda)$ *be indecomposable and let S be a torsionless nonprojective Λ-module. Then*

(a) $(\overline{M}, \overline{S}) = 0$ *if M is not isomorphic to S.*

(b) $(\overline{S}, \overline{S}) = (S, S)$.

(c) *If M and N are not simple torsionless, M is not projective and N is not injective, then* $(\underline{M}, \underline{N}) = (\overline{M}, \overline{N})$.

PROOF. (a) Assume M is not isomorphic to S. Let $f\colon M \to S$ be a nonzero map and let $E \to S \to 0$ be minimal right almost split. We know by Proposition 1.3 that E is injective. The map $f\colon M \to S$ is not a splittable epimorphism because M and S are not isomorphic, so f factors through E, which is injective. Therefore $\bar{f} = 0$, so $(\overline{M}, \overline{S}) = 0$.

(b) To prove that $(\overline{S}, \overline{S}) = (S, S)$ we have to see that if a map $f\colon S \to S$ factors through an injective then $f = 0$. So we assume that $0 \neq f\colon S \to S$ factors through the injective module E; then S is a direct summand of E and is, therefore, injective; a contradiction, because S is torsionless nonprojective, hence cotorsionless noninjective (Proposition 1.3).

(c) Assume now that M and N are not simple torsionless, M is not projective and N is not injective. We shall see that a map $f\colon M \to N$ factors through a projective module if and only if it factors through an injective module. So, we assume that P is projective and that there is a commutative diagram

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "\alpha"'] & & N \\ & P \arrow[ur, "\beta"'] & \end{tikzcd}$$

As M is in $\mathrm{mod}_P(\Lambda)$, $\mathrm{Im}(\alpha)$ cannot contain a projective summand, so $\mathrm{Im}(\alpha) = \coprod_{i=1}^{r} S_i$, where the S_i are torsionless nonprojective simple Λ-modules. Then $\alpha = \sum_{i=1}^{r} \alpha_i$, with $\alpha_i\colon M \to S_i$. Since M is not simple torsionless M is not isomorphic to S_i and by (a) we know that $(\overline{M}, \overline{S_i}) = 0$, so $\bar{\alpha}_i = 0$, $i = 1, \ldots, r$, and then $\bar{\alpha} = 0$. Therefore $\bar{f} = \bar{\beta}\bar{\alpha} = 0$, i.e., f factors through an injective module.

Assume now that $f\colon M \to N$ factors through an injective. Then $D(f)\colon D(N) \to D(M)$ factors through a projective. $D(N)$ is not projective because N is not injective, $D(M)$ is not injective because M is not projective and $D(N)$ and $D(M)$ are not simple torsionless. It follows then that $D(f)$ factors through an injective and therefore that f factors through a projective. This ends the proof of (b).

We will need the following consequence of Proposition 2.4.

LEMMA 3.2. *Let M and N be indecomposable modules and assume that there is a map $f\colon N \to M$ that is not a splittable epimorphism and such that $\underline{f} \in (\underline{N}, \underline{M})$ is not zero. If $(,\underline{M})$ has finite length there is a chain of irreducible maps of indecomposable modules*

$$N = C_0 \xrightarrow{f_1} C_1 \to \cdots \xrightarrow{f_i} C_i = M$$

such that $\underline{f}_i \cdots \underline{f}_1 \neq 0$.

PROOF. Since f is not a splittable epimorphism then the image of the composition

$$(,N) \xrightarrow{(,f)} (,M) \to (,\underline{M})$$

is contained in $\mathrm{r}(,\underline{M})$. Since the functor $(,\underline{M})$ has finite length there is some $i \geqslant 1$ such that $\mathrm{Im}(,f)$ is contained in $\mathrm{r}^i(,\underline{M})$ but not in $\mathrm{r}^{i+1}(,\underline{M})$. So the simple functor $(,N)/\mathrm{r}(,N)$ is isomorphic to a direct summand of $\mathrm{r}^i(,\underline{M})/\mathrm{r}^{i+1}(,\underline{M})$. Since $(,M)$ is indecomposable and the canonical epimorphism $(,M) \to (,\underline{M})$ is a projective cover, we know by Proposition 2.4 that there is a chain of irreducible maps

$$N = C_0 \xrightarrow{f_1} C_1 \to \cdots \xrightarrow{f_i} C_i = M$$

such that the image of the composition

$$(,N) = (,C_0) \to \cdots \to (,C_i) = (,M) \to (,\underline{M})$$

is not zero. Therefore $\underline{f}_i \cdots \underline{f}_1 \neq 0$.

PROPOSITION 3.3. *Let M and N be indecomposable Λ-modules such that $(D\,\mathrm{Tr})^n M$ is torsionless for some $n > 0$. If there is a homomorphism $f\colon N \to M$ that is not an isomorphism and such that $\underline{f} \neq 0$, then there is an integer $m > 0$ such that $(D\,\mathrm{Tr})^m N$ is torsionless and $\alpha(N) < \alpha(M)$.*

PROOF. Assume that there is a map $f\colon N \to M$ that is not an isomorphism and such that $\underline{f} \neq 0$. Then f is not a splittable epimorphism, because N is indecomposable. From Lemma 3.2 we know that there is a chain of irreducible maps of indecomposable modules $N = C_0 \xrightarrow{f_1} C_1 \to \cdots \xrightarrow{f_i} C_i = M$ such that $\underline{f}_i \cdots \underline{f}_1 \neq 0$. Then all the C_i are not projective, so by repeated application of Proposition 2.2 we have that for every $0 \leqslant j \leqslant i$ there is $n_j > 0$ such that $(D\,\mathrm{Tr})^{n_j} C_j$ is torsionless and $\alpha(N) = \alpha(C_0) < \alpha(C_1) < \cdots < \alpha(C_i) = \alpha(M)$. So $\alpha(N) < \alpha(M)$, and this ends the proof of the proposition.

As a consequence of this and Proposition 2.7:

COROLLARY 3.4. *Let M and N be indecomposable modules. Then*

(a) *If $(\ ,\underline{M})$ has finite length and $(\underline{N}, \underline{M}) \neq 0$, then $(\ ,\underline{N})$ has finite length.*

(b) *If $(\overline{N},\ )$ has finite length and $(\overline{N}, \overline{M}) \neq 0$, then $(\overline{M},\ )$ has finite length.*

PROOF. (a) is consequence of Proposition 3.3. (b) is obtained from (a) by duality.

Another immediate consequence of Proposition 3.3 is

COROLLARY 3.5. *Let M and N be indecomposable modules and assume that $(D\,\mathrm{Tr})^n N$ is torsionless for some $n > 0$. Then*

(a) *If there are maps $f\colon M \to N$ and $g\colon N \to M$ such that $0 \neq \underline{f} \in (\underline{M}, \underline{N})$ ana $0 \neq \underline{g} \in (\underline{N}, \underline{M})$, then f and g are isomorphisms.*

(b) *If there are maps $f\colon M \to N$ and $g\colon N \to M$ such that $0 \neq \bar{f} \in (\overline{M}, \overline{N})$ and $0 \neq \bar{g} \in (\overline{N}, \overline{M})$, then f and g are isomorphisms.*

PROOF. (a) The existence of $f\colon M \to N$ such that $\underline{f} \neq 0$ implies, by Proposition 3.3, that $(D\,\mathrm{Tr})^m M$ is torsionless for some $m > 0$, so we can apply Proposition 3.3 also to M and the map $g\colon N \to M$. We cannot have simultaneously that $\alpha(N) < \alpha(M)$ and $\alpha(M) < \alpha(N)$, so f and g are both isomorphisms.

(b) If $0 \neq \bar{f} \in (\overline{M}, \overline{N})$, $0 \neq \bar{g} \in (\overline{N}, \overline{M})$, then $0 \neq \underline{\mathrm{Tr}D(f)} \in (\underline{\mathrm{Tr}DM}, \underline{\mathrm{Tr}DN})$ and $0 \neq \underline{\mathrm{Tr}D(g)} \in (\underline{\mathrm{Tr}DN}, \underline{\mathrm{Tr}DM})$, since $\mathrm{Tr}\,D\colon \overline{\mathrm{mod}}_I\Lambda \to \underline{\mathrm{mod}}_P\Lambda$ is an equivalence of categories. (b) follows now from (a).

For a noninjective module M, $\overline{\mathrm{End}}(M) = (\overline{M}, \overline{M}) \neq 0$ so we obtain as a particular case

COROLLARY 3.6. *Let M be an indecomposable Λ-module.*

(a) *If M is not injective and $(D\,\mathrm{Tr})^n M$ is torsionless for some $n \geqslant 0$ then $\overline{\mathrm{End}}_\Lambda(M)$ is a division ring. In particular, if Λ is of finite representation type then $\overline{\mathrm{End}}_\Lambda(M)$ is a division ring for every indecomposable noninjective Λ-module M.*

(b) *If M is not projective and $(\mathrm{Tr}\,D)^n M$ is cotorsionless for some $n \geqslant 0$, then $\underline{\mathrm{End}}_\Lambda(M)$ is a division ring. In particular, if Λ is of finite representation type*

then $\underline{\mathrm{End}}_\Lambda(M)$ *is a division ring for every indecomposable nonprojective* Λ-*module* M.

We now give sufficient conditions on the modules M and N that imply that $\mathrm{Ext}^1_\Lambda(M, N) = 0$.

PROPOSITION 3.7. *Let* M *and* N *be indecomposable* Λ-*modules such that* $(D\ \mathrm{Tr})^n M$ *is torsionless for some* $n > 0$. *If* $(\ ,\underline{N})$ *has infinite length or* $\alpha(M) < \alpha(N)$ *then* $\mathrm{Ext}^1_\Lambda(M, D\ \mathrm{Tr}\ N) = 0$. *In particular, when* Λ *is of finite representation type* $\mathrm{Ext}^1_\Lambda(M, M) = 0$ *for every indecomposable* Λ-*module* M.

PROOF. We know that there is a functorial isomorphism

$$\mathrm{Ext}^1_\Lambda(M, D\ \mathrm{Tr}\ N) \cong D\big(\mathrm{Tor}^\Lambda_1(\mathrm{Tr}\ N, M)\big) \cong D(\underline{N}, \underline{M})$$

(see [3] and [11, p. 119]). If $\mathrm{Ext}^1_\Lambda(M, D\ \mathrm{Tr}\ N) \neq 0$ then $(\underline{N}, \underline{M}) \neq 0$, so, by Proposition 3.3, there is $m > 0$ such that $(D\ \mathrm{Tr})^m N$ is torsionless and $\alpha(N) < \alpha(M)$ or $N \cong M$; in any case $\alpha(N) \leqslant \alpha(M)$. This contradicts the hypothesis. So $\mathrm{Ext}^1_\Lambda(M, D\ \mathrm{Tr}\ N) = 0$. In particular if M is not injective, since $\alpha(M) < \alpha(\mathrm{Tr}\ DM)$, we have that $\mathrm{Ext}^1_\Lambda(M, M) = 0$.

As a consequence of this result and Proposition 2.2 we have

PROPOSITION 3.8. *Let* M *be an indecomposable* Λ-*module. Then*

(a) *If* M *is in* $\mathrm{mod}_P\Lambda$, $(D\ \mathrm{Tr})^n M$ *is torsionless for some* $n \geqslant 0$ *and*

$$0 \to D\ \mathrm{Tr}\ M \to E \xrightarrow{f} M \to 0$$

is the almost split sequence, then $\mathrm{Ext}^1_\Lambda(M, E) = 0$.

(b) *If* M *is in* $\mathrm{mod}_I\Lambda$, $(\mathrm{Tr}\ D)^n M$ *is cotorsionless for some* $n \geqslant 0$ *and*

$$0 \to M \to E \to \mathrm{Tr}\ DM \to 0$$

is the almost split sequence, then $\mathrm{Ext}^1_\Lambda(E, M) = 0$.

