WEAK CHEBYSHEV SUBSPACES AND CONTINUOUS SELECTIONS FOR THE METRIC PROJECTION

BY

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Abstract. Let G be an n-dimensional subspace of C[a,b]. It is shown that there exists a continuous selection for the metric projection if for each f in C[a,b] there exists exactly one alternation element g, i.e., a best approximation for f such that for some a < x_0 < \cdots < x_n < b,

\[ \varepsilon(-1)^i(f - g_i)(x_i) = \|f - g_i\|, \quad i = 0, \ldots, n, \varepsilon = \pm 1. \]

Further it is shown that this condition is fulfilled if and only if G is a weak Chebyshev subspace with the property that each g in G, g \neq 0, has at most n distinct zeros. These results generalize in a certain sense results of Lazar, Morris and Wulbert for n = 1 and Brown for n = 5.

If G is a nonempty subset of a normed linear space E then for each f in E, we define \( P_G(f) := \{ g_0 \in G : \|f - g_0\| = \inf(\|f - g\| : g \in G) \} \). \( P_G \) defines a set-valued mapping of E into \( 2^G \) which in the literature is called the metric projection onto G. A continuous mapping \( s \) of E into G is called a continuous selection for the metric projection \( P_G \) (or, more briefly, continuous selection) if \( s(f) \) is in \( P_G(f) \) for each \( f \) in E. In this paper we treat the problem of the existence of continuous selections for n-dimensional subspaces G of C[a,b], with C[a,b] as usual the Banach space of real-valued continuous functions on [a,b] under the uniform norm.

A. Lazar, P. Morris and D. Wulbert [4] have characterized the 1-dimensional subspaces of C(X) with X compact Hausdorff, which admit a continuous selection. They raised the problem of characterizing the corresponding n-dimensional subspaces. The only known result for higher dimensional subspaces has been given by A. Brown [1], who has shown the existence of continuous selections for certain 5-dimensional subspaces of C[-1,1].

To obtain continuous selections, Lazar, Morris and Wulbert [4] and Brown [1] proceeded as follows: For each f in C[a,b] they considered all g in \( P_G(f) \) which can be written as \( g = a_1 g_1 + \cdots + a_n g_n \), where \( g_1, \ldots, g_n \) is a basis
of $G$, and chose the unique element $g$ in $P_G(f)$ with maximal coefficient $a_n$. This works in the cases $n = 1$ and $n = 5$.

Using this kind of selection it does not seem possible to get a general theorem for $n$-dimensional subspaces in $C[a,b]$. With new methods, however, and in the setting of weak Chebyshev subspaces we can give a sufficient condition for the existence of continuous selections.

R. Jones and L. Karlovitz [2, Theorem 4] have shown that an $n$-dimensional subspace $G$ of $C[a,b]$ is weak Chebyshev if and only if for each $f$ in $C[a,b]$ there exists at least one alternation element $g_f$ (see Definition 1 below) in $P_G(f)$. We show that if for each $f$ in $C[a,b]$ there exists exactly one alternation element $g_f$ in $P_G(f)$, then $s(f) = g_f$ defines a continuous selection (Proposition 2). From Theorem 8 and Theorem 11, which together represent the main result of this paper, it follows that for an $n$-dimensional weak Chebyshev subspace $G$ each $f$ in $C[a,b]$ has exactly one alternation element in $P_G(f)$ if and only if each $g \in G$, $g \neq 0$, has at most $n$ distinct zeroes. (In particular, $g$ may not vanish on intervals.)

Using this result and Proposition 2, we immediately get an existence theorem for continuous selections for $n$-dimensional subspaces (Corollary 9). Brown [1] uses essentially stronger conditions to guarantee the existence of continuous selections for 5-dimensional subspaces of $C[-1,1]$. Brown’s result disproves a claim of Lazar, Morris and Wulbert [4], who tried to show that for $n$-dimensional subspaces $G$ in $C(X)$ ($X$ a connected, compact, Hausdorff space) such that 1 is in $G$ and each $g \in G$, $g \neq 0$, does not vanish on an open set in $X$, there does not exist a nontrivial continuous selection.

Finally, Let us remark that from P. Schwartz [8] it follows that under the assumption of Corollary 9 the continuous selection is unique.