PROOF. (a) Let E_1 be an indecomposable summand of E. If E_1 is injective then $\mathrm{Ext}^1_\Lambda(M, E_1) = 0$. So we assume that E_1 is not injective. Therefore M is not simple torsionless, so $(D\ \mathrm{Tr})^n M$ is torsionless with $n > 0$. Since there is an irreducible map $M \to \mathrm{Tr}\ DE_1$ we have, by Proposition 2.7, that $(D\ \mathrm{Tr})^n(\mathrm{Tr}\ DE_1)$ is torsionless for some $m > 0$. We can apply now Proposition 2.2 to the irreducible map $M \to \mathrm{Tr}\ DE_1$ and we obtain that M is projective or $\alpha(M) < \alpha(\mathrm{Tr}\ DE_1)$. In any case $\mathrm{Ext}^1_\Lambda(M, D\ \mathrm{Tr}(\mathrm{Tr}\ DE_1)) = \mathrm{Ext}^1_\Lambda(M, E_1) = 0$ (Proposition 3.7). Since this is the case for all indecomposable summands E_1 of E, then $\mathrm{Ext}^1_\Lambda(M, E) = 0$.

(b) Follows from (a) by duality.

4. A property of $\bigcap_{i\geqslant 0}\mathrm{r}^i(\ ,\underline{M})$. We recall that a contravariant functor F from $\mathrm{mod}(\Lambda)$ to the category Ab of abelian groups is said to be *locally finite* if every finitely generated subfunctor of F has finite length.

For a functor F: $\mathrm{mod}\ \Lambda \to \mathrm{Ab}$, the *locally finite part* $\mathrm{lf}(F)$ is defined as the

unique locally finite subfunctor of F having the property that if $f\colon F' \to F$ is a morphism of functors and F' is locally finite, then $\mathrm{Im}(f) \subseteq \mathrm{lf}(F)$. (See [2].)

We are going to use the results of the preceding sections to see that $\mathrm{lf}((\ ,\underline{M})) \subseteq \bigcap_{i\geq 0} \mathrm{r}^i(\ ,\underline{M})$, for every indecomposable Λ-module M such that the functor $(\ ,\underline{M})$ has infinite length. We prove first:

LEMMA 4.1. *Let M, N be two indecomposable Λ-modules and assume that there is a map $f\colon M \to N$ such that the image of the map $(\ ,\underline{f})\colon (\ ,\underline{M}) \to (\ ,\underline{N})$ is not zero and has finite length. Then the functor $(\ ,\underline{M})$ has finite length.*

PROOF. The map $\underline{f}\colon M \to N$ is not zero, so $M, N \in \mathrm{mod}_P(\Lambda)$ and the map

$$\overline{D\mathrm{Tr}f}\colon D\mathrm{Tr}M \to D\mathrm{Tr}N$$

is not zero. We shall see first that the length of the image of the induced map $(\ ,\overline{D\mathrm{Tr}M}) \to (\ ,\overline{D\mathrm{Tr}N})$ is finite. Let X be a nonzero noninjective indecomposable module such that $(\mathrm{Im}((\ ,\overline{D\mathrm{Tr}f})))(X) \neq 0$. Then $\mathrm{Im}((\ ,\underline{f}))(\mathrm{Tr}DX) \neq 0$, because $\mathrm{Tr}D\colon \overline{\mathrm{mod}}(\Lambda) \to \underline{\mathrm{mod}}(\Lambda)$ is an equivalence of categories. Since $\mathrm{Im}(\ ,\underline{f})$ has finite length there are only a finite number of nonisomorphic indecomposable modules Y such that $\mathrm{Im}((\ ,\underline{f}))(Y) \neq 0$. On the other hand, the number of injective indecomposable modules is finite; therefore there are only a finite number of nonisomorphic indecomposable modules X such that $\mathrm{Im}((\ ,\overline{D\mathrm{Tr}f}))(X) \neq 0$. Therefore $\mathrm{Im}((\ ,\overline{D\mathrm{Tr}f}))$ has finite length.

Let $\pi\colon P \to D\ \mathrm{Tr}\ M$ be a projective cover. The composition

$$P \to D\mathrm{Tr}M \xrightarrow{D\mathrm{Tr}f} D\mathrm{Tr}N$$

is not zero. To prove the lemma we consider two cases:

Case 1. $\overline{D\mathrm{Tr}f \cdot \pi} = 0$. Let I be an injective Λ-module such that $D\ \mathrm{Tr}\, f \cdot \pi$ factors through I and assume that there is a commutative diagram

$$\begin{tikzcd} P \arrow[rr, "D\ \mathrm{Tr}\, f \cdot \pi"] \arrow[dr, "\alpha"'] & & D\ \mathrm{Tr}\ N \\ & I \arrow[ur, "\beta"'] & \end{tikzcd}$$

Then $\mathrm{Im}(\beta)$ cannot contain injective summands because $D\ \mathrm{Tr}\ N \in \mathrm{mod}_I(\Lambda)$. So $\mathrm{Im}(\beta) = \coprod_{i=1}^r S_i$ is a sum of nonprojective torsionless, hence simple, modules S_i (Proposition 1.3). Let $E_i \to^{\varepsilon_i} S_i \to 0$ be minimal right almost split; by Proposition 1.3 we know that the module E_i is injective. We can write the map $D\ \mathrm{Tr}\, f\colon D\mathrm{Tr}\ M \to \mathrm{Im}(\beta) = \coprod_{i=1}^r S_i$ as a sum $D\mathrm{Tr}\, f = \sum_{i=1}^r h_i$, where $h_i\colon D\mathrm{Tr}\ M \to S_i$. If h_i is not a splittable epimorphism there is a map $\theta_i\colon D\mathrm{Tr}\ M \to E_i$ such that $h_i = \varepsilon_i\theta_i$, for $i = 1, \ldots, r$. Then $D\mathrm{Tr}\, f = \sum_{i=1}^r h_i = \sum_{i=1}^r \varepsilon_i\theta_i$; so $D\mathrm{Tr}\, f$ factors through the injective module $E = \coprod_{i=1}^n E_i$; this contradicts the fact that $\overline{D\mathrm{Tr}f} \neq 0$. Thus h_i is a splittable epimorphism, for some $i \leq r$, so

$r = 1$ and $D\mathrm{Tr}\, M$ is isomorphic to the torsionless module S_1. Hence, by Proposition 2.7, $(\ ,\underline{M})$ has finite length.

Case 2. $\overline{D\mathrm{Tr} f \cdot \pi} \neq 0$. Let P_1 be an indecomposable direct summand of P such that $\overline{D\mathrm{Tr} f \cdot \pi | P_1} \neq 0$. Then $(\ ,D\mathrm{Tr}\, f \cdot \pi | P_1)$ defines a nonzero morphism

$$(\ ,P_1) \to \mathrm{Im}\big((\ ,\overline{D\mathrm{Tr} f})\big)$$

and therefore, as $\mathrm{Im}((\ ,\overline{D\mathrm{Tr} f}))$ has finite length, $(\ ,P_1)/\mathrm{r}(\ ,P_1)$ is isomorphic to a direct summand of $\mathrm{r}^i\mathrm{Im}((\ ,\overline{D\mathrm{Tr} f}))/\mathrm{r}^{i+1}\mathrm{Im}((\ ,\overline{D\mathrm{Tr} f}))$, for some $i \geqslant 0$. On the other hand, $(\ ,D\mathrm{Tr}\, M)$ is a projective indecomposable functor that maps onto $\mathrm{Im}((\ ,\overline{D\mathrm{Tr} f}))$ and is, therefore, the projective cover of $\mathrm{Im}((\ ,\overline{D\mathrm{Tr} f}))$. We know then by Proposition 2.4 that there is a chain of irreducible maps of indecomposable Λ-modules $P_1 = C_k \to \cdots \to C_0 = D\mathrm{Tr} M$. By Proposition 2.7 we have now that there is an $m > 0$ such that $(D\mathrm{Tr})^m D\mathrm{Tr} M = (D\mathrm{Tr})^{m+1} M$ is torsionless. Then, again by Proposition 2.7, the length of $(\ ,\underline{M})$ is finite.

PROPOSITION 4.2. *Let M, N be two indecomposable Λ-modules. Assume that $(\ ,\underline{M})$ has finite length and $(\ ,\underline{N})$ has infinite length. Then the simple functor $(\ ,M)/\mathrm{r}(\ ,M)$ is not isomorphic to a direct summand of $\mathrm{r}^i(\ ,\underline{N})/\mathrm{r}^{i+1}(\ ,\underline{N})$, for any $i \geqslant 0$.*

PROOF. Assume $(\ ,M)/\mathrm{r}(\ ,M)$ is isomorphic to a direct summand of $\mathrm{r}^i(\ ,\underline{N})/\mathrm{r}^{i+1}(\ ,\underline{N})$, with $i \geqslant 0$. As $(\ ,N) \to (\ ,\underline{N})$ is a projective cover we know by Proposition 2.4 that there is a chain of irreducible maps of indecomposable Λ-modules $M = C_k \to \cdots \to C_0 = N$. If $(\ ,\underline{M})$ has finite length we know by Proposition 2.7 that there is a chain of irreducible maps of indecomposable Λ-modules $P = C_m \to \cdots \to C_k = M$, with P projective. So we have a chain

$$P = C_m \to \cdots \to C_k \to \cdots \to C_0 = N$$

and therefore the length of $(\ ,\underline{N})$ is finite. Contradiction.

COROLLARY 4.3. *Let N be an indecomposable Λ-module such that the length of $(\ ,\underline{N})$ is infinite. Then* $\mathrm{lf}((\ ,\underline{N})) \subseteq \bigcap_{i \geqslant 0} \mathrm{r}^i(\ ,\underline{N})$.

PROOF. Suppose $F \subseteq \mathrm{lf}((\ ,\underline{N}))$, i.e., $F \subseteq (\ ,\underline{N})$ and F has finite length. If $F \not\subset \bigcap_{i \geqslant 0} \mathrm{r}^i(\ ,\underline{N})$, let i be the smallest positive integer such that $F \not\subset \mathrm{r}^i(\ ,\underline{N})$. If $(\ ,M) \to^p F$ is a projective cover of F then the composition

$$(\ ,M) \xrightarrow{p} F \to \mathrm{r}^{i-1}(\ ,\underline{N})/\mathrm{r}^i(\ ,\underline{N})$$

is not zero. Let M_1 be an indecomposable direct summand of M such that the composition

$$(\ ,M_1) \xrightarrow{p|(\ ,M_1)} F \to \mathrm{r}^{i-1}(\ ,\underline{N})/\mathrm{r}^i(\ ,\underline{N})$$

is not zero. Then the simple fucntor $(\ ,M_1)/\mathrm{r}(\ ,M_1)$ is isomorphic to a direct

summand of $\mathrm{r}^{i-1}(\ ,\underline{N})/\mathrm{r}^{i}(\ ,\underline{N})$. Since the length of $(\ ,\underline{N})$ is infinite we know by the preceding proposition that $(\ ,\underline{M}_1)$ has infinite length. We now show that this leads to a contradiction.

Since the functor $(\ ,M_1)$ is projective, the composition α: $(\ ,M_1) \to F \subseteq (\ ,\underline{N})$ can be factored through $(\ ,N)$, that is, we have a commutative diagram

$$\begin{tikzcd} & (\ ,M_1) \arrow[dl, "\rho"'] \arrow[d, "\alpha"] & \\ (\ ,N) \arrow[r, "\pi"'] & (\ ,\underline{N}) \arrow[r] & 0 \end{tikzcd}$$

where π is the canonical map. Let f: $M_1 \to N$ be such that $(\ ,f) = \rho$. Then $\alpha = (\ ,f)$: $(M_1) \to (\ ,\underline{N})$, where $\mathrm{Im}(\ ,f)$ is contained in F and has, therefore, finite length. Hence, by Lemma 4.1, since $\alpha \neq 0$, the length of $(\ ,\underline{M}_1)$ is finite, contradiction. This proves that $F \subseteq \bigcap_{i\geqslant 0}\mathrm{r}^{i}(\ ,\underline{N})$ and therefore $\mathrm{lf}((\ ,\underline{N})) \subseteq \bigcap_{i\geqslant 0}\mathrm{r}^{i}(\ ,\underline{N})$.

COROLLARY 4.4. *Let Λ be an hereditary artin algebra. If M is an indecomposable module such that $(\ ,M)$ has infinite length, then* $\mathrm{lf}((\ ,M)) \subseteq \bigcap_{i\geqslant 0}\mathrm{r}^{i}(\ ,M)$.

5. **The isomorphism** c. Throughout this section we will use the following notations: $\mathfrak{a}$ denotes the sum of the nonprojective simples of the socle of Λ and $\mathfrak{b}$ the left annihilator of $\mathfrak{a}$ in Λ. $\mathfrak{a}$ is a two sided ideal such that $\mathfrak{a}^2 = 0$. If Γ is a ring and M is a Γ-module, we denote by $\mathrm{Gr}(\Gamma)$ the Grothendieck group of Γ and by $[M]$ the class of M in $\mathrm{Gr}(\Gamma)$. Let $G = \mathrm{Gr}(\Lambda/\mathfrak{a}) \times \mathrm{Gr}(\Lambda/\mathfrak{b})$. If M is a Λ-module, we denote by $\langle M\rangle$ the element $([M/\mathfrak{a}M], [\mathfrak{a}M]) \in G$.

In [7] we defined, for an hereditary artin algebra Λ, a group isomorphism c: $\mathrm{Gr}(\Lambda) \to \mathrm{Gr}(\Lambda)$, with the property that $c([M]) = [D\mathrm{Tr}\, M]$, for every indecomposable nonprojective Λ-module M. When Λ is stably equivalent to an hereditary ring it is in general not possible to define a group homomorphism $\mathrm{Gr}(\Lambda) \to \mathrm{Gr}(\Lambda)$ with this property. However, we can define an isomorphism c from $\mathrm{Gr}(\Lambda/\mathfrak{a}) \times \mathrm{Gr}(\Lambda/\mathfrak{b})$ into itself such that for every indecomposable nonprojective module M we have

$$c(\langle M\rangle) = \begin{cases} \langle D\mathrm{Tr}M\rangle & \text{if } D\mathrm{Tr}M \text{ is torsion or projective.} \\ (0, [D\mathrm{Tr}M]) & \text{otherwise.} \end{cases}$$

Similar results to those obtained in [7] for the hereditary case can be proven here. In particular, if M and N are indecomposable modules such that $l((\ ,\underline{M})) < \infty$ and $\langle M\rangle = \langle N\rangle$ then M and N are isomorphic. We can also prove that Λ is of finite representation type if and only if $c^n = \mathrm{Id}_G$, for some integer $n > 0$.

In [15], W. Müller proved for a weakly-symmetric-self-dual artin ring Λ

with radical square zero (which includes artin algebras of radical square zero), that Λ is of finite representation type if and only if for every simple right or left Λ-module S there is an integer n such that $(D\mathrm{Tr})^n S = S$ or $(D\mathrm{Tr})^n S$ is projective. To prove this he defines a map from $\mathrm{Gr}(\Lambda/\mathfrak{r}) \times \mathrm{Gr}(\Lambda/\mathfrak{r})$ to itself. These results can be obtained for artin algebras of radical square zero as a consequence of the results of [7] for hereditary rings in the following way. Consider the ring

$$\Gamma = \begin{pmatrix} \Lambda/\mathfrak{r} & 0 \\ \mathfrak{r} & \Lambda/\mathfrak{r} \end{pmatrix}.$$

Γ is hereditary and stably equivalent to Λ and $\mathrm{Gr}(\Gamma) = \mathrm{Gr}(\Lambda/\mathfrak{r}) \times \mathrm{Gr}(\Lambda/\mathfrak{r})$; the isomorphism c associated to the hereditary ring Γ is the isomorphism defined by Müller in [15], and the criterion for the ring Λ being of finite representation type can now be obtained from the results proven for hereditary rings.

A similar argument can be applied in general for an artin algebra stably equivalent to an hereditary ring: it is possible to prove that the ring

$$\Gamma = \begin{pmatrix} \Lambda/\mathfrak{a} & 0 \\ \mathfrak{a} & \Lambda/\mathfrak{b} \end{pmatrix}$$

is hereditary and stably equivalent to Λ, and $\mathrm{Gr}(\Gamma) = \mathrm{Gr}(\Lambda/\mathfrak{a}) \times \mathrm{Gr}(\Lambda/\mathfrak{b})$. The map associated above to the ring Λ is precisely the isomorphism associated to $\left(\begin{smallmatrix} \Lambda/\mathfrak{a} & 0 \\ \mathfrak{a} & \Lambda/\mathfrak{b} \end{smallmatrix}\right)$ defined in [7] for hereditary rings. Even though the results of this section can also be obtained from the hereditary case considering the stable equivalence between Λ and the hereditary ring Γ, we are going to give an independent treatment. This has the advantage of being more explicit as well as giving a unified approach to the radical square zero and the hereditary cases.

We recall that Λ is stably equivalent to an hereditary ring if and only if it satisfies the two conditions:

(1) Each indecomposable submodule of an indecomposable projective Λ-module is projective or simple.

(2) If S is a nonprojective torsionless simple Λ-module then S is cotorsionless.

We shall first prove some properties of modules over rings stably equivalent to an hereditary algebra that will be needed later. Some of the results are true when the ring Λ satisfies only one of the properties (1) and (2) stated above.

LEMMA 5.1. *Let Λ be an arbitrary artin algebra. If P is a projective Λ-module, then $\mathfrak{a}P = \tau_{\mathfrak{a}}(P)$, that is, $\mathfrak{a}P$ is the sum of the nonprojective simple submodules of P.*

PROOF. $\mathfrak{a}P$ is semisimple; if S is a simple contained in $\mathfrak{a}P$ then $S \subseteq \mathfrak{a}\Lambda = \mathfrak{a}$, so S is nonprojective and therefore $\mathfrak{a}P \subseteq \tau_{\mathfrak{a}}P$.

Now let $S \subseteq \tau_{\mathfrak{a}}P$. If P is indecomposable there is an idempotent e and an isomorphism $\sigma\colon P \to \Lambda \cdot e$. As $S \subseteq \tau_{\mathfrak{a}}P$, $\sigma(S)$ is a nonprojective torsionless simple, so $\sigma(S) \subseteq \mathfrak{a}$. Then $\sigma(S) = \sigma(S) \cdot e \subseteq \sigma(S) \cdot \Lambda \cdot e = \sigma(S)\sigma(P) \subseteq \mathfrak{a}\sigma(P)$ and, therefore $S \subseteq \mathfrak{a}P$. Now let $P = P_1 \amalg \cdots \amalg P_n$, where the P_i are indecomposable Λ-modules and let $\pi_i\colon P \to P_i$ be the projection. $\pi_i(S) \subseteq P_i$ and $\pi_i(S)$ is a nonprojective torsionless simple or it is zero; in any case $\pi_i(S) \subseteq \mathfrak{a}P_i$ and then $S \subseteq \mathfrak{a}P$.

LEMMA 5.2. *Suppose* Λ *satisfies* (1). *Let* M *be in* $\operatorname{mod}(\Lambda)$, *let* P *be a finitely generated projective* Λ*-module,* $\pi\colon P \to M$ *an epimorphism and* $K = \operatorname{Ker}(\pi)$. *Then* $K = V \amalg Q$, *where* $V \subseteq \mathfrak{a}P$ *and* Q *is projective. For any decomposition of* K *of this type, the sequence* $0 \to Q/\mathfrak{a}Q \to P/\mathfrak{a}P \to M/\mathfrak{a}M \to 0$ *is exact.*

PROOF. K is a submodule of the projective module P and Λ satisfies (1), so $K = V \amalg Q$, where V is semisimple with no projective summands and Q is projective. By Lemma 5.1 we know that $V \subseteq \mathfrak{a}P$. The sequence

$$K/\mathfrak{a}K = \left(V/\mathfrak{a}V \amalg Q/\mathfrak{a}Q\right) \xrightarrow{\bar{i}} P/\mathfrak{a}P \to M/\mathfrak{a}M \to 0$$

is exact, where $\bar{i}$ is the map induced by the inclusion $i\colon K \to P$. $\bar{i}(V/\mathfrak{a}V) = 0$, because $V \subseteq \mathfrak{a}P$. By Lemma 5.1 we know that $\mathfrak{a}P \cap Q = \mathfrak{a}Q$, so the restriction map $\bar{i}\colon Q/\mathfrak{a}Q \to P/\mathfrak{a}P$ is a monomorphism. Combining this with the fact that $\bar{i}(V/\mathfrak{a}V) = 0$ we have that the sequence $0 \to Q/\mathfrak{a}Q \to P/\mathfrak{a}P \to M/\mathfrak{a}M \to 0$ is exact.

The projective modules in $\operatorname{mod}(\Lambda/\mathfrak{a})$ are precisely those of the form $P/\mathfrak{a}P$, where P is a projective in $\operatorname{mod}(\Lambda)$. If $0 \to L \xrightarrow{i} P/\mathfrak{a}P$ is a submodule of the projective module $P/\mathfrak{a}P$ and $N = \operatorname{Coker}(i)$ we have, by Lemma 5.2, an exact sequence

$$0 \to V \amalg Q \to P \to N \to 0,$$

with Q projective such that $0 \to Q/\mathfrak{a}Q \to P/\mathfrak{a}P \to N/\mathfrak{a}N = N \to 0$ is exact. Therefore $L \cong Q/\mathfrak{a}Q$ is projective in $\operatorname{mod}(\Lambda/\mathfrak{a})$. This proves

PROPOSITION 5.3. *If* Λ *satisfies* (1) *then* $\Lambda/\mathfrak{a}$ *is hereditary.*

LEMMA 5.4. *Assume* Λ *satisfies the property* (2). *Let* P, Q *be projective* Λ*-modules such that* $Q \subseteq \mathfrak{r}P$; *then the simple summands of* $Q/\mathfrak{r}Q$ *are torsion or projective.*

PROOF. Let $S \subseteq Q/\mathfrak{r}Q$ be simple and assume that S is nonprojective torsionless. As Λ satisfies (2) S is cotorsionless, so there is an injective E and an epimorphism $g\colon E \to S$. So we have a diagram

$$\begin{tikzcd} E \arrow[r, "g"] & S \\ & Q \arrow[u, "\pi"'] \end{tikzcd}$$

with Q projective and $\pi \neq 0$.

Let h: $Q \to E$ be such that $gh = \pi$. The module E is injective, so if i: $Q \to P$ is the inclusion map there is a map t: $P \to E$ such that $ti = h$. Then the composition $Q \to^{i} P \to^{gt} S$ is equal to π and is therefore nonzero. This contradicts the fact that $Q \subseteq \mathrm{r}P$.

In Lemma 5.1 we saw that if Λ is an arbitrary artin algebra then $\tau_{\mathfrak{a}}P = \mathfrak{a}P$, for every projective Λ-module P. If Λ is stably equivalent to an hereditary ring we can prove a more general result:

LEMMA 5.4′. *Let M be in* mod(Λ). *If S is a nonprojective torsionless submodule of M, then either S is a direct summand of M or $S \subseteq \mathfrak{a}M$.*

PROOF. Let $0 \to S \to^{f} P$ be the minimal left almost split map. The module P is projective (Proposition 1.3). If S is not a direct summand of M the inclusion i: $S \to M$ is not a splittable monomorphism. Thus there is a morphism g: $P \to M$ such that $gf = i$. As S is nonprojective torsionless, $f(S) \subseteq \tau_{\mathfrak{a}}(P) = \mathfrak{a}P$, and then $i(S) = gf(S) \subseteq g(\mathfrak{a}P) \subseteq \mathfrak{a}M$.

We shall need the following result about the injective envelope of the simple Λ-modules.

LEMMA 5.5. *Assume that Λ satisfies* (1) *and let $I_0(S)$ be the injective envelope of the simple Λ-module S. Then $\mathfrak{a}I_0(S) = S$ if S is nonprojective torsionless, and $\mathfrak{a}I_0(S) = 0$ otherwise.*

PROOF. Assume first that S is nonprojective torsionless. Since $\mathfrak{a}I_0(S)$ is semisimple it is contained in $\mathrm{soc}(I_0(S)) = S$. So we only have to prove that $\mathfrak{a}I_0(S) \neq 0$. S is torsionless, so let P be a projective such that there is a monomorphism $0 \to S \to^{\alpha} P$. Since $I_0(S)$ is injective, the inclusion map i: $S \to I_0(S)$ can be extended to a map θ: $P \to I_0(S)$.