In the following let $G$ be an $n$-dimensional subspace of $C[a,b]$.

1. Definition. If $f$ is in $C[a,b]$, then $g$ in $P_G(f)$ is called an alternation element (A-element) of $f$ if there exist $n + 1$ distinct points $a < x_0 < \cdots < x_n < b$ such that

$$\varepsilon (-1)^i(f - g)(x_i) = \|f - g\|, \quad i = 0, \ldots, n, \quad \varepsilon = \pm 1.$$ 

The points $a < x_0 < \cdots < x_n < b$ are called alternating extreme points of $f - g$.

First, we want to show that when each $f$ has a unique $A$-element then we can always define a continuous selection.

2. Proposition. Suppose for each $f$ in $C[a,b]$ there exists exactly one $A$-element $g_f$ in $P_G(f)$. Define $s : C[a,b] \to G$ by $s(f) = g_f$ for each $f \in C[a,b]$. Then, $s$ is a continuous selection for $P_G : C[a,b] \to 2^G$.

Proof. We suppose $s$ is not continuous.
Because of the finite dimensionality of $G$, there exist $f \in C[a,b]$, $g \in G$ and a sequence $(f_m) \subset C[a,b]$ so that $f_m \to f$, $s(f_m) \to g$, but $g \neq s(f)$.

We will show that $g$ is an $A$-element of $f$ and this will contradict the uniqueness of the $A$-element.

By definition, $s(f_m)$ is an $A$-element of $f_m$, $m \in N$. Therefore, there are extreme points $a < x_0^{(m)} < x_1^{(m)} < \cdots < x_n^{(m)} < b$ of $f_m - s(f_m)$.

We can assume that

$$(-1)^i (f_m - s(f_m)) (x_i^{(m)}) = \|f_m - s(f_m)\|, \quad i = 0, \ldots, n, \ m \in N.$$  

Here it may be necessary to choose a subsequence of $(f_m)$ and perhaps work with $-f$ and $-f_m$ in place of $f$ and $f_m$. We can also assume (again choosing a subsequence if necessary) that $\lim_{m \to \infty} x_i^{(m)} = x_i$ exists, $i = 0,1,\ldots, n$. Now, since $\lim_{m \to \infty} f_m = f$ and $\lim_{m \to \infty} s(f_m) = g$, we have

$$\|f - g\| = \lim_{m \to \infty} \|f_m - s(f_m)\| = (-1)^i \lim_{m \to \infty} (f_m - s(f_m))(x_i^{(m)}) = (-1)^i (f - g)(x_i)$$

where in the second equality we used (1) and the uniform convergence. This shows that $g$ is an $A$-element, which is the desired contradiction.

Jones and Karlovitz [2] have characterized those $n$-dimensional subspaces of $C[a,b]$ which have at least one $A$-element for each $f$ in $C[a,b]$. For this characterization, we need the following definition:

**3. Definition.** $G$ is called weak Chebyshev if each $g$ in $G$ has at most $n - 1$ changes of sign, i.e., there do not exist points $a < x_0 < \cdots < x_n < b$ such that $g(x_i) \cdot g(x_{i+1}) < 0$, $i = 0, \ldots, n - 1$.

Jones-Karlovitz [2] have proved the following theorem:

**4. Theorem.** $G$ is weak Chebyshev if and only if for each $f$ in $C[a,b]$ there exists at least one $A$-element in $P_G(f)$.

To get a continuous selection under application of Proposition 2, we examine what additional conditions a weak Chebyshev subspace has to fulfill in order that each $f$ in $C[a,b]$ has exactly one $A$-element.

We need the following standard definition:

**5. Definition.** A zero $x_0$ of $f$ in $C[a,b]$ is said to be a simple zero if $f$ changes sign at $x_0$ or if $x_0 = a$ or $x_0 = b$.

A zero $x_0$ of $f$ in $C[a,b]$ is said to be a double zero if $f$ does not change sign at $x_0$ and $x_0 \neq a, x_0 \neq b$.

In the following, we count simple zeroes as one zero and double zeroes as two zeroes. To prove the following results we need the lemma below.