But S is not projective, so $\alpha(S) \subseteq \tau_{\mathfrak{a}}(P) = \mathfrak{a}P$, by Lemma 5.1. Therefore $i(S) = \theta\alpha(S) \subseteq \theta(\mathfrak{a}P) = \mathfrak{a}\theta(P) \subseteq \mathfrak{a}I_0(S)$; then $\mathfrak{a}I_0(S) \neq 0$.

Assume now that S is torsion or projective and let $P \to^{f} I_0(S) \to 0$ be a projective cover of $I_0(S)$. Then $f|\mathfrak{a}P$: $\mathfrak{a}P \to \mathfrak{a}I_0(S)$ is an epimorphism. $\mathfrak{a}I_0(S)$ is semisimple, so if it is not zero, $\mathfrak{a}I_0(S) = S$ and we have an epimorphism $f|\mathfrak{a}P$: $\mathfrak{a}P \to S$. This is a contradiction, because $\mathfrak{a}P$ is a semisimple, sum of nonprojective torsionless modules, and S is torsion or projective. Hence $\mathfrak{a}I_0(S) = 0$.

We assume in all that follows that Λ is stably equivalent to an hereditary artin algebra.

PROPOSITION 5.6. *Let M be an indecomposable Λ-module, $P \to M$ a projective cover for M and $0 \to V \amalg Q \to P \to M \to 0$ exact with $V \subseteq \mathfrak{a}P$ and Q projective. Then $V \cong D\mathrm{Tr}M$ and $Q = 0$ if $D\mathrm{Tr}M$ is nonprojective torsionless; $V \cong \mathfrak{a}D\mathrm{Tr}M$ otherwise.*

PROOF. We know by [3, Proposition 5.3] that $(V \amalg Q)/\mathrm{r}(V \amalg Q) \cong \mathrm{soc}(D\mathrm{Tr}M)$. Thus $V \amalg Q/\mathrm{r}Q \cong \mathrm{soc}(D\mathrm{Tr}M)$. If $D\mathrm{Tr}M$ is nonprojective torsionless then it is simple, so $D\mathrm{Tr}M \cong V \amalg Q/\mathrm{r}Q$. Since $P \to M$ is a projective cover, $Q \subseteq \mathrm{r}P$, so $Q/\mathrm{r}Q$ does not contain any nonprojective torsionless simples (Lemma 5.3). Hence $Q/\mathrm{r}Q = 0$, so $Q = 0$ and $V \cong D\mathrm{Tr}M$.

If $D\mathrm{Tr}M$ is torsion or projective and $S \subseteq V$, then S cannot be a direct summand of $D\mathrm{Tr}M$. Then, since $S \subseteq V \subseteq D\mathrm{Tr}M$ we know by the preceding lemma that $S \subseteq \mathfrak{a}D\mathrm{Tr}M$. So $V \subseteq \mathfrak{a}D\mathrm{Tr}M$. On the other hand, $\mathfrak{a}D\mathrm{Tr}M \subseteq \mathrm{soc}(D\mathrm{Tr}M) \cong V \amalg Q/\mathrm{r}Q$, so $\mathfrak{a}D\mathrm{Tr}M \subseteq V$ because $Q/\mathrm{r}Q$ does not contain nonprojective torsionless simples. Thus $\mathfrak{a}D\mathrm{Tr}M \cong V$.

We consider now the group $G = \mathrm{Gr}(\Lambda/\mathfrak{a}) \times \mathrm{Gr}(\Lambda/\mathfrak{b})$. We recall that if M is a Λ-module then $\langle M \rangle$ denotes the element $([M/\mathfrak{a}M], [\mathfrak{a}M]) \in G$. If M is a $\Lambda/\mathfrak{a}$-module we denote by $P_{\Lambda/\mathfrak{a}}(M)$ and by $I_{\Lambda/\mathfrak{a}}(M)$ the projective cover and the injective envelope of M respectively; M is also a Λ-module and $P_{\Lambda/\mathfrak{a}}(M) = P_0(M) \otimes_{\Lambda'} \Lambda/\mathfrak{a} \cong P_0(M)/\mathfrak{a}P_0(M)$ where $P_0(M)$ denotes the projective cover of the Λ-module M. Let $I_0(M)$ denote the injective envelope of the Λ-module M. Let $S_1, \ldots, S_n$ be a complete set of nonisomorphic simple Λ-modules and assume that $S_1, \ldots, S_r$ are the nonprojective torsionless. Since $\Lambda/\mathfrak{a}$ is hereditary the sets $\{P_{\Lambda/\mathfrak{a}}(S_i),\ i = 1, \ldots, n\}$, $\{I_{\Lambda/\mathfrak{a}}(S_i),\ i = 1, \ldots, n\}$ are bases for $\mathrm{Gr}(\Lambda/\mathfrak{a})$ (see [7]). Therefore the sets $\mathfrak{B} = \{([P_{\Lambda/\mathfrak{a}}(S_i)], [\mathfrak{a}P_0(S_i)]) = \langle P_0(S_i) \rangle, (0, [S_j]),\ i = 1, \ldots, n;\ j = 1, \ldots, r\}$ and $\mathfrak{B}' = \{([I_{\Lambda/\mathfrak{a}}(S_i)], 0), \langle I_0(S_j) \rangle,\ i = 1, \ldots, n;\ j = 1, \ldots, r\}$ are two bases of $\mathrm{Gr}(\Lambda/\mathfrak{a}) \times \mathrm{Gr}(\Lambda/\mathfrak{b}) = G$, since, by Lemma 5.5, $\langle I_0(S_j) \rangle = ([I_0(S_j)/S_j], [S_j])$, $j = 1, \ldots, r$.

DEFINITION. Let $c\colon G \to G$ be the group homomorphism defined by

$$c(\langle P_0(S_i) \rangle) = -\langle I_{\Lambda/\mathfrak{a}}(S_i) \rangle, \quad i = 1, \ldots, n,$$

$$c((0, [S_j]) = -\langle I_0(S_j) \rangle \quad j = 1, \ldots, r.$$

c is an isomorphism since it carries a basis to a basis. Let M be an indecomposable nonprojective Λ-module; we will prove that $c(\langle M \rangle) = \langle D\mathrm{Tr}M \rangle$ if M is torsion or projective, and $c(\langle M \rangle) = (0, [D\mathrm{Tr}M])$ otherwise. So we consider an exact sequence

$$0 \to V \amalg Q \to P \to M \to 0,$$

with $P \to M$ a projective cover, $V \subseteq \mathfrak{a}P$ and Q projective (Lemma 5.2). From

the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d, "|"] & 0 \arrow[d] & \\
0 \arrow[r] & \mathfrak{a}Q \amalg V \arrow[r] \arrow[d] & Q \amalg V \arrow[r] \arrow[d] & Q/\mathfrak{a}Q \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathfrak{a}P \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & P/\mathfrak{a}P \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathfrak{a}M \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M/\mathfrak{a}M \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

we have

$$\begin{aligned}
c(\langle M\rangle) &= c([P/\mathfrak{a}P],[\mathfrak{a}P]) - c\big([Q/\mathfrak{a}Q],\big[\mathfrak{a}Q \amalg V\big]\big) \\
&= c(\langle P\rangle) - c(\langle Q\rangle) - c((0,[V]) \\
&= -\langle I_{\Lambda/\mathfrak{a}}(P/\mathfrak{r}P)\rangle + \langle I_{\Lambda/\mathfrak{a}}(Q/\mathfrak{r}Q)\rangle + \langle I_0(V)\rangle.
\end{aligned}$$

This can also be written in the form

(1) $c(\langle M\rangle) = \big([I_{\Lambda/\mathfrak{a}}(Q/\mathfrak{r}Q)] - [I_{\Lambda/\mathfrak{a}}(P/\mathfrak{r}P)] + [I_0(V)/V],[V]\big).$

The computations that follow are devoted to evaluate the right-hand side. With this purpose we will prove that there is an exact sequence

$$0 \to D\mathrm{Tr}M \to I_{\Lambda/\mathfrak{a}}(Q/\mathfrak{r}Q) \amalg I_0(V) \to I_{\Lambda/\mathfrak{a}}(P/\mathfrak{r}P) \to 0;$$

or, what is equivalent, an exact sequence

(2) $$0 \to P^*/\mathfrak{a}P^* \to P_0(V)^* \amalg Q^*/\mathfrak{a}Q^* \to \mathrm{Tr}M \to 0,$$

that is obtained by dualizing the first sequence, since, for a simple Λ-module S, $D(P_0(S)^*) \cong I_0(S)$ and $P_0(S)^*/\mathfrak{a}P_0(S)^* \cong P_{(\Lambda/\mathfrak{a})^{\mathrm{op}}}(D(S))$; so

$$D(P^*/\mathfrak{a}P^*) \cong I_{\Lambda/\mathfrak{a}}(P/\mathfrak{r}P), \quad D(Q^*/\mathfrak{a}Q^*) \cong I_{\Lambda/\mathfrak{a}}(Q/\mathfrak{r}Q) \quad \text{and}$$
$$D(P_0(V)^*) \cong I_0(V).$$

Therefore what follows is devoted to prove that there is an exact sequence as indicated in (2).

Let M be an indecomposable Λ-module, let $\pi\colon P \to M$ be a projective cover of M and let $0 \to V \amalg Q \to^{i} P \to^{\pi} M \to 0$ be exact with $i(V) \subseteq \mathfrak{a}P$ and Q projective (Lemma 5.2). Let $p\colon P_0(V) \to V$ be the projective cover of V and let β be the composition $P_0(V) \to^{p} V \to^{i|V} P$; then

$$P_0(V) \amalg Q \xrightarrow{\beta + i|Q} P \xrightarrow{\pi} M \to 0$$

is a minimal projective presentation for M and the sequence

$$P^* \xrightarrow{\beta^* + (i|Q)^*} P_0(V)^* \amalg Q^* \to \mathrm{Tr}M \to 0$$

is exact. Since $\mathrm{Im}(\beta) \subseteq \mathfrak{a}P$ and $\mathfrak{a}^2 = 0$ we have that $\beta^*|\mathfrak{a}P^* = 0$. Therefore β^* induces a map $\overline{\beta^*}$: $P^*/\mathfrak{a}P^* \to P_0(V)^*$. On the other hand $(i|Q)^*$: $P^* \to Q^*$ induces a map $\overline{(i|Q)^*}$: $P^*/\mathfrak{a}P^* \to Q^*/\mathfrak{a}Q^*$. Thus we have a sequence

$$0 \to P^*/\mathfrak{a}P^* \xrightarrow{\overline{\beta^*} \amalg \overline{(i|Q)^*}} P_0(V)^* \amalg Q^*/\mathfrak{a}Q^* \to \mathrm{Tr}M \to 0.$$

Statements (b) and (d) of the following lemma imply that this sequence is exact.

LEMMA 5.7. *With the above notations*:

(a) *If P_1 is projective and $f \in P_1^*$ is such that* $\mathrm{Im}(f) \subseteq \mathfrak{a}$, *then $f \in \mathfrak{a}P_1^*$.*

(b) $(i|Q)^*|\mathfrak{a}P^*$: $\mathfrak{a}P^* \to \mathfrak{a}Q^*$ *is an epimorphism.*

(c) $\mathrm{Im}(\pi^*: M^* \to P^*) \subseteq \mathfrak{a}P^*$.

(d) $\overline{\beta^*} \amalg \overline{(i|Q)^*}$: $P^*/\mathfrak{a}P^* \to P_0(V)^* \amalg Q^*/\mathfrak{a}Q^*$ *is a monomorphism.*

PROOF. (a) Let $f \in P_1^*$ be such that $\mathrm{Im}(f) \subseteq \mathfrak{a}$. We may assume that P_1 is indecomposable and of the form $\Lambda \cdot e$ for some idempotent e in Λ. Let $f(e) = a \in \mathfrak{a}$. Then $f(p) = f(p \cdot e) = p \cdot a$; so, if j: $\Lambda e \to \Lambda$ is the inclusion, $f = a \cdot j$, so $f \in \mathfrak{a}P^*$.