**6. Lemma.** If $f$ is in $C[a,b]$ and if there exist $n + 1$ points $a < x_0 < \cdots <
\( x_n \leq b \) such that
\[ \epsilon (-1)^i f(x_i) > 0, \quad i = 0, \ldots, n, \epsilon = \pm 1, \]
then \( f \) has at least \( n \) zeroes \( y_i \) such that
\[ x_0 < y_0 < x_1 < y_1 < \cdots < x_{n-1} < y_{n-1} < x_n. \]

7. Lemma. If \( G \) is an \( n \)-dimensional weak Chebyshev subspace of \( C[a,b] \) such that there exists a \( g \) in \( G \), \( g \neq 0 \), with at least \( n + 2 \) zeroes, then there exists a \( \tilde{g} \) in \( G \), \( \tilde{g} \neq 0 \), with at least \( n + 1 \) distinct zeroes.

Proof. Let \( g \) be in \( G \), \( g \neq 0 \), with at least \( n + 2 \) zeroes in \([a,b]\), but only \( r \), \( r < n \), distinct zeroes. Suppose first that \( g(a) = g(b) = 0 \), and set \( \bar{x} = \max\{x \in [a,b] | g(x) = 0\} \).

Let \( a < x_1 < \cdots < x_s \leq \bar{x} \) be the simple zeroes of \( g \).

(a) \( s + n - 1 \) is an even number.

We choose \( n - 1 - s \) points \( \bar{x} < x_{s+1} < \cdots < x_{n-1} < b \). Since \( G \) is weak Chebyshev, by Jones and Karlovitz [2, p. 140] there exists a \( \tilde{g} \in G \), \( \tilde{g} \neq 0 \), with
\[ \epsilon (-1)^i \tilde{g}(x) > 0, \quad x_{i-1} < x < x_i, \quad i = 1, \ldots, n, \epsilon = \pm 1, \]
where \( x_0 = a, x_n = b \). By Lemma 6, \( \tilde{g} \) has at least \( n - 1 \) distinct zeroes. We choose \( \epsilon \) such that \( \text{sgn}(g(x) \cdot \tilde{g}(x)) > 0 \) if \( x \in [a,x_{s+1}] \). Let \( a = y_1 < \cdots < y_s = b \) be the distinct zeroes of \( g \) in \([a,b]\).

Then
\[ M := \min_{i=2,\ldots,n} \| g \|_{[y_{i-1},y_i]} > 0. \]

We define \( \hat{g} := M\tilde{g}/(2\|\tilde{g}\|) \).

The function \( \hat{g} \) has at least two further distinct zeroes in \([a,b]\), otherwise the function \( g - \hat{g} \) would have at least \( n \) changes of sign. This would be a contradiction.

(b) \( s + n - 1 \) is an odd number.

We choose \( x_0 = a \) and \( n - s - 2 \) points
\[ \bar{x} < x_{s+1} < \cdots < x_{n-2} < b. \]

Since \( G \) has an \((n - 1)\)-dimensional weak Chebyshev subspace (see Sommer and Strauss [11, Theorem 2.6]), by Jones and Karlovitz [2, p. 140] there exists a \( \tilde{g} \in G \), \( \tilde{g} \neq 0 \) with \( \epsilon (-1)^i \tilde{g}(x) > 0, x_{i-1} < x < x_i, \ i = 1, \ldots, n - 1, \epsilon = \pm 1 \) where \( x_{n-1} = b \).

As before let \( \text{sgn}(g(x) \cdot \tilde{g}(x)) > 0 \) if \( x \in [a,x_{s+1}] \).

Following (a) we construct a function \( \hat{g} \).

As before it follows that either the function \( \hat{g} \) or the function \( g - \hat{g} \) has \( n + 1 \) distinct zeroes in \([a,b]\).
If not \( g(a) = g(b) = 0 \), the assertion can be shown in an analogous way. This completes the proof.

8. **Theorem.** If \( G \) is an \( n \)-dimensional weak Chebyshev subspace of \( C[a,b] \) such that each \( g \) in \( G, g \neq 0 \), has at most \( n \) distinct zeroes, then each \( f \) in \( C[a,b] \) has exactly one \( A \)-element \( g \) in \( P_G(f) \).

**Proof.** *Assumption.* There exists a function \( f \) in \( C[a,b] \) which has two \( A \)-elements \( g_1 \) and \( g_2 \) in \( P_G(f) \).