(b) Let $f = \Sigma a_i g_i$, $a_i \in \mathfrak{a}$, $g_i \in Q^*$. Then $\mathrm{Im}(f) \subseteq \mathfrak{a}$. The ideal $\mathfrak{a}$ is a sum of torsionless nonprojective simples. Since torsionless nonprojective simples are cotorsionless noninjective, there is an injective module E and an epimorphism g: $E \to \mathfrak{a}$.

Since Q is projective, the map f: $Q \to \mathfrak{a}$ can be lifted to a map h: $Q \to E$ such that $gh = f$. So we have

$$\begin{tikzcd} 0 \arrow[r] & Q \arrow[r, "i|Q"] \arrow[d, "h"'] & P \\ & E & \end{tikzcd}$$

Since E is injective, there exists θ: $P \to E$ such that $\theta(i|Q) = h$. Then $f = gh = g\theta(i|Q) = (i|Q)^*(g\theta)$, where the image of $g\theta$: $P \to \Lambda$ is contained in $\mathfrak{a}$. By (a) we know that $g\theta \in \mathfrak{a}P^*$, so $f = (i|Q)^*(g\theta) \in (i|Q)^*(\mathfrak{a}P^*)$, i.e., $(i|Q)^*|\mathfrak{a}P^*$: $\mathfrak{a}P^* \to \mathfrak{a}Q^*$ is an epimorphism.

(c) Let $h \in \mathrm{Im}(\pi^*: M^* \to P^*)$. Then $h = \alpha\pi$, for some α: $M \to \Lambda$. $\mathrm{Im}(\alpha)$ is torsionless and cannot contain a projective summand because M is in $\mathrm{mod}_P(\Lambda)$. Therefore $\mathrm{Im}(\alpha) \subseteq \mathfrak{a}$ and consequently, $\mathrm{Im}(h) = \mathrm{Im}(\alpha\pi) \subseteq \mathfrak{a}$. We know by (a) that $h \in \mathfrak{a}P^*$.

(d) We want to prove that $\bar{\beta}^* \amalg \overline{(i|Q)^*}$ is a monomorphism. Let $\bar{f} \in$

$P^*/\mathfrak{a}P^*$ be such that $(\overline{\beta^*}\amalg(\overline{i|Q})^*)(\bar{f}) = 0$. Then $\overline{\beta^*}(\bar{f}) = 0$, i.e., $f\cdot\beta = f\cdot i|V\cdot p = 0$. So $f\cdot i|V = 0$, because $p\colon P_0(V)\to V$ is an epimorphism. On the other hand, $\overline{(i|Q)^*(f)} = 0$, so $(i|Q)^*(f) = f(i|Q) \in \mathfrak{a}Q^*$. We know by (b) that $(i|Q)^*\colon \mathfrak{a}P^*\to\mathfrak{a}Q^*$ is an epimorphism. Let $h\in\mathfrak{a}P^*$ be such that $(i|Q)^*(h) = (i|Q)^*(f)$. Since $h\in\mathfrak{a}P^*$, $\mathrm{Im}(h)\subseteq\mathfrak{a}$, and $V\subseteq\mathfrak{a}P$, then $h|V = 0$. Therefore $f-h\colon P\to\Lambda$ has the property that $(f-h)|V = f|V - h|V = 0$ and $(f-h)\cdot(i|Q) = 0$. Therefore $f-h$ factors through M, i.e., there is $\rho\colon M\to\Lambda$ such that $f-h = \rho\pi = \pi^*(\rho)\in\pi^*(M^*)\subseteq\mathfrak{a}P^*$.

Then $f-h\in\mathfrak{a}P^*$. But $h\in\mathfrak{a}P^*$, so $f\in\mathfrak{a}P^*$, and then $0 = \bar{f}\in P^*/\mathfrak{a}P^*$. This finishes the proof of (d). So we have proven

PROPOSITION 5.8. *There is an exact sequence*

$$0\to D\mathrm{Tr}M\to I_{\Lambda/\mathfrak{a}}(Q/\mathfrak{r}Q)\amalg I_0(V)\to I_{\Lambda/\mathfrak{a}}(P/\mathfrak{r}P)\to 0.$$

The map $D\mathrm{Tr}M\to I_{\Lambda/\mathfrak{a}}(Q/\mathfrak{r}Q)\amalg I_0(V)$ induces a monomorphism $f\colon \mathfrak{a}D\mathrm{Tr}M\to\mathfrak{a}I_0(V) = V$ (Lemma 5.5).

If $D\mathrm{Tr}M$ is torsion or projective, then, by Proposition 5.6, $\mathfrak{a}D\mathrm{Tr}M\cong V$, so f is an isomorphism. If $D\mathrm{Tr}M$ is nonprojective torsionless the image of $D\mathrm{Tr}M\to I_{\Lambda/\mathfrak{a}}(Q/\mathfrak{r}Q)\amalg I_0(V)$ is contained in $\tau_{\mathfrak{a}}(Q/\mathfrak{r}Q\amalg V)$. By Lemma 5.4 $\tau_{\mathfrak{a}}(Q/\mathfrak{r}Q) = 0$. On the other hand, we know by Proposition 5.6 that in this case $V\cong D\mathrm{Tr}M$. Therefore the image of $D\mathrm{Tr}M\to I_{\Lambda/\mathfrak{a}}(Q/\mathfrak{r}Q)\amalg I_0(V)$ is V and we have

PROPOSITION 5.9. *Let M be an indecomposable Λ-module. Then, with the above notations*:

(a) *If $D\mathrm{Tr}M$ is torsion or projective there is an exact sequence of $\Lambda/\mathfrak{a}$-modules*

$$0\to D\mathrm{Tr}M/\mathfrak{a}D\mathrm{Tr}M\to I_{\Lambda/\mathfrak{a}}(Q/\mathfrak{r}Q)\amalg I_0(V)/V\to I_{\Lambda/\mathfrak{a}}(P/\mathfrak{r}P)\to 0.$$

(b) *If $D\mathrm{Tr}M$ is nonprojective torsionless then there is an isomorphism of $\Lambda/\mathfrak{a}$-modules $I_{\Lambda/\mathfrak{a}}(Q/\mathfrak{r}Q)\amalg I_0(V)/V\to I_{\Lambda/\mathfrak{a}}(P/\mathfrak{r}P)$.*

COROLLARY 5.10. (a) *If $D\mathrm{Tr}M$ is torsion or projective then in* $\mathrm{Gr}(\Lambda)$, $[D\mathrm{Tr}M/\mathfrak{a}D\mathrm{Tr}M] = [I_{\Lambda/\mathfrak{a}}(Q/\mathfrak{r}Q)] + [I_0(V)] - [V] - [I_{\Lambda/\mathfrak{a}}(P/\mathfrak{r}P)]$.

(b) *If $D\mathrm{Tr}M$ is nonprojective torsionless, then* $[I_{\Lambda/\mathfrak{a}}(Q/\mathfrak{r}Q)] + [I_0(V)] - [V] - [I_{\Lambda/\mathfrak{a}}(P/\mathfrak{r}P)] = 0$.

DEFINITION. An element x in G is said to be *positive* (*negative*) if and only if it has nonnegative (nonpositive) coordinates in the basis $\{([S_i], 0), (0, [S_j]), i = 1,\dots,n; j = 1,\dots,r\}$ of G.

We can now state:

PROPOSITION 5.11. (a) $c(\langle M\rangle) = -\langle I_{\Lambda/\mathfrak{a}}(M/\mathfrak{r}M)\rangle$ *if M is a projective Λ-module.*

If M is an indecomposable module then

(b) $c(\langle M\rangle) = (0, [D\mathrm{Tr}M])$ *if* $D\mathrm{Tr}M$ *is nonprojective torsionless.*

(c) $c(\langle M\rangle) = \langle D\mathrm{Tr}M\rangle$ *if* $D\mathrm{Tr}M$ *is torsion or projective.*

(d) M *is projective if and only if* $c(\langle M\rangle)$ *is negative.*

(e) $D\mathrm{Tr}M$ *is nonprojective torsionless if and only if the image of* $c(\langle M\rangle)$ *under the projection* $\mathrm{Gr}(\Lambda/\mathfrak{a}) \times \mathrm{Gr}(\Lambda/\mathfrak{b}) \to \mathrm{Gr}(\Lambda/\mathfrak{a})$ *is zero.*

PROOF. (a) follows from the definition. (b) and (c) are a consequence of formula (1) and Corollary 5.10 and (d) and (e) follow from (a), (b) and (c).

As a consequence of this proposition we have the following useful corollary:

COROLLARY 5.12. *Let M and N be indecomposable Λ-modules with $\langle M\rangle = \langle N\rangle$. Then:*

(a) *M is projective if and only if N is projective.*

(b) *If M is projective then $M \cong N$.*

(c) *$D\mathrm{Tr}M$ is nonprojective torsionless if and only if $D\mathrm{Tr}N$ is nonprojective torsionless.*

(d) *If $D\mathrm{Tr}M$ is nonprojective torsionless then $M \cong N$.*

(e) *If $D\mathrm{Tr}M$ is torsion or projective then $\langle D\mathrm{Tr}M\rangle = \langle D\mathrm{Tr}N\rangle$.*

PROOF. (a) By part (e) of Proposition 5.11, M is projective if and only if $c(\langle M\rangle)$ is negative. But $\langle M\rangle = \langle N\rangle$ implies $c(\langle M\rangle) = c(\langle N\rangle)$. Thus M is projective if and only if N is projective. (c) and (e) follow in a similar way from (f) and (d) of Proposition 5.11.

(b) If M is projective then, by (a), N is projective. From $\langle M\rangle = \langle N\rangle$ we have that $[M/\mathfrak{a}M] = [N/\mathfrak{a}N]$. But $M/\mathfrak{a}M$ and $N/\mathfrak{a}N$ are indecomposable projective modules over the hereditary ring $\Lambda/\mathfrak{a}$. It is not hard to see that then $M/\mathfrak{a}M \cong N/\mathfrak{a}N$ (see [7]). Then $M/\mathfrak{r}M \cong N/\mathfrak{r}N$ and therefore $M \cong N$.

(d) If $D\mathrm{Tr}M$ is nonprojective torsionless then $D\mathrm{Tr}N$ is also nonprojective torsionless, so $(0, [D\mathrm{Tr}M]) = c(\langle M\rangle) = c(\langle N\rangle) = (0, [D\mathrm{Tr}N])$. Thus $[D\mathrm{Tr}M] = [D\mathrm{Tr}N]$ and therefore $D\mathrm{Tr}M \cong D\mathrm{Tr}N$, because both are simple modules. Then $M \cong N$.

PROPOSITION 5.13. *Let M, N be indecomposable Λ-modules and assume that the length of $(,\underline{M})$ is finite. If $\langle M\rangle = \langle N\rangle$ then $M \cong N$.*

PROOF. If M is projective the proposition is true, so we may assume that there is an $n > 0$ such that $(D\mathrm{Tr})^n M$ is torsionless. Let m be the smallest positive integer such that $(D\mathrm{Tr})^m M$ is torsionless; if $(D\mathrm{Tr})^r N$ is torsionless with $0 < r < m$ then $(,\underline{N})$ has finite length. Thus we may assume that $(D\mathrm{Tr})^r N$ is not torsionless for $0 < r < m$. From $\langle M\rangle = \langle N\rangle$, by repeated application of Corollary 5.12 we get $\langle (D\mathrm{Tr})^{m-1}M\rangle = \langle (D\mathrm{Tr})^{m-1}N\rangle$. If

$(D\mathrm{Tr})^m M$ is not projective, then since it is torsionless we have by (d) of the preceding corollary that $M \cong N$. If $(D\mathrm{Tr})^m M$ is projective then $\langle (D\mathrm{Tr})^m M\rangle = c(\langle (D\mathrm{Tr})^{m-1}M\rangle) = c(\langle (D\mathrm{Tr})^{m-1}N\rangle)$. If $(D\mathrm{Tr})^m N$ is nonprojective torsionless, then $c(\langle (D\mathrm{Tr})^{m-1}N\rangle) = (0, [D\mathrm{Tr}N]) \neq \langle D\mathrm{Tr}^m M\rangle$, contradiction. Therefore, $c(\langle (D\mathrm{Tr})^{m-1}N\rangle) = \langle (D\mathrm{Tr})^m N\rangle$, so $\langle (D\mathrm{Tr})^m M\rangle = \langle (D\mathrm{Tr})^m N\rangle$. Since $(D\mathrm{Tr})^m M$ is projective, then by (b) of Corollary 5.12 we have that $(D\mathrm{Tr})^m M \cong (D\mathrm{Tr})^m N$ and therefore $M \cong N$.