Let \( a < x_0 < \cdots < x_n < b \) be \( n + 1 \) alternating extreme points of \( f - g_1 \) and let \( a < y_0 < \cdots < y_n < b \) be \( n + 1 \) alternating extreme points of \( f - g_2 \).

We distinguish two cases:

**First case.**

\[
(-1)^i(f - g_1)(x_i) = \|f - g_1\|, \quad i = 0, \ldots, n,
\]

\[
(-1)^i(f - g_2)(y_i) = \|f - g_2\|, \quad i = 0, \ldots, n.
\]

Then

\[
(-1)^i(g_2 - g_1)(x_i) > 0, \quad i = 0, \ldots, n,
\]

\[
(-1)^i(g_2 - g_1)(y_i) < 0, \quad i = 0, \ldots, n.
\]

We treat only the case

\[
(\text{ii}) \quad x_{i-2} < y_i < x_{i+2}, \quad i = 0, \ldots, n,
\]

where the points \( x_i \) for \( i = -2, -1, n + 1, n + 2 \) are omitted. In the other case, if \( y_i < x_{i-2} \) for some \( i \), we choose the points \( y_0, \ldots, y_i, x_{i-2}, \ldots, x_n \) fulfilling

\[
(-1)^j(g_2 - g_1)(y_j) < 0, \quad j = 0, \ldots, i,
\]

\[
(-1)^j(g_2 - g_1)(x_{j-3}) < 0, \quad j = i + 1, \ldots, n + 3.
\]

By Lemma 6, \( g_2 - g_1 \) has at least \( n + 3 \) zeroes. Applying Lemma 7 we get a contradiction of the hypothesis that elements of \( G \) have at most \( n \) distinct zeroes.

A similar argument works for \( x_{i+2} < y_i \).

Now we prove by induction that \( g_1 - g_2 \) has at least \( n + 1 \) distinct zeroes. This is a contradiction of the hypothesis on \( G \). If \( x_i = y_i, i = 0, \ldots, n \), then

\[
(g_1 - g_2)(x_i) = 0, \quad i = 0, \ldots, n.
\]

We may assume \( x_i < y_i \) for some \( i = 0, \ldots, n \).

We show: \( x_j < y_j, j = 0, \ldots, n \).

If \( y_j < x_j \) for any \( j_0 \in \{0, \ldots, n\} \) we choose

\[
y_0, \ldots, y_{j_0}, x_{j_0}, \ldots, x_i, y_i, \ldots, y_n \quad \text{if} j_0 < i
\]

and
Because of (i) in both cases $g_1 - g_2$ has at least $n + 2$ zeroes by Lemma 6. Applying Lemma 7 we get a contradiction of the hypothesis on $G$.

Now we show by induction that $g_1 - g_2$ has at least $n + 1$ distinct zeroes in $[x_0, y_n]$: $n = 1$.

If $x_0 < y_0 < x_1 < y_1$ (respectively $x_0 < y_0 = x_1 < y_1$ or $x_0 < x_1 < y_0 < y_1$), then $g_1 - g_2$ has one zero in each interval $[x_0, y_0]$, $[x_1, y_1]$ (respectively $[x_0, y_0)$, $(x_1, y_1]$ or $[x_0, x_1]$, $[y_0, y_1]$).

Let the statement be true for $n - 1$.

If $y_{n-1} < x_n < y_n$, then by assumption $g_1 - g_2$ has $n$ distinct zeroes in $[x_0, y_{n-1}]$ and a further zero in $[x_n, y_n]$.

If $y_{n-1} = x_n < y_n$, then by assumption $g_1 - g_2$ has $n$ distinct zeroes in $[x_0, y_{n-1}]$ and a further zero in $(x_n, y_n)$.

Finally we consider the case $x_n < y_{n-1} < y_n$:

Since $(-1)^n(g_2 - g_1)(y_n) > 0$, $(-1)^n(g_2 - g_1)(y_{n-1}) > 0$, and $y_{n-2} < x_n$ we conclude as in the case $y_{n-1} < x_n < y_n$.

**Second case.**

\[
(-1)^i(f - g_1)(x_i) = ||f - g_1||, \quad i = 0, \ldots, n.
\]

\[
-(-1)^i(f - g_2)(y_i) = ||f - g_2||, \quad i = 0, \ldots, n.
\]

We treat only the case that $f - g_1$ and $f - g_2$ have exactly $n + 1$ alternating extreme points.