COROLLARY 5.14. *If $c^m = \mathrm{Id}_G$ for some $m > 0$, then for every injective nonprojective indecomposable Λ-module I there exists a positive integer $t < m$ such that $(D\mathrm{Tr})^t I$ is torsionless.*

PROOF. I is not torsionless, because it is injective and nonprojective. Assume that $(D\mathrm{Tr})^j I$ is not torsionless for all $0 \leqslant j \leqslant m-1$. If $(D\mathrm{Tr})^m I$ is nonprojective torsionless then $\langle I\rangle = c^m(\langle I\rangle) = (0, [(D\mathrm{Tr})^m I])$, contradiction. Thus $(D\mathrm{Tr})^m I$ is torsion or projective and $c^m(\langle I\rangle) = \langle (D\mathrm{Tr})^m I\rangle = \langle I\rangle$. Then $\langle D(I)\rangle = \langle \mathrm{Tr}(D\mathrm{Tr})^{m-1}I\rangle$ and, since $D(I)$ is projective, then by Corollary 5.12(b) we have that $D(I) \cong \mathrm{Tr}(D\mathrm{Tr})^{m-1}I$. But this is a contradiction, because $\mathrm{Tr}((D\mathrm{Tr})^{m-1}I)$ is in $\mathrm{mod}_P(\Lambda)$. Therefore there is a $t < m$ such that $(D\mathrm{Tr})^t I$ is torsionless.

LEMMA 5.15. *Assume that for every injective nonprojective indecomposable Λ-module I there is a positive integer t such that $(D\mathrm{Tr})^t I$ is torsionless. Then Λ is of finite representation type.*

PROOF. Let I be an injective nonprojective indecomposable Λ-module; then, by hypothesis, there is $t > 0$ such that $(D\mathrm{Tr})^t I$ is torsionless. Therefore, by Proposition 2.7 the functor $(\ ,\underline{I})$ has finite length.

Let S be a simple Λ-module. If S is projective then $l(\ ,S) < \infty$. Assume S is not projective. To prove that $l(\ ,S) < \infty$ we consider separately two cases:

Case 1. S is torsion. Let i: $S \to I_0(S)$ be the injective envelope of S. Then $0 \to (\ ,S) \to (\ ,I_0(S))$. We shall see that for every X in $\mathrm{mod}_P(\Lambda)$ the map $0 \to (X, S) \to (X, I_0(S))$ is a monomorphism; this will prove that $(\ ,S)$ has finite length. Let π: $X \to S$ be such that the composition $X \to^{\pi} S \to^{i} I_0(S)$ factors through a projective module P, i.e., there are α, β such that the diagram

$$\begin{tikzcd} X \arrow[rr, "i\pi"] \arrow[dr, "\beta"'] & & I_0(S) \\ & P \arrow[ur, "\alpha"'] & \end{tikzcd}$$

commutes. If X is in $\mathrm{mod}_P(\Lambda)$ then $\mathrm{Im}(\beta)$ is semisimple, so $\alpha(\mathrm{Im}(\beta)) \subseteq \mathrm{soc}(I_0(S)) = S$. Then, if $\pi \neq 0$, $\alpha(\mathrm{Im}(\beta)) = S$ and therefore the semisimple

module $\mathrm{Im}(\beta)$ contains a copy of S. Therefore $S \subseteq \mathrm{Im}(\beta) \subseteq P$ and this contradicts the fact that S is torsion. Thus $\pi = 0$.

Case 2. S is nonprojective torsionless. Let $E \to S \to 0$ be minimal right almost split. By Proposition 1.3, E is injective. Let X be a nonprojective indecomposable module not isomorphic to S. Then, by Lemma 2.8, $(\underline{X}, \underline{S}) = (X, S)$. We know by Lemma 2.5 that the cokernel F of the map $(\ ,E) \to (\ ,S)$ is a simple functor. From $(X, E) \to (X, S) \to F(X) \to 0$ and $(\underline{X}, \underline{S}) = (X, S)$ we get an exact sequence $(\underline{X}, \underline{E}) \to (X, S) \to F(X) \to 0$. $(\ ,\underline{E})$ has finite length because E is injective, and F is simple. So there are only a finite number of nonisomorphic indecomposable nonprojective modules X such that $(X, S) \neq 0$. Therefore, since the number of nonisomorphic indecomposable projective Λ-modules is finite, the functor $(\ ,S)$ has finite length.

We can now prove:

THEOREM 5.16. *Λ is of finite representation type if and only if there is a positive integer m such that $c^m = \mathrm{Id}_G$.*

PROOF. Assume that there is an integer $m > 0$ such that $c^m = \mathrm{Id}_G$. Then, by Corollary 5.14 we have that for every nonprojective injective Λ-module I there is a positive integer $t < m$ such that $(D\mathrm{Tr})^t I$ is torsionless. Then, by Lemma 5.15, Λ is of finite representation type.

Assume now that Λ is of finite representation type. Let M be an indecomposable Λ-module. If M is projective then $c(\langle M\rangle) = -\langle I_0(M/\mathrm{r}M)\rangle$ and $I_0(M/\mathrm{r}M)$ is indecomposable. If M is not projective then $c(\langle M\rangle) = (0, [D\mathrm{Tr}M])$ if $D\mathrm{Tr}M$ is nonprojective torsionless and $c(\langle M\rangle) = \langle D\mathrm{Tr}M\rangle$ otherwise. If S is nonprojective torsionless then $c(0, [S]) = -\langle I_0(S)\rangle$ and $I_0(S)$ is indecomposable.

Let $\mathfrak{D} = \{\langle M\rangle, -\langle N\rangle, \pm(0, [S]), N, M$ are indecomposable Λ-modules, S is a nonprojective torsionless simple$\}$. Since Λ is of finite representation type $\mathfrak{D}$ is finite and c transforms $\mathfrak{D}$ into itself. So the group generated by c acts as a permutation group of a finite set and is, therefore, finite. This proves that there is $m > 0$ such that $c^m = \mathrm{Id}_G$.

If M is an indecomposable Λ-module we say that M is $D\mathrm{Tr}$-*periodic* if $(D\mathrm{Tr})^n M = M$ for some $n > 0$.

COROLLARY 5.17. *Suppose that Λ is of finite representation type. Let $I_1, \ldots, I_n$ be a complete set of nonisomorphic indecomposable injective Λ-modules. Then, for every $i = 1, \ldots, n$ there is an $n_i \geqslant 0$ such that $(D\mathrm{Tr})^{n_i} I_i$ is projective. Let $\mathfrak{D} = \{(D\mathrm{Tr})^s I_i,\ 0 \leqslant s \leqslant n_i\}$. The modules in $\mathfrak{D}$ are pairwise nonisomorphic and, if M is an indecomposable Λ-module that is not isomorphic to an element of $\mathfrak{D}$, then M is $D\mathrm{Tr}$-periodic. Moreover, $M \cong (\mathrm{Tr}D)^r(S)$, for some $r > 0$ and some nonprojective torsionless module S.*

PROOF. If M is not $D\mathrm{Tr}$-periodic then there is some $t \geqslant 0$ such that $(D\mathrm{Tr})^t(M)$ is projective, because Λ is of finite representation type. Therefore, for every injective indecomposable module I_i there is an integer $n_i \geqslant 0$ such that $(D\mathrm{Tr})^{n_i}I_i$ is projective. It is easily seen that the elements of $\mathfrak{P}$ are pairwise nonisomorphic, so $\{(D\mathrm{Tr})^{n_i}I_i,\ i = 1, \ldots, n\}$ is a complete set of nonisomorphic projective Λ-modules. Let now M be an indecomposable module. If M is not $D\mathrm{Tr}$-periodic then there is a nonnegative integer t such that $(D\mathrm{Tr})^t(M)$ is projective, so, for some i, $(D\mathrm{Tr})^tM = (D\mathrm{Tr})^{n_i}I_i$. If $t > n_i$, then $(D\mathrm{Tr})^{t-n_i}M = I_i$ with $t - n_i > 0$. This is a contradiction, because $(D\mathrm{Tr})^{t-n_i}M$ is in $\mathrm{mod}_I(\Lambda)$. Therefore $t \leqslant n_i$. Then $M \cong (D\mathrm{Tr})^{n_i-t}I_i$, so $M \in \mathfrak{P}$ since $0 \leqslant n_i - t \leqslant n_i$. Now suppose that M is $D\mathrm{Tr}$-periodic and let S be a nonprojective torsionless module such that $(D\mathrm{Tr})^rM = S$, for some $r > 0$. Then S is $D\mathrm{Tr}$-periodic and $M = (\mathrm{Tr}D)^rS$.

PROPOSITION 5.18. *Assume that Λ is of finite representation type and let $m > 0$ be such that $c^m = \mathrm{Id}_G$. If M is an indecomposable nonprojective Λ-module and s is the smallest positive integer such that $(D\mathrm{Tr})^sM$ is torsionless, then $s < m$. Let n be the number of nonisomorphic simple modules, t the number of nonisomorphic projective injective indecomposable modules and r the number of nonisomorphic torsionless nonprojective simples. Then the number of nonisomorphic indecomposable Λ-modules is not greater than $(n - t)\cdot m + t + (m - 1)\cdot r$.*

PROOF. We write $s = \lambda m + t$, with λ, t nonnegative integers and $t < m$. Then

$$c^t(\langle M\rangle) = c^s(\langle M\rangle) = \begin{cases} \langle (D\mathrm{Tr})^sM\rangle & \text{if } (D\mathrm{Tr})^sM \text{ is projective,} \\ \left(0, \left[(D\mathrm{Tr})^sM\right]\right) & \text{otherwise.} \end{cases}$$

If $\lambda > 0$, then $t + 1 < s$, since $c \neq \mathrm{Id}_G$, so $m \neq 1$. So $c^{t+1}(\langle M\rangle) = \langle (D\mathrm{Tr})^{t+1}M\rangle$; but $c^{t+1}(\langle M\rangle) = c^{s+1}(\langle M\rangle)$ is, in any case, negative. This is a contradiction, therefore $\lambda = 0$, so $s < m$.

So the indecomposable modules are of the form $(\mathrm{Tr}D)^sN$, for some torsionless module N and some s satisfying $0 < s < m$ if N is nonprojective, $0 \leqslant s < m$ otherwise; if N is projective injective, $s = 0$. Then the total number of indecomposable modules cannot exceed $(n - t)\cdot m + t + (m - 1)\cdot r$.

We give now two examples to illustrate how calculations can be done using the preceding results.

EXAMPLE 5.19. Let K be a field, Λ the subring of $M_{3\times 3}(K)$,

$$\Lambda = \left\{ \begin{bmatrix} \alpha & 0 & 0 \\ a_1 & a_2 & 0 \\ a_3 & a_4 & \alpha \end{bmatrix}, a_i \in K, \alpha \in K \right\}.$$

The indecomposable projective Λ-modules are $P_1 = \Lambda \cdot e$, $P_2 = \Lambda \cdot (1 - e)$, where

$$e = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let $S_1 = P_1/\mathfrak{r}P_1$, $S_2 = P_2/\mathfrak{r}P_2$. The only proper submodules of P_1 are P_2 and S_1 and the only proper submodule of P_2 is S_1. So Λ is not an hereditary ring; besides, $\mathfrak{r}^2 \neq 0$. But S_1 is cotorsionless, so Λ is stably equivalent to an hereditary ring (see [**8** Example 3.1]).