Otherwise we can apply the first case.

Then

\[
(-1)^i(g_2 - g_1)(x_i) > 0, \quad i = 0, \ldots, n,
\]

\[
(-1)^i(g_2 - g_1)(y_i) > 0, \quad i = 0, \ldots, n.
\]

It is now enough to treat the case

\[
(x_{i-1} < y_i < x_{i+1}, \quad i = 0, \ldots, n),
\]

where the points $x_{i-1}$ and $x_{i+1}$ are omitted. Otherwise we can conclude as in the first case. Applying the first case to the points $x_0, \ldots, x_{n-1}, y_1, \ldots, y_n$ because of (v) $g_1 - g_2$ has $n$ distinct zeroes $z_1, \ldots, z_n$ in $[x_0, y_n]$.

We first prove: $z_1, \ldots, z_n \in (a, b)$. If $z_1 = x_0$, then $y_0 < x_0$. Otherwise $f - g_2$ has $n + 2$ alternating extreme points $x_0, y_0, \ldots, y_n$. This is a contradiction to (iii). Therefore $z_1 > a$.

If $z_n = y_n$, then $x_n > y_n$. Otherwise $f - g_1$ has $n + 2$ alternating extreme points $x_0, \ldots, x_n, y_n$. This is a contradiction to (iii). Therefore $z_n < b$.

If $g_1 - g_2$ has $n + 1$ distinct zeroes, then we would get a contradiction of the hypothesis on $G$.  

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Therefore we know \( g_1 - g_2 \) has no further zero in \([a, b]\). Because of \( a < z_1 < \cdots < z_n < b \) and \( G \) weak Chebyshev \( g_1 - g_2 \) has at most \( n - 1 \) changes of sign under the points \( z_i \). We show \( g_1 - g_2 \) has at most \( n - 2 \) changes of sign:

- If \( g_1 - g_2 \) has \( n - 1 \) changes of sign under the points \( z_i \), then there exists exactly one zero \( z_j \in (a, b) \) such that \( g_1 - g_2 \) does not change sign at \( z_j \).
- Then it holds: If \( z_j > x_0 \), then because of (iv)
  \[
  (-1)^0(g_2 - g_1)(x) > 0 \quad \text{if } a < x < z_1 \\
  (-1)^n(g_2 - g_1)(x) < 0 \quad \text{if } z_n < x < b.
  \]
- If \( z_j = x_0 \), then \( x_0 < x_0 \) and (vi) is also valid.
- Now we get a contradiction to (iv):
  \[
  (1) \ x_n > y_n.
  \]
  Then \( z_n < x_n \) and because of (iv) \((-1)^n(g_2 - g_1)(x_n) > 0 \). This is a contradiction.
  \[
  (2) \ x_n < y_n.
  \]
- If \( z_n < y_n \) we also get a contradiction because of (iv). But if \( z_n = y_n \), then \( x_n > y_n \) is always valid because of (iii).
- We have shown:
  - If \( g_1 - g_2 \) has exactly \( n \) distinct zeroes, then \( g_1 - g_2 \) has at most \( n - 2 \) changes of sign. But in this case there exist \( n + 2 \) zeroes of \( g_1 - g_2 \) because of \( a < z_1, z_n < b \).
  - Applying Lemma 7 we get a contradiction to the assumption.

Schwartz [8] has shown that for an \( n \)-dimensional subspace \( G \) of \( C(X) \) with the property that no \( g \) in \( G \), \( g \neq 0 \), vanishes identically on an open subset of \( X \), the set of functions in \( C(X) \) having unique best approximation in \( G \) is dense in \( C(X) \). Therefore there exists at most one continuous selection. By this result, Proposition 2 and Theorem 8 the next corollary follows immediately.

9. Corollary. If \( G \) is an \( n \)-dimensional weak Chebyshev subspace such that each \( g \) in \( G \), \( g \neq 0 \), has at most \( n \) distinct zeroes, then there exists a unique continuous selection \( s: C[a, b] \to G \) for \( P_G: C[a, b] \to 2^G \).

Now we will give some nontrivial examples of subspaces \( G \) in \( C[a, b] \) fulfilling the assumption of Corollary 9.

10. Examples. (a) \( G: = \langle x, x^2, \ldots, x^n \rangle \subset C[0, 1] \). \( G \) is Chebyshev in \((0, 1)\) and therefore the assumption of Corollary 9 is fulfilled, but \( G \) is not Chebyshev in \([0, 1]\).

(b) For \( n > 2 \) and \( n \) even, we define \( G: = \langle 1, x(1 - x^2), x^2, x^3(1 - x^2), x^4, \ldots, x^{n-1}(1 - x^2), x^n \rangle \subset C[-1, 1] \). The dimension of \( G \) is \( n + 1 \). Each
function \( g \) in \( G \) is a polynomial of degree \( \leq n + 1 \) and has therefore at most \( n + 1 \) zeroes in \([-1, 1]\). Such a function \( g \) can be written as \( g = g_1 - g_2 \) where

\[
g_1(x) = \sum_{i=0}^{n/2} a_{2i} x^{2i} \quad \text{and} \quad g_2(x) = x(1 - x^2) \sum_{i=1}^{n/2} a_{2i-1} x^{2i-2}.
\]

Because of the behaviour of \( g_1(x) \) and \( g_2(x) \) for \( x \to \pm \infty \) it can be shown that \( g_1 - g_2 \) has a zero in \((-\infty, -1] \cup [1, \infty)\). Therefore \( G \) is Chebyshev in \((-1, 1)\).

\( G \) is not Chebyshev in \([-1, 1]\) because there exists a function

\[
g_0(x) = x(1 - x^2) \sum_{i=1}^{n/2} a_{2i-1} x^{2i-2} \quad \text{in} \ G, \ g \neq 0,
\]

having exactly \( n + 1 \) zeroes in \([-1, 1]\).

A similar example has been given by Brown [1] in the case \( n = 5 \).

(c) \( G' = \langle |x|, x^3 \rangle \subset C[-1, 1] \). \( G \) is weak Chebyshev and each \( g \in G, \ g \neq 0, \) has at most 2 distinct zeroes, but \( G \) is not Chebyshev in \([-1, 1) \) or \((-1, 1]\).

Finally we ask how strong the assumption of Theorem 7 is for the uniqueness of \( A \)-elements and we show that this is the weakest condition because the converse of Theorem 7 is true.

11. Theorem. If \( G \) is an \( n \)-dimensional weak Chebyshev subspace of \( C[a, b] \) such that for each \( f \in C[a, b] \) there exists exactly one \( A \)-element in \( P_G(f) \) then each \( g \in G, \ g \neq 0, \) has at most \( n \) distinct zeroes.

Proof. Assumption. There exists a \( \tilde{g}_0 \) in \( G, \ \tilde{g}_0 \neq 0, \) with at least \( n + 1 \) distinct zeroes.

We define: \( g_0 := \tilde{g}_0/\|\tilde{g}_0\| \). Then \( \|g_0\| = 1 \).

Since \( G \) is weak Chebyshev, \( g_0 \) has at most \( n - 1 \) changes of sign. Therefore \( n + 1 \) distinct zeroes \( x_0, \ldots, x_n \) of \( g_0 \) exist such that \( e_i g_0(x) > 0, \ x \in [x_i, x_{i+1}], i = -1, 0, \ldots, n, \ e_i = \pm 1, x_{-1} = a, x_{n+1} = b. \)

We construct a function \( f \) in \( C[a, b] \), having two \( A \)-elements in \( P_G(f) \). We define \( f \) in the following way:

1. \( \varepsilon_{-1}(-1)^i f(x_i) = 1, \ i = 0, \ldots, n, \)
2. \( \|f\| = 1, \)
3. \( 0, g_0 \ \text{in} \ P_G(f). \)

Then \( g_0 \) and \( 0 \) are \( A \)-elements of \( f \).

Construction of \( f \):
(a) We may assume \( g > 0 \) for \( x \in [a, x_0] \).