In this case,

$$\mathfrak{a} = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_3 & a_4 & 0 \end{bmatrix}, a_3, a_4 \in K \right\},$$

so

$$\Lambda/\mathfrak{a} \cong \begin{pmatrix} a_1 & 0 \\ a_2 & a_3 \end{pmatrix},$$

and $G = Z \times Z \times Z$, where we identify the canonical basis of $Z \times Z \times Z$ with the basis $\{([S_1], 0), ([S_2], 0), (0, [S_1])\}$ of G. Then $\langle P_1 \rangle = ([P_1/\mathfrak{a}P_1], [\mathfrak{a}P_1]) = (1,1,1)$; $\langle P_2 \rangle = (0,1,1)$; $\langle I_0(S_1) \rangle = (1,1,1)$; $\langle I_{\Lambda/\mathfrak{a}}(S_1) \rangle = (1,0,0)$; $\langle I_{\Lambda/\mathfrak{a}}(S_2) \rangle = (1,1,0)$. Since c is defined by $c(\langle P_0(S_i) \rangle) = \langle I_{\Lambda/\mathfrak{a}}(S_i) \rangle$, $i = 1, 2$; $c((0, [S_1]) = -\langle I_0(S_1) \rangle)$ we have:

$$c(1,1,1) = -(1,0,0), \qquad c(0,1,1) = -(1,1,0), \qquad c(0,0,1) = -(1,1,1).$$

Therefore,

$$c = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}.$$

Since $c^4 = I$, Λ is of finite representation type. We describe now all the indecomposable modules. As Λ is of finite representation type an indecomposable module M is completely determined by the element $\langle M \rangle \in G$. So we find now all the $\langle M \rangle$ such that M is indecomposable. According to the last proposition, the indecomposable modules are of the form $(D\mathrm{Tr})^r (I_0(S_i))$, with $0 \leqslant r < 4$, and $(D\mathrm{Tr})^s(S_1)$, with $0 < s < 4$.

$$c(\langle I_0(S_1) \rangle) = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \quad \text{is negative, since } I_0(S_1) \text{ is projective.}$$

$$c(\langle I_0(S_2) \rangle) = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \langle D\mathrm{Tr}\, I_0(S_2) \rangle.$$

$$c^2(\langle I_0(S_2)\rangle) = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} \quad \begin{array}{l}\text{is negative,} \\ \text{since } D\mathrm{Tr}I_0(S_2) \text{ is projective.}\end{array}$$

$$c(\langle S_1\rangle) = c(1,0,0) = (0,1,0) = \langle D \text{ Tr } S_1\rangle.$$

So D Tr $S_1 = S_2$. $c^2(\langle S_1\rangle) = c(\langle S_2\rangle) = (0,0,1)$; since the projection of the vector $(0,0,1) \in \mathrm{Gr}(\Lambda/\mathfrak{a}) \times \mathrm{Gr}(\Lambda/\mathfrak{b})$ on $\mathrm{Gr}(\Lambda/\mathfrak{a})$ is zero, then $c(\langle S_2\rangle) = (0, [D\mathrm{Tr}S_2])$, so D Tr $S_2 = S_1$.

Therefore the periodic modules are S_1 and S_2, and the other indecomposable modules are given by $\langle I_0(S_1)\rangle = (1,1,1)$, $\langle I_0(S_2)\rangle = (1,1,0)$, $\langle D\mathrm{Tr}I_0(S_2)\rangle = (0,1,1)$.

EXAMPLE 5.20. Let K be a field, F a finite extension of K of degree n, Λ the quotient of

$$\Lambda_1 = \left\{\begin{bmatrix} f_1 & 0 & 0 \\ f_2 & k_1 & 0 \\ f_3 & k_2 & k_3 \end{bmatrix}, f_i \in F, k_i \in K\right\}$$

$$\text{by the ideal } I = \left\{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ f & 0 & 0 \end{bmatrix}, f \in F\right\}.$$

Λ is of square radical zero. The projectives are

$$P_1 = \Lambda \cdot \overline{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}, \qquad P_2 = \Lambda \cdot \overline{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}},$$

$$P_3 = \Lambda \cdot \overline{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}.$$

Let $S_i = P_i/\mathrm{r}P_i$, $i = 1, 2, 3$. Here

$$\mathfrak{a} = \left\{\overline{\begin{bmatrix} 0 & 0 & 0 \\ f & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}, f \in F\right\},$$

so $\mathfrak{a} \cong S_2^n$, and $G \cong Z^4$, identifying the basis $\{([S_i], 0), (0, [S_2]), i = 1, 2, 3\}$ with the canonical basis of Z^4. Then

$$c(\langle P_1\rangle) = c(1,0,0,n) = -(1,0,0,0), \qquad c(\langle P_2\rangle) = c(0,1,1,0) = -(0,1,0,0),$$

$$c(\langle P_3\rangle) = c(0,0,1,0) = -(0,1,1,0), \qquad c(0, [S_2]) = c(0,0,0,1) = -(1,0,0,1).$$

So

$$c = \begin{bmatrix} n-1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ n & 0 & 0 & -1 \end{bmatrix}.$$

If $c^m = I$ for some $m > 0$ then all the eigenvalues are roots of unity of some order. It can be seen that if this is the case then $n \leqslant 4$. If $n = 4$ the $c^m \neq I$ for all $m > 0$, so for $n \geqslant 4$ the ring Λ is of infinite representation type. For $n = 1$, $c^3 = I$ and using the bound for the number of indecomposable modules given in Proposition 5.18 we have that there are at most 9 nonisomorphic indecomposable modules. If $n = 2$, then $c^4 = 1$ and there are at most 13 nonisomorphic indecomposable modules.

When $n = 3$, $c^6 = 1$. We describe the set of indecomposable modules in this case. We write $I_i = I_0(S_i)$. $c(\langle I_1\rangle) = c(1,0,0,0) = (2,0,0,3)$; $c^2(\langle I_1\rangle) = (1,0,0,3) = \langle P_1\rangle$. $c(\langle I_2\rangle) = (1,0,0,2)$; $c^2(\langle I_2\rangle) = c(\langle D\mathrm{Tr} I_2\rangle) = (0,0,0,1) = (0, [(D\mathrm{Tr})^2 I_2]$ so $(D\mathrm{Tr})^2 I_2 = S_2$. $c(\langle S_2\rangle) = (0,0,1,0)$; $c^2(\langle S_2\rangle) = (0, -1, -1, 0)$ is negative, since $D\mathrm{Tr} S_2 = S_3$ is projective. The module $I_3 = P_2$ is projective, so there are exactly 9 nonisomorphic indecomposable modules and none of them is periodic.

6. **The bilinear form associated to** Λ. We keep the notations of the preceding section: $\mathfrak{a}$ denotes the sum of the nonprojective simples of the socle of Λ and $\mathfrak{b}$ is the left annihilator of $\mathfrak{a}$ in Λ. $G = \mathrm{Gr}(\Lambda/\mathfrak{a}) \times \mathrm{Gr}(\Lambda/\mathfrak{b})$, Z denotes the ring of integers and Q the field of rational numbers.

We define a bilinear form B: $G \times G \to Q$. When Λ is hereditary $\mathfrak{a} = 0$, $\mathfrak{b} = \Lambda$ and B is the same bilinear form: $\mathrm{Gr}(\Lambda) \times \mathrm{Gr}\,\Lambda \to Q$ defined in [7]. We will prove that, for an appropriate indexing of the elements of the basis of $G \times G$, the Coxeter transformation associated to the bilinear form B is the isomorphism c defined in the preceding section (see [7], [13]). And we will prove that Λ is of finite representation type if and only if B is positive definite. To prove these results we will use a different approach to that followed in the preceding sections: we will prove the results using that they are known for hereditary rings (see [7]) and considering the explicit description of an hereditary ring stably equivalent to Λ that we mentioned at the beginning of §5 and we recall now.

Let Γ be the triangular matrix ring $\left(\begin{smallmatrix} \Lambda/\mathfrak{a} & 0 \\ \mathfrak{a} & \Lambda/\mathfrak{b} \end{smallmatrix}\right)$. Then Γ is hereditary and stably equivalent to Λ. We describe a functor F: $\mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Gamma$ that induces a stable equivalence.

The Γ modules can be considered as triples (A, B, f), where A is a $\Lambda/\mathfrak{a}$-module, B is a $\Lambda/\mathfrak{b}$-module and f: $\mathfrak{a} \otimes A \to B$ is a $\Lambda/\mathfrak{b}$-homomorphism. A map

$$g\colon (A, B, f) \to (A', B', f')$$

is a pair (g_1, g_2), where $g_1\colon A \to A'$ is a $\Lambda/\mathfrak{a}$-homomorphism, $g_2\colon B \to B'$ is a $\Lambda/\mathfrak{b}$-homomorphism and the diagram

$$\begin{tikzcd} \mathfrak{a} \otimes A \arrow[r, "\mathrm{id} \otimes g_1"] \arrow[d, "f"] & \mathfrak{a} \otimes A' \arrow[d, "f'"] \\ B \arrow[r, "g_2"] & B' \end{tikzcd}$$

is commutative. If

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \begin{pmatrix} \Lambda/\mathfrak{a} & 0 \\ \mathfrak{a} & \Lambda/\mathfrak{b} \end{pmatrix},$$

then $\mathfrak{a} = (1 - e) \cdot \Gamma \cdot e$ and the equivalence between mod Γ and the category of triples is given by $g(X) = (eX, (1 - e)X, f)$, where $f\colon \mathfrak{a} \otimes eX \to (1 - e)X$ is defined by $f((1 - e) \cdot \lambda \cdot e \otimes e \cdot m) = (1 - e) \cdot \lambda \cdot e \cdot m$.

We define $F\colon \operatorname{mod} \Lambda \to \operatorname{mod} \Gamma$ by $F(M) = (M/\mathfrak{a}M, \mathfrak{a}M, f)$, where $f\colon M/\mathfrak{a}M \to \mathfrak{a}M/\mathfrak{a}^2M = \mathfrak{a}M$ is the multiplication map. If $h\colon M \to N$ is a morphism in mod Λ, then $F(h) = (\bar{h}, h|M)$, where $\bar{h}\colon M/\mathfrak{a}M \to N/\mathfrak{a}N$ is the map induced by h.

F defines a full dense functor $\operatorname{mod}_P\Lambda \to \operatorname{mod}_P\Gamma$ and the induced functor $F\colon \underline{\operatorname{mod}}\ \Lambda \to \underline{\operatorname{mod}}\ \Gamma$ is an equivalence of categories. The proof of these results will not be included here, since it will be published in another paper.

The Grothendieck group of Γ is isomorphic to $\operatorname{Gr}(\Lambda/\mathfrak{a}) \times \operatorname{Gr}(\Lambda/\mathfrak{b}) = G$. We define the bilinear form $B_\Lambda\colon G \times G \to Q$ associated to Λ to be the bilinear form $B_\Gamma\colon G \times G \to Q$ associated to the hereditary ring Γ (see [7, §3]). We recall the definition now. Since the center of an hereditary indecomposable artin algebra is a field, to define B_Γ we may assume that the center of Γ is the field K. Then B_Γ is the symmetric bilinear form associated to the form $B_{1,\Gamma}\colon G \times G \to Q$ defined by

$$B_{1,\Gamma}([X], [Y]) = \dim_K(X, Y) - \dim_K \operatorname{Ext}^1_\Gamma(X, Y),$$

for any pair of Γ-modules X and Y. That is,

$$B_\Gamma([X], [Y]) = \tfrac{1}{2}(B_{1,\Gamma}([X], [Y]) + B_{1,\Gamma}([Y], [X])).$$

As in the preceding section, for a Λ-module M we denote by $\langle M \rangle$ the element $([M/\mathfrak{a}M], [\mathfrak{a}M]) \in G$. Then $\langle M \rangle = [F(M)]$. Therefore

$$\begin{aligned} B_{1,\Lambda}(\langle M \rangle, \langle N \rangle) &= B_{1,\Gamma}([F(M)], [F(N)]) \\ &= \dim_K \operatorname{Hom}_\Gamma(F(M), F(N)) - \dim_K \operatorname{Ext}^1_\Gamma(F(M), F(N)). \end{aligned}$$

The following calculations are devoted to describe B_Λ only in terms of Λ.

$\operatorname{Hom}_\Gamma(F(M), F(N)) = \operatorname{Hom}_\Gamma((M/\mathfrak{a}M, \mathfrak{a}M, f), (N/\mathfrak{a}N, \mathfrak{a}N, f')) \cong \operatorname{Hom}_\Lambda(M, N)/\operatorname{Hom}_\Lambda(M, \mathfrak{a}N)$, for M, N in mod Λ. On the other hand,

$D\,\mathrm{Ext}^1_\Gamma(X, Y) \cong \mathrm{Hom}_\Gamma(\underline{\mathrm{Tr}DY}, \underline{X})$, if $X \in \mathrm{mod}\,\Gamma$, $Y \in \mathrm{mod}_I\Gamma$. (See [3, Proposition 2.2] and [11, p. 119].) So we have

$$D\,\mathrm{Ext}^1_\Gamma(F(M), F(N)) \cong \mathrm{Hom}_\Gamma(\underline{\mathrm{Tr}\,D\,F(N)}, \underline{F(M)}). \tag{1}$$

Using Proposition 5.9 and an appropriate description of the injective Γ-modules one can prove that $\mathrm{Tr}\,D\,F(N) = F(\mathrm{Tr}\,D\,N)$ if N is torsion or projective, and $\mathrm{Tr}\,D(F(N)) = (0, \mathrm{Tr}DN, 0)$ otherwise. When N is torsion this result follows also from the following result of [6]: Let G: $\underline{\mathrm{mod}}(\Lambda_1) \to \underline{\mathrm{mod}}(\Lambda_2)$ be a stable equivalence between two artin algebras Λ_1 and Λ_2. If M in $\mathrm{mod}_P\Lambda_1$ is indecomposable and $0 \to D\mathrm{Tr}M \to E \to M \to 0$ is the almost split sequence, then if E is not projective $G(D\mathrm{Tr}M) = D\mathrm{Tr}G(M)$.

Therefore from (1) we obtain

$$\begin{aligned} D\,\mathrm{Ext}^1_\Gamma(F(M), F(N)) &\cong (\underline{F(\mathrm{Tr}DN)}, \underline{F(M)}) \\ &\cong (\underline{\mathrm{Tr}DN}, \underline{M}) \cong \mathrm{Ext}^1_\Lambda(M, N) \end{aligned}$$

if N is torsion or projective; and $D\,\mathrm{Ext}^1_\Gamma(F(M), F(N)) = ((\underline{0, \mathrm{Tr}DN, 0}), F(\underline{M})) = 0$ otherwise, because $(0, \mathrm{Tr}DN, 0)$ is a projective Γ-module.

We can give now an explicit formula for $B(\langle M\rangle, \langle N\rangle)$ in terms of Λ.

PROPOSITION 6.1. *Let M and N be indecomposable Λ-modules. Then*:

$$B_{1,\Lambda}(\langle M\rangle, \langle N\rangle) = \begin{cases} \dim_K \mathrm{Hom}_\Lambda(M, N) \\ \quad \textit{if } N \textit{ is nonprojective torsionless}, \\ \dim_K (\mathrm{Hom}_\Lambda(M, N)/\mathrm{Hom}_\Lambda(M, \mathfrak{a}N)) \\ \quad - \dim_K \mathrm{Ext}^1_\Lambda(M, N) \quad \textit{otherwise}. \end{cases} \tag{a}$$

(b) $B_{1,\Lambda}((\langle M\rangle), (0, [S])) = -\dim_K(S, \tau_{\mathfrak{a}}(D\mathrm{Tr}M))$.
(c) $B_{1,\Lambda}((0, [S]), \langle M\rangle) = \dim_K(S, \mathfrak{a}M)$.
(d) $B_{1,\Lambda}((0, [S]), (0, [S'])) = \dim_K(S, S')$.

PROOF. We already proved (a).

$$\begin{aligned} B_{1,\Lambda}(\langle M\rangle, (0, [S])) &= B_{1,\Gamma}(\langle M\rangle, (0, [S])) \\ &= \dim_K \mathrm{Hom}_\Gamma(F(M), (0, S, 0)) - \dim_K \mathrm{Ext}^1_\Lambda(F(M), (0, S, 0)). \end{aligned} \tag{b}$$

$$\mathrm{Hom}_\Gamma(F(M), (0, S, 0)) = 0$$

and

$$\mathrm{Ext}^1_\Lambda(F(M), (0, S, 0)) = \mathrm{Hom}_\Gamma((\overline{0, S, 0}), \overline{D\mathrm{Tr}F(M)})$$

$$= \begin{cases} \mathrm{Hom}_\Gamma(\overline{(0, S, 0)}, \overline{F(D\mathrm{Tr}M)}) = (\bar{S}, \overline{\mathfrak{a}\cdot D\mathrm{Tr}M}) \\ \qquad\qquad\qquad\qquad \text{if } D\mathrm{Tr}M \text{ is torsion or projective}, \\ \mathrm{Hom}_\Gamma((\overline{0, S, 0}), (\overline{0, D\mathrm{Tr}M, 0})) = (\bar{S}, \overline{D\mathrm{Tr}M}) \quad \text{otherwise}. \end{cases}$$

(b) follows now from:

LEMMA 6.2. *If S is a nonprojective torsionless simple Λ-module and N is in* $\mathrm{mod}_I\Lambda$ *then* $(\bar{S}, \bar{N}) \cong (\bar{S}, \overline{\tau_{\mathfrak{a}}(N)})$.

PROOF. Let $f\colon S \to N$ be a homomorphism. Then $\mathrm{Im}(f) \subseteq \tau_{\mathfrak{a}}(N)$, since S is torsionless nonprojective. Thus the map $\varphi\colon (\bar{S}, \overline{\tau_{\mathfrak{a}}(N)}) \to (\bar{S}, \bar{N})$ induced by the inclusion $\tau_{\mathfrak{a}}(N) \subseteq N$ is an epimorphism. If $f\colon S \to N$ factors through an injective E, i.e., $f = \beta\alpha$, $\alpha\colon S \to E$, $\beta\colon E \to N$, then $\mathrm{Im}(\beta)$ cannot contain injective summands because $N \in \mathrm{mod}_I(\Lambda)$. So $\mathrm{Im}(\beta)$ is a sum of nonprojective torsionless Λ-modules and is therefore contained in $\tau_{\mathfrak{a}}(N)$. This proves that φ is also a monomorphism.

(c) $B_{1,\Lambda}((0, [S]), \langle M\rangle) = \dim_K \mathrm{Hom}_\Gamma((0, S, 0), F(M)) = \dim_K (S, \mathfrak{a}M)$, since $(0, S, 0)$ is projective and then $\mathrm{Ext}^1_\Gamma((0, S, 0), F(M)) = 0$.

(d) follows from the definition.

We saw in §5 that $\Lambda/\mathfrak{a}$ is an hereditary ring. Let $\mathcal{S}$ be a complete set of nonisomorphic simple $\Lambda/\mathfrak{a}$-modules. We recall from [7] that $\mathcal{S}$ is partially ordered writing $S < S'$ if and only if $P_0(S) \subseteq P_0(S')$, and that an *admissable indexing* of the elements of $\mathcal{S}$ is an order preserving map $a\colon \mathcal{S} \to \{1, 2, \ldots, n\}$. So, if we write $a^{-1}(i) = S_i$, then

$$i < j \Rightarrow \big(P_0(S_j), P_0(S_i)\big) = 0.$$

We saw in [7] that admissible indexings always exist. We will say that an indexing $S_1, \ldots, S_n$ of the simple Λ-modules is admissible if considering $S_1, \ldots, S_n$ as $\Lambda/\mathfrak{a}$-modules the indexing is admissible.

We recall briefly the definition of the Coxeter transformation (see, for example, [13], [7]). If $B\colon Z^m \times Z^m \to Q$ is a bilinear form, $e_1, \ldots, e_m$ is the canonical basis of Z^m and $B(e_i, e_i) \neq 0$ for $i = 1, \ldots, m$, we denote by σ_i the symmetry with respect to the vector e_i. That is,

$$\sigma_i(x) = x - 2\frac{B(x, e_i)}{B(e_i, e_i)} \cdot e_i.$$

For $i = 1, \ldots, m$, $\sigma_i\colon Z^m \to Q^m$ can be extended uniquely to a map, that we call also σ_i, in $\mathrm{Gl}(m, Q)$. The subgroup of $\mathrm{Gl}(m, Q)$ generated by these maps is called the *Weyl group*, and $C = \sigma_m \cdots \sigma_1$ is the *Coxeter transformation*.

Let $S_1, \ldots, S_n$ be a complete set of nonisomorphic simple Λ-modules and assume that $S_{i_1}, \ldots, S_{i_t}$ are the nonprojective torsionless simples. We identify $\mathrm{Gr}(\Lambda/\mathfrak{a}) \times \mathrm{Gr}(\Lambda/\mathfrak{b})$ with Z^{n+t} by means of the isomorphism that transforms $(0, [S_{i_k}])$ into e_k, $k = 1, \ldots, t$ and $\langle S_i\rangle$ into e_{n+i}, $i = 1, \ldots, n$. This identification depends on the indexing of the simple Λ-modules. But Coxeter transformations associated to two admissible indexings of the simples of Λ are equal. And we will prove that the isomorphism c defined in §5 is precisely

the Coxeter transformation C associated to an admissible indexing of the simples of Λ.

Therefore we assume in all that follows that the indexing $S_1, \ldots, S_n$ of the simple Λ-modules is admissible. As before, let $S_{i_1}, \ldots, S_{i_t}$ be the nonprojective torsionless. Then $\mathcal{S} = \{(0, S_{i_k}, 0), F(S_j), k = 1, \ldots, t; j = 1, \ldots, n\}$ is a complete system of nonisomorphic simple Γ-modules and $a: \mathcal{S} \to \{1, \ldots, n + t\}$ given by $a((0, S_{i_k}, 0)) = k$, $k = 1, \ldots, t$; $a(F(S_j)) = t + j$, $j = 1, \ldots, n$, is an admissible indexing of the elements of $\mathcal{S}$.

Since we are going to deal with two different rings, to indicate that c and C are, respectively, the map defined in §5 and the Coxeter transformation associated to the ring Λ, we will write c_Λ, C_Λ.

Then $C_\Lambda = \sigma_{n+t} \cdots \sigma_1 = C_\Gamma$. On the other hand it follows from the definition of c and the description of the indecomposable injective Γ-modules that $c_\Lambda = c_\Gamma$. Since Γ is an hereditary ring we know by [7, §3] that $c_\Gamma = C_\Gamma$. So $c_\Lambda = c_\Gamma = C_\Gamma = C_\Lambda$, and we have proven:

PROPOSITION 6.3. *The isomorphism c: $G \to G$ defined in §5 is the Coxeter transformation associated to an admissible indexing of the simple Λ-modules.*

As an easy consequence of this proposition and of the results proven for hereditary rings in [7] we have:

THEOREM 6.2. *The following conditions are equivalent*:

(a) Λ *is of finite representation type.*

(b) $c^m = \mathrm{Id}_G$ *for some* $m > 0$.

(c) B *is positive definite.*

PROOF. We proved in §5 that (a) and (b) are equivalent. It is known and not hard to prove that when B is positive definite the Weyl group is finite and therefore $c^m = \mathrm{Id}_G$ for some $m > 0$, so (c) $\Rightarrow$ (b). (See, for example, [13].) That (a) and (b) imply (c) follows easily using the fact that the result is true when the ring is hereditary: assume that Λ is of finite representation type; then

$$\Gamma = \begin{pmatrix} \Gamma/\mathfrak{a} & 0 \\ \mathfrak{a} & \Lambda/\mathfrak{b} \end{pmatrix},$$

that is stably equivalent to Λ, is also of finite representation type. Since Γ is hereditary we know (by [7, Theorem 4.1]) that the bilinear form B_Γ associated to Γ is positive definite. Then B, that is equal to B_Γ, is also positive definite.

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