We define: \( f(x) = 1 \) if \( x \in [a, x_0], (-1)^i f(x_i) = 1, i = 0, \ldots, n. \)
(b) Definition of $f$ in $[x_0, x_1]$

First case. $g_0(x) > 0$ if $x \in [x_0, x_1]$

Let $\tilde{x} = (x_0 + x_1)/2$ and $f(\tilde{x}) = 0$

Let $f$ be linear in $[x_0, \tilde{x}]

\[
f(x) = g_0(x) - g_0(\tilde{x}) + 2(g_0(\tilde{x}) - 1) \frac{x - \tilde{x}}{x_1 - x_0} \quad \text{if } x \in [\tilde{x}, x_1].\]

Second case. $g_0(x) < 0$ for $x \in [x_0, x_1]$

\[
f(x) = g_0(x) - g_0(\tilde{x}) + (1 + g_0(\tilde{x})) \frac{\tilde{x} - x}{\tilde{x} - x_0} \quad \text{if } x \in [x_0, \tilde{x}],
\]

\[
f(\tilde{x}) = 0.
\]

Let $f$ be linear in $[\tilde{x}, x_1]$

This construction of $f$ is continued in an analogous way for the intervals $[x_1, x_2], \ldots, [x_{n-1}, x_n], [x_n, b]$. Obviously $f$ is continuous in $[a, b]$

We show: $|f(x)| < 1$ if $x \in [x_0, x_1]$

In the first case:

\[
-1 < g_0(x) - g_0(\tilde{x}) + g_0(\tilde{x}) - 1
\]

\[
< g_0(x) - g_0(\tilde{x}) + 2(g_0(\tilde{x}) - 1) \frac{x - \tilde{x}}{x_1 - x_0}
\]

\[
= f(x) < g_0(x) - g_0(\tilde{x}) < g_0(x) < 1 \quad \text{if } x \in [\tilde{x}, x_1].
\]

In the second case:

\[
-1 < g_0(x) < g_0(x) - g_0(\tilde{x})
\]

\[
< g_0(x) - g_0(\tilde{x}) + (1 + g_0(\tilde{x})) \frac{\tilde{x} - x}{\tilde{x} - x_0}
\]

\[
= f(x) < g_0(x) - g_0(\tilde{x}) + (1 + g_0(\tilde{x}))
\]

\[
< g_0(x) + 1 < 1 \quad \text{if } x \in [x_0, \tilde{x}].
\]

Therefore $|f(x)| < 1$ if $x \in [x_0, x_1]$

We can show in an analogous way: $|f(x) - g_0(x)| < 1$ if $x \in [x_0, x_1]$

These estimations hold in each interval because of the construction of $f$

Therefore $f - 0$ and $f - g_0$ have $x_0, \ldots, x_n$ alternating extreme points.

If $0$ and $g_0$ are not in $P_G(f)$, then there would exist a function $g$ in $G$ such that $\|f - g\| < \|f\|$.

Therefore $f - 0$ and $f - g_0$ have $x_0, \ldots, x_n$ alternating extreme points.

This completes the proof.

Finally we show in Proposition 14 that a large class of weak Chebyshev
subspaces in $C[a, b]$ whose nonzero functions have only finitely many zeroes fulfill the assumption of Corollary 9 and therefore admit a unique continuous selection.

We need the following definition (cf. Singer [10, p. 126]):

12. Definition. A linear subspace $G$ of a normed linear space $E$ is called $k$-Chebyshev (where $k$ is an integer with $0 < k < \infty$), if for each $f$ in $E$ we have $0 \leq \dim P_G(f) < k$.

Finite-dimensional $k$-Chebyshev subspaces in $C(X)$ ($X$ compact) are characterized in Singer [10, p. 240]:

13. Theorem. If $G$ is an $n$-dimensional subspace of $C(X)$ ($X$ compact) and $k$ an integer with $0 < k < n - 1$. Then $G$ is a $k$-Chebyshev subspace if and only if there do not exist $n - k$ distinct points $x_1, \ldots, x_{n-k}$ in $X$ and $k + 1$ linearly independent functions $g_0, g_1, \ldots, g_k$ in $G$, such that

$$g_i(x_j) = 0, \quad j = 1, \ldots, n - k, \quad i = 0, 1, \ldots, k.$$  

Using the methods in the proof of Lemma 7 and Theorem 13 we can show in a straightforward manner that the following Proposition holds:

14. Proposition. If $G$ is an $n$-dimensional, weak Chebyshev subspace which is $(n - 1)$-Chebyshev and if each $g$ in $G$, $g \neq 0$, has only finitely many zeroes, then each $g$ in $G$, $g \neq 0$, has at most $n$ distinct zeroes.

References


